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The Polynomial Ring over a Goldie Ring Need Not Be a Goldie Ring

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Rings satisfying chain conditions have been of interest for quite some time. The famous Wedderburn Theorem states that semisimple Artinian rings must be finite direct sums of matrix rings. Goldie's theorem links rings with ascending chain conditions to rings with descending chain conditions, thus extending Wedderburn's result. A ring R is said to be a (right) Goldie ring if R satisfies the ascending chain condition on (right) annihilator ideals and R contains no infinite direct sum of (right) ideals. From Goldie's theorem we know that a semiprime (right) Goldie ring must be an order in a semisimple (right) Artinian ring.

A natural question to arise is whether the matrix and polynomial rings over (right) Goldie rings are also necessarily (right) Goldie rings. Under certain additional hypotheses such as if the ring is an order in a (right) Artinian ring [4] or if the ring contains a certain type of uncountable set in its center [1], the answer is yes. Moreover, the second Goldie condition is always preserved [3]. Hence, as in the case for the matrix ring counter-example [2], we must explore the ascending chain condition on (right) annihilator ideals.

In the next section we construct a commutative Goldie ring $\mathbf{R}$ whose polynomial ring $\mathbf{R}[\mathbf{t}]$ contains two infinite sets of polynomials, $\{\mathbf{p}_i(\mathbf{t})\}$ and $\{\mathbf{q}_j(\mathbf{t})\}$, such that $\mathbf{p}_i(\mathbf{t})\,\mathbf{q}_j(\mathbf{t})=\mathbf{0}$ iff $i\neq j$. This condition forces $\mathbf{q}_k(\mathbf{t})$ to be in $\mathrm{Ann}(\{\mathbf{p}_i(\mathbf{t})\colon i>k\})$ and $\mathbf{q}_k(\mathbf{t})$ to be excluded from $\mathrm{Ann}(\{\mathbf{p}_i(\mathbf{t})\colon i>k-1\})$. Thus $\mathbf{R}[\mathbf{t}]$ has an infinite ascending chain of annihilator ideals

$$\ldots\mathrm{Ann}(\{\mathbf{p}_i(\mathbf{t})\colon i>k-1\})\subset\mathrm{Ann}(\{\mathbf{p}_i(\mathbf{t})\colon i>k\})\ldots$$

and hence, $\mathbf{R}[\mathbf{t}]$ fails the Goldie criteria.

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We obtain **R** in two stages. First we define a polynomial ring $Z_2[\mathbb{A}]$ and a prime ideal $\mathbb{P}$. The factor ring $Z_2[\mathbb{A}]/\mathbb{P}$ is our domain D. We then consider $R = D[U]$, the polynomials in $\{u_0, u_1, u_2, u_3\}$ over D. In R we determine two ideals, P and I. The prime ideal P plays an important role in all that follows. The ring R/I is **R**. Now we begin.

Let $\mathbb{A} = \{\mathbb{a}_{ij}$, for i in Z^+ and $j = 0, 1, 2, 3\}$; $U = \{u_0, u_1, u_2, u_3\}$; $U_{-1} = \{u_0^{-1}, u_1^{-1}, u_2^{-1}, u_3^{-1}\}$; $X = \{x_i$, for i in $Z^+\}$; and $\{c\}$ be sets of commuting indeterminates over Z_2. Let $\mathbb{y}, \mathbb{z}$ be elements in $Z_2[\mathbb{A}, U, U_{-1}, X, \{c\}]$. Define

$\deg(\mathbb{a}_{ij}) = (1, 0) = \deg(x_i)$;

$\deg(u_j) = (0, 1)$;

$\deg(c) = (0, 0)$;

$\deg_{\mathbb{A}}(\mathbb{y})$ = the total degree of $\mathbb{y}$ in the $\mathbb{a}_{ij}$;

$\deg_U(\mathbb{z})$ = the total degree of $\mathbb{z}$ in the u_j.

Consider the doubly graded Z_2-algebra homomorphism

$$\mathfrak{f}: Z_2[\mathbb{A}] \to Z_2[X, c, U, U_{-1}];$$
$$\mathbb{a}_{ij} \to c^j x_i u_0 u_j^{-1}.$$

Clearly $Z_2[X, c, U, U_{-1}]$ is a domain, so $\mathbb{P}$, the kernel of $\mathfrak{f}$, must be a prime homogeneous ideal. It is easy to see that the kernel of such a graded homomorphism is generated by elements of the form (monomial—monomial). For example,

$$(\mathbb{a}_{ij}\mathbb{a}_{km} + \mathbb{a}_{im}\mathbb{a}_{kj}) \text{ is a generator of } \mathbb{P}. \tag{1}$$

This element comes up again later. A quick glance at $\mathfrak{f}$ yields

$$\{\mathbb{y} \text{ such that } \deg_{\mathbb{A}}(\mathbb{y}) < 2\} \cap \mathbb{P} = 0. \tag{2}$$

Let y, a_{ij}, A be the images of $\mathbb{y}, \mathbb{a}_{ij}, \mathbb{A}$ in $D = Z_2[\mathbb{A}]/\mathbb{P}$. Extend $\mathfrak{f}$ to $D[U]$; that is,

$$f: D[U] \to Z_2[X, c, U, U_{-1}];$$
$$a_{ij} \to c^j x_i u_0 u_j^{-1};$$
$$u_j \to u_j.$$

Let $P = \ker f$ and denote by P_i the set of elements in P whose total degree in the u_j's is i. So $P_0 = (\ker f) \cap D = \ker \mathfrak{f} = 0$.

Note that in addition to the usual grading by degrees, we have another grading—a "weighted" grading W—defined on $D[U]$ by $W(\beta \prod_j u_j^{n_j}) = \sum_j jn_j$, where β is a monomial in the a_{ij}'s; on $Z_2[X, c, U, U_{-1}]$ by $W(\delta c^J \prod_j u_j^{n_j}) = J + \sum_j jn_j$, where δ is a monomial in the x_i's. The homomorphism f respects this weighted grading also, so $P = \ker f$ is a prime, homogeneous ideal with respect to both gradings.

Next we examine the elements that generate the R_0-modules P_1 and P_2. Using the weighted grading, the only elements of degree 1 in U we need to consider have the form βu_r where $\deg_U(\beta) = 0$. Obviously u_r is not in the prime ideal P. This forces β to be in P. But P_0 is 0. Thus $P_1 = 0$.

Straightforward calculations yield

$$\begin{aligned} \alpha_{2ik} &= a_{i0} a_{k2} u_0 u_2 + a_{i1} a_{k1} u_1^2 && \text{in } P; \\ \alpha_{3ik} &= a_{i0} a_{k3} u_0 u_3 + a_{i1} a_{k2} u_1 u_2 && \text{in } P; \\ \alpha_{4ik} &= a_{i1} a_{k3} u_1 u_3 + a_{i2} a_{k2} u_2^2 && \text{in } P. \end{aligned} \tag{3}$$

So clearly P_2 will not be trivial. We shall see that $\{\alpha_{2ik}, \alpha_{3ik}, \alpha_{4ik}\}$ generate the elements in P_2 of $\deg_A = 2$, and that $\boldsymbol{\alpha}_{2ik}$, $\boldsymbol{\alpha}_{3ik}$ turn out to be the coefficients of $\mathbf{t}^2$, $\mathbf{t}^3$ in the polynomial $\mathbf{p}_i(\mathbf{t})\,\mathbf{q}_k(\mathbf{t})$ in $\mathbf{R}[\mathbf{t}]$.

Suppose y_2, a difference of two monomials, is in P_2. Let $y_2 = (\prod_{i,j} a_{ij}^{n_{ij}})\, u_r u_v + (\prod_{i,j} a_{ij}^{m_{ij}})\, u_s u_w \neq 0$. Using the weighted grading, we can assume that $r + v = s + w$. Because $P_0 = 0 = P_1$ and P is prime, we immediately conclude $\{r, v\} \cap \{s, w\} = \varnothing$. There are only three possibilities for $\{(r, s), (v, w)\}$:

$$\{(0, 2), (1, 1)\}, \qquad \{(0, 3), (1, 2)\}, \qquad \{(1, 3), (2, 2)\}.$$

Furthermore, by examining the X and U parts of $f(y_2)$ and recalling (1), we see that if $\deg_A(y_2) < 3$, then y_2 must be of the forms described in (3).

Now we are ready to define I as the homogeneous (with respect to the degree grading) ideal generated by

$$\begin{aligned} G = \{&\text{all elements in } P \text{ of } \deg_A > 2;\ \alpha_{2jk},\ \alpha_{3jk}, \text{ for all } k \neq j; \\ &\alpha_{2ii} - \alpha_{3jj} \text{ for all } i, j;\ u_1 u_3,\ u_2^2,\ u_0^2,\ u_i u_j u_s \text{ for all } i, j, s\}. \end{aligned}$$

Let R be the domain $D[U]$ and $\mathbf{R}$ be the graded ring R/I. Let $\mathbf{y}$ denote the image of y in $\mathbf{R}$. Consider $\mathbf{p}_i(\mathbf{t}) = \mathbf{a}_{i0}\mathbf{u}_0 + \mathbf{a}_{i1}\mathbf{u}_1\mathbf{t}$, $\mathbf{q}_k(\mathbf{t}) = \mathbf{a}_{k0}\mathbf{u}_0 + \mathbf{a}_{k1}\mathbf{u}_1\mathbf{t} + \mathbf{a}_{k2}\mathbf{u}_2\mathbf{t}^2 + \mathbf{a}_{k3}\mathbf{u}_3\mathbf{t}^3$. Then the product $\mathbf{p}_i(\mathbf{t})\,\mathbf{q}_k(\mathbf{t}) = \alpha_{\mathbf{2ik}}\mathbf{t}^2 + \alpha_{\mathbf{3ik}}\mathbf{t}^3$ is zero precisely when $i \neq k$. Hence from our discussion in Section 1, we see that $\mathbf{R}[\mathbf{t}]$ is not a Goldie ring. Next we show $\mathbf{R}$ is a Goldie ring.

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In order to show that $\mathbf{R}$ has no infinite ascending chain of annihilators, we will explicitly determine its annihilator ideals. First note that $\mathbf{R}$ may be written as $\mathbf{R}_0+\mathbf{R}_1+\mathbf{R}_2$, where $\mathbf{R}_i$ is the ith component with respect to degree in $\mathbf{U}$. Denote by $\mathbf{y}_i$ the ith component of the element $\mathbf{y}$. Let R_P denote R localized at P and let $\mathscr{S}_1$ denote $\{\mathbf{y}_1 \in \mathbf{R}_1$ such that $\mathbf{y}_1=\sum_{i=0}^{3} \mathbf{c}_i\mathbf{u}_i$ or $\mathbf{y}_1=\mathbf{c}_0\mathbf{u}_0+\mathbf{c}_1\mathbf{u}_1$ or $\mathbf{y}_1=\mathbf{c}_0\mathbf{u}_0+\mathbf{c}_2\mathbf{u}_2$ where $c_i \in R_0$ and $c_i \neq 0\}$. We define equivalence classes on $\mathscr{S}_1$ of $\mathbf{R}_1$ by:

$\sum_{i=0}^{3} \mathbf{c}_i\mathbf{u}_i$ is equivalent to $\sum_{i=0}^{3} \mathbf{d}_i\mathbf{u}_i$ if $c_0/c_1=d_0/d_1$ and for $i, j=0, 1, 2$, $(c_iu_i)/(c_{i+1}u_{i+1})=(d_ju_j)/(d_{j+1}u_{j+1})$ in R_p;

$\mathbf{c}_0\mathbf{u}_0+\mathbf{c}_1\mathbf{u}_1$ is equivalent to $(\mathbf{d}_0\mathbf{u}_0+\mathbf{d}_1\mathbf{u}_1)$ if $c_0/c_1=d_0/d_1$ in R_P;

$\mathbf{c}_0\mathbf{u}_0+\mathbf{c}_2\mathbf{u}_2$ is equivalent to $(\mathbf{d}_0\mathbf{u}_0+\mathbf{d}_2\mathbf{u}_2)$ if $c_0/c_2=d_0/d_2$ in R_P.

Denote by $\mathscr{C}(\mathbf{y}_1)$ the equivalence class of $\mathbf{y}_1$.

THEOREM 1. (i) *Given $\mathbf{y}_1 \in \mathscr{S}_1$, the number of ideals in* $\{\mathrm{Ann}(\mathbf{z})$ *such that* $\mathbf{z}_0=\mathbf{0}$ *and* $\mathbf{z}_1 \in \mathscr{C}(\mathbf{y}_1)\}$ *is no more than two.*

(ii) *If* $\mathbf{y}_1, \mathbf{z}_1 \in \mathscr{S}_1$ *such that* $\mathrm{Ann}(\mathbf{y}, \mathbf{z}) \neq \mathbf{R}_2, \mathbf{0}$, *or* $\mathbf{P}$, *then* $\mathbf{y}_0=\mathbf{z}_0=\mathbf{0}$ *and* $\mathbf{z}_1 \in \mathscr{C}(\mathbf{y}_1)$.

(iii) *If* $\mathbf{y}_1$ *is not in* $\mathscr{S}_1$ *or if* $\mathbf{y}_0 \neq \mathbf{0}$, *then* $\mathrm{Ann}(\mathbf{y})$ *is in the following finite set of ideals* $\{\mathbf{0}, \mathbf{R}, \mathbf{R}_2, \mathbf{P}, \mathbf{R}_1\mathbf{R}, \mathbf{A}\mathbf{R}_0+\mathbf{R}_1\mathbf{R}_0+\mathbf{R}_2, \mathbf{u}_i\mathbf{R}$ *for* $i=0, 1, 2, 3\}$. *Furthermore,*

$$\mathrm{Ann}(\mathbf{A}\mathbf{R}_0)=\mathbf{P};$$
$$\mathrm{Ann}(\mathbf{P})=\mathbf{A}\mathbf{R}_0+\mathbf{R}_1\mathbf{R}_0+\mathbf{R}_2;$$
$$\mathrm{Ann}(\mathbf{u}_0)=\mathbf{u}_0\mathbf{R};$$
$$\mathrm{Ann}(\mathbf{u}_1)=\mathbf{u}_3\mathbf{R};$$
$$\mathrm{Ann}(\mathbf{u}_2)=\mathbf{u}_2\mathbf{R};$$
$$\mathrm{Ann}(\mathbf{u}_3)=\mathbf{u}_1\mathbf{R}.$$

Recall that the annihilator ideal of a set $\mathbf{S}$ is the intersection of the annihilators of the elements in $\mathbf{S}$. Theorem 1 implies that $\mathrm{Ann}\,\mathbf{S}$ will be an ideal other than those listed in (iii) only when all of the degree 1 parts of the elements in $\mathbf{S}$ fall into one equivalence class. Let $\mathscr{A}_y=\{\mathrm{Ann}\,\mathbf{S}$ such that if $\mathbf{z} \in \mathbf{S}$ then $\mathbf{z}_1$ is equivalent to $\mathbf{y}_1\}$. Parts (i) and (ii) of Theorem 1 imply $\mathscr{A}_y$ is a finite set. Note if $\mathrm{Ann}(\mathbf{S}_1) \subset \mathrm{Ann}(\mathbf{S}_2)$, then $\mathrm{Ann}(\mathbf{S}_1)=\mathrm{Ann}(\mathbf{S}_1 \cup \mathbf{S}_2)$, so we may assume $\mathbf{S}_2 \subset \mathbf{S}_1$. Thus an infinite chain cannot exist. The rather difficult proof of Theorem 1 appears in Section 4.

We show $\mathbf{R}$ has finite Goldie rank by determining $\mathbf{H}$, an essential ideal of rank 6. Let $\mathbf{J}=\mathbf{R}\mathbf{u}_0\mathbf{u}_1+\mathbf{R}\mathbf{u}_0\mathbf{u}_2+\mathbf{R}\mathbf{u}_0\mathbf{u}_3+\mathbf{R}\mathbf{u}_2\mathbf{u}_3+\mathbf{R}\mathbf{u}_3^2$. Each of the above direct summands of $\mathbf{J}$ is isomorphic as an $\mathbf{R}_0$-module to the domain

$\mathbf{R}_0$, and thus has rank 1. Hence $\mathbf{J}$ has rank 5. Note that $\mathbf{P}=\{\mathbf{0}, \boldsymbol{\alpha}\}$, where $\boldsymbol{\alpha}$ is the image of α_{2ii} in $\mathbf{R}$. Let $\mathbf{H}=\mathbf{J}+\mathbf{P}$. Then $\mathbf{H}$ has rank 6.

We need only show that $\mathbf{H}$ is essential. Let $\mathbf{K}$ be an ideal in $\mathbf{R}$. Then without loss of generality, there exists $\mathbf{k}$ in $\mathbf{K}$ such that $\mathbf{k}$ is in $\mathbf{R}_2$. (If $\mathbf{k}=\mathbf{k}_0+\mathbf{k}_1+\mathbf{k}_2$ with $\mathbf{k}_0\neq\mathbf{0}$, then $\mathbf{0}\neq\mathbf{u}_1\mathbf{u}_2\mathbf{k}_0$ is in $\mathbf{K}$. If $\mathbf{k}=\mathbf{k}_1+\mathbf{k}_2$ with $\mathbf{k}_1\neq\mathbf{0}$, then by part (iii) of Theorem 1, there exists an $\mathbf{u}_i$ such that $\mathbf{u}_i\mathbf{k}_1$ is a nonzero element of $\mathbf{K}$.) So $\mathbf{K}$ contains a nonzero element of the form $\mathbf{k}=\sum_{ij}\mathbf{k}_{ij}\mathbf{u}_i\mathbf{u}_j$, where $\mathbf{k}_{ij}$ is in $\mathbf{R}_0$. If $\mathbf{a}_{i1}\mathbf{a}_{j1}\mathbf{a}_{i2}\sum_{ij}\mathbf{k}_{ij}\mathbf{u}_i\mathbf{u}_j$, is zero, then by the proof of Theorem 1(iii), $\sum_{ij}\mathbf{k}_{ij}\mathbf{u}_i\mathbf{u}_j$ must be in $\mathbf{P}$, in which case we would be done. Otherwise from $\mathbf{A}\boldsymbol{\alpha}_{2ij}$, $\mathbf{A}\boldsymbol{\alpha}_{3ii}=\mathbf{0}$, we have

$$\begin{aligned}\mathbf{a}_{i1}\mathbf{a}_{j1}\mathbf{a}_{i2}\sum_{ij}\mathbf{k}_{ij}\mathbf{u}_i\mathbf{u}_j &= (\text{element in } \mathbf{H})+\mathbf{a}_{j1}\mathbf{k}_{12}(\mathbf{a}_{i1}\mathbf{a}_{i2}\mathbf{u}_1\mathbf{u}_2)\\ &\quad+\mathbf{a}_{i2}\mathbf{k}_{11}(\mathbf{a}_{i1}\mathbf{a}_{j1}\mathbf{u}_1^2)\\ &= (\text{element in } \mathbf{H})+\mathbf{a}_{j1}\mathbf{k}_{12}(\mathbf{a}_{i0}\mathbf{a}_{i3}\mathbf{u}_0\mathbf{u}_3)\\ &\quad+\mathbf{a}_{i2}\mathbf{k}_{12}(\mathbf{a}_{i0}\mathbf{a}_{j2}\mathbf{u}_0\mathbf{u}_2),\end{aligned}$$

which is in H. Therefore, H is essential.

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We now proceed with the proof of Theorem 1. Recall that $\mathbf{R}=R/I$ may be written as $\mathbf{R}_0+\mathbf{R}_1+\mathbf{R}_2$, where $\mathbf{R}_i$ is the ith component with respect to degree in $\mathbf{U}$. Denote by $\mathbf{y}_i$ the ith component of the element $\mathbf{y}$ (the image of y in $\mathbf{R}$). For $\mathbf{z}$ in $\mathbf{R}$, the product $\mathbf{yz}$ is $\mathbf{0}$ iff

$$y_0z_0\in I_0=P_0; \tag{4}$$

$$y_0z_1+y_1z_0\in I_1=P_1; \tag{5}$$

$$y_0z_2+y_1z_1+y_2z_0\in I_2\subset P_2+(R_0u_0^2+R_0u_1u_3+R_0u_2^2). \tag{6}$$

First we show that in the case $y_0\neq 0$, the annihilator of $\mathbf{y}$ must be either $\mathbf{0}$ or $\mathbf{P}$. Because P is prime and y_0 is not in P, using (4) we have z_0 in $P_0=0$. Substituting 0 for z_0 in (5), and using a similar argument, we have $z_1=0$. Substituting these values into (6) yields y_0z_2 in I_2. Now $z_2=\sum_{ij}z_{ij}u_iu_j$, for some $z_{ij}\in R_0$. Note that without loss of generality, we may assume $(i,j)\neq(0,0),(1,3),(2,2)$. Then $y_0z_2\in P_2\subset P$, hence $z_2\in P$. Thus, $\mathbf{z}_2$ is in $\mathbf{P}=\{0,\boldsymbol{\alpha}\}$, where $\boldsymbol{\alpha}$ is the image of α_{2ii} in $\mathbf{R}$. That is, $\operatorname{Ann}\mathbf{y}\subset\mathbf{P}$. Clearly if $\mathbf{y}_0\in\mathbf{AR}_0$ (where $\mathbf{AR}_0$ is the ideal in $\mathbf{R}_0$ generated by the $\mathbf{a}_{ij}$'s), then $\operatorname{Ann} \mathbf{y} \supset \mathbf{P}$ and so, $\operatorname{Ann} \mathbf{y} = \mathbf{P}$. Otherwise $\mathbf{y}_0 = \mathbf{1} + (\text{elt in } \mathbf{AR}_0)$ and $\operatorname{Ann} \mathbf{y} = \mathbf{0}$.

Now suppose $y_0 = 0$. Then if $z_0 \neq 0$, by a symmetrical argument $\mathbf{y}$ is in $\mathbf{P}$. In the nontrivial case $\mathbf{y} = \boldsymbol{\alpha}$, we have $\operatorname{Ann} \mathbf{y} = \mathbf{AR}_0 + \mathbf{R}_1\mathbf{R}_0 + \mathbf{R}_2$. Hence we have

LEMMA 2. (i) *If* $\mathbf{y}_0$ *is not in* $\mathbf{AR}_0$, *then* $\operatorname{Ann}(\mathbf{y}) = \mathbf{0}$;

(ii) *if* $\mathbf{y}_0$ *is nonzero and in* $\mathbf{AR}_0$, *then* $\operatorname{Ann}(\mathbf{y}) = \mathbf{P}$;

(iii) $\operatorname{Ann}(\mathbf{P}) = \mathbf{AR}_0 + \mathbf{R}_1\mathbf{R}_0 + \mathbf{R}_2$.

Next we consider the more complicated case, $y_0 = 0 = z_0$. Note that $q_k(1)\, p_i(1) = (\sum_{j=0}^{3} a_{kj} u_j 1^j)(\sum_{t=0}^{1} a_{it} u_t 1^t) = \alpha_{2ik} + \alpha_{3ik}$.

If $i \neq k$, then $\mathbf{q}_k(\mathbf{1})\,\mathbf{p}_i(\mathbf{1})$ is clearly $\mathbf{0}$; if $i = k$, then $\mathbf{q}_k(\mathbf{1})\,\mathbf{p}_i(\mathbf{1})$ is $\boldsymbol{\alpha} + \boldsymbol{\alpha} = 2\boldsymbol{\alpha} = \mathbf{0}$ since $\operatorname{ch} \mathbf{R}$ is 2. So there exist many nontrivial annihilators for elements in $\mathbf{R}_1 + \mathbf{R}_2$.

Let $y_1 = \sum_i c_i u_i$, $z_1 = \sum_i d_i u_i$. Then since $y_0 = 0 = z_0$, we see that (4), (5), and (6) become

$$y_1 z_1 = \sum_{i<j} (c_i d_j + c_j d_i)\, u_i u_j + c_0 d_0 u_0^2 + c_1 d_1 u_1^2 + c_2 d_2 u_2^2 + c_3 d_3 u_3^2 \in I. \tag{7}$$

From the weighted grading on P and the generators of I, we know no u_3^2 term and no $u_2 u_3$ term may exist in P_2, and hence not in I. So $c_3 d_3 = 0$, which implies c_3 or d_3 is zero. We assume c_3 is zero. Then the coefficient of $u_2 u_3$ becomes $c_2 d_3$, which must also be zero. Thus either c_2 or d_3 is zero.

Suppose c_2 is not zero. Then from $d_3 = 0$, (7), the generators of I, and the homogeneity of P with respect to the weighted grading, we have

$$(c_0 d_1 + c_1 d_0)\, u_0 u_1 \in P; \qquad \text{i.e., } c_0 d_1 + c_1 d_0 = 0; \tag{8}$$

$$(c_0 d_2 + c_2 d_0)\, u_0 u_2 + c_1 d_1 u_1^2 \in P; \tag{9}$$

$$(c_1 d_2 + c_2 d_1)\, u_1 u_2 \in P; \qquad \text{i.e., } c_1 d_2 + c_2 d_1 = 0. \tag{10}$$

If $c_2 \neq 0$ and $c_1 = 0$, then (10) implies d_1 is 0. In this case we have $\mathbf{y}_1 = \mathbf{c}_0\mathbf{u}_0 + \mathbf{c}_2\mathbf{u}_2$ and $\mathbf{z}_1 = \mathbf{d}_0\mathbf{u}_0 + \mathbf{d}_2\mathbf{u}_2$, so $\mathbf{y}_1\mathbf{z}_1$ is $\mathbf{0}$ iff $c_0 d_2 + c_2 d_0 = 0$.

Next we show that if c_2 is nonzero, then c_1 must be zero. For if c_1 is nonzero (with c_2 and z_1 nonzero), then d_1 is nonzero. (Note: If $d_1 = 0$, then (8) implies $d_0 = 0$ and (10) implies $d_2 = 0$. Similarly if $d_2 = 0$, then $z_1 = 0$.) So from $c_2, c_1 \neq 0$, we have d_1, d_2 are nonzero and the formulas in (8) and (10) yield

$$\begin{aligned} c_0/c_1 &= d_0/d_1; \\ c_1/c_2 &= d_1/d_2. \end{aligned} \tag{11}$$

From the equations in (11) we have $c_0/c_2 = d_0/d_2$, or equivalently, $(c_0d_2 + c_2d_0)\,u_0u_2$ is in P. Applying (9), this forces $c_1d_1u_1^2$ to be in P. So c_1 or d_1 must be zero, which is a contradiction. Hence the only nontrivial case for $c_2 \neq 0$ is where $c_1 = 0$, $d_1 = 0$ and $c_0d_2 + c_2d_0 = 0$.

Now suppose c_2 is zero. Then from (7), and from the generators of I, we have

$$(c_0d_1 + c_1d_0)\,u_0u_1 \in P; \qquad \text{i.e., } c_0d_1 + c_1d_0 = 0; \tag{8'}$$

$$c_0d_2u_0u_2 + c_1d_1u_1^2 \in P; \tag{9'}$$

$$c_0d_3u_0u_3 + c_1d_2u_1u_2 \in P. \tag{10'}$$

In R_P, the above formulas yield

$$\begin{aligned} c_0/c_1 &= d_0/d_1; \\ c_0u_0/c_1u_1 &= d_1u_1/d_2u_2; \\ c_0u_0/c_1u_1 &= d_2u_2/d_3u_3. \end{aligned} \tag{11'}$$

Hence we have proved

LEMMA 3. *Suppose* $\mathbf{y}, \mathbf{z} \in \mathbf{R}_1\mathbf{R}$. *If* $\mathbf{yz} \in \mathbf{P}$, *then one of the following cases holds.*

(i) *The product* $\mathbf{yz} = \mathbf{0}$ *if* $y_1 \in u_iR$ *and* $z_1 \in u_jR$, *where* $(i, j) \in \{(0,0), (2,2), (1,3)\}$.

(ii) *If* $y_1 = c_0u_0 + c_2u_2$ *with* c_0, c_2 *nonzero, then the product* $\mathbf{yz} \in \mathbf{P}$ *iff* $z_1 = d_0u_0 + d_2u_2$, *and* $c_0/c_2 = d_0/d_2$.

(iii) *If* $y_1 = c_0u_0 + c_1u_1$ *with* c_0, c_1 *nonzero, then the product* $\mathbf{yz} \in \mathbf{P}$ *iff* $z_1 = d_0u_0 + d_1u_1 + d_2u_2 + d_3u_3$, *and* (11′) *holds.*

(iv) *If* $z_1 = d_0u_0 + d_1u_1 + d_2u_2 + d_3u_3$ *with* d_i *nonzero, then the product* $\mathbf{yz} \in \mathbf{P}$ *iff* $y_1 = c_0u_0 + c_1u_1$ *and* (11′) *holds.*

Remark 4. Lemmas 2 and 3 imply if $\mathbf{y}_1, \mathbf{x}_1 \in \mathscr{S}_1$ such that $\mathrm{Ann}(\mathbf{y}, \mathbf{x}) \neq \mathbf{R}_2$, $\mathbf{0}$, or $\mathbf{P}$, then $\mathbf{y}_0 = \mathbf{x}_0 = \mathbf{0}$, $\mathbf{x}_1 \in \mathscr{C}(\mathbf{y}_1)$. Furthermore if $\mathbf{x}_1 \in \mathscr{C}(\mathbf{y}_1)$, $\mathbf{w}_1 \in \mathscr{C}(\mathbf{z}_1)$ with $\mathbf{y}_1\mathbf{z}_1 \in \mathbf{P}$ then $\mathbf{x}_1\mathbf{w}_1 \in \mathbf{P}$.

To complete the proof of Theorem 1, we shall show

LEMMA 5. *Given* $\mathbf{y}_1 \in \mathscr{S}_1$, *the number of ideals in* $\{\mathrm{Ann}(\mathbf{x})$ *such that* $\mathbf{x}_0 = \mathbf{0}$ *and* $\mathbf{x}_1 \in \mathscr{C}(\mathbf{y}_1)\}$ *is no more than two.*

Proof. Suppose $x_0 = y_0 = z_0 = 0$ and $x_1 = b_0u_0 + b_1u_1$, $y_1 = c_0u_0 + c_1u_1$, $z_1 = \sum_{i=0}^{3} d_iu_i$, and $\mathbf{y}_1\mathbf{z}_1 = \boldsymbol{\alpha}$. We shall show that $\mathscr{C}$, the equivalence class of $\mathbf{y}_1$, can be written as the union of $\mathscr{C}_1$ and $\mathscr{C}_2$, two disjoint subsets, such that if $\mathbf{x}_1 \in \mathscr{C}_1$ then $\operatorname{Ann} \mathbf{x} = \operatorname{Ann} \mathbf{y}$ and if $\mathbf{x}_1 \in \mathscr{C}_2$ then $\operatorname{Ann} \mathbf{x} = \operatorname{Ann} \mathscr{C}_2$. A similar argument applies to the other elements in $\mathscr{S}_1$.

By Remark 4, if $\mathbf{x}_1$ is equivalent to $\mathbf{y}_1$ then we have $\mathbf{xz} \in \mathbf{P}$. Since $\boldsymbol{\alpha}$ is obtained from the lowest $\deg_A$ part of its preimage in R, it is only the lowest degree parts of x_1 and z_1 that determine whether $\mathbf{xz} = \boldsymbol{\alpha}$ or $\mathbf{0}$. After a moment's thought we realize that for our purposes, we may assume the b_i's, c_i's, and d_i's are homogeneous by disregarding all but the lowest $\deg_A$ parts. Because $\mathbf{x}_1$ and $\mathbf{y}_1$ are equivalent, we have $b_0 c_1 = b_1 c_0$. The two subsets $\mathscr{C}_1$ and $\mathscr{C}_2$, will be determined by $\deg_A b_1$ and $\deg_A c_1$.

Since $\mathbf{y}_1 \mathbf{z}_1 = \boldsymbol{\alpha}$, part of $c_0 d_2 u_0 u_2 + c_1 d_1 u_1^2 + c_0 d_3 u_0 u_3 + c_1 d_2 u_1 u_2$ has degree 2 in A. Without loss of generality, assume the $\deg_A = 2$ part occurs in $c_0 d_2 u_0 u_2 + c_1 d_1 u_1^2$. Note that $\deg_A b_1$ cannot be less than $\deg_A c_1$. (Otherwise from $b_0 c_1 = b_1 c_0$, we would have $\deg_A b_0$ less than $\deg_A c_0$ and so $\deg_A(b_0 d_2 u_0 u_2 + b_1 d_1 u_1^2)$ is less than $\deg_A(c_0 d_2 u_0 u_2 + c_1 d_1 u_1^2) = 2$. This is impossible since the weight 2 part of $x_1 z_1$ $W_2(x_1 y_1) = b_0 d_2 u_0 u_2 + b_1 d_1 u_1^2 \in P$.) If $\deg_A b_1$ exceeds $\deg_A c_1$, then by a similar argument we see that every part of $b_0 d_2 u_0 u_2 + b_1 d_1 u_1^2 + b_0 d_3 u_0 u_3 + b_1 d_2 u_1 u_2$ has $\deg_A$ greater than 2. Hence, from $\mathbf{xz} \in \mathbf{P}$, we have $\mathbf{xz}$ is zero if $\deg_A b_1 > \deg_A c_1$. Next we show that if $\deg_A b_1 = \deg_A c_1$, then the weight 2 part of $\mathbf{xz}$ is the same as the weight 2 part of $\mathbf{yz}$. Moreover in this case $b_1 = c_1$ and $b_0 = c_0$, so $\operatorname{Ann} \mathbf{x} = \operatorname{Ann} \mathbf{y}$.

Assume $\deg_A b_1 = \deg_A c_1$ and the $\deg_A = 2$ part of yz is of weight 2; i.e., it occurs in $c_0 d_2 u_0 u_2 + c_1 d_1 u_1^2$. From $b_0 c_1 = b_1 c_0$ we have $b_1 y_1 z_1 = c_1 x_1 z_1$ and

$$W_2(b_1 y_1 z_1) = b_1 \sum_{(i,j) \in \mathscr{J}} \alpha_{2ij} = b_1 \sum_{(i,j) \in \mathscr{J}} \{a_{i0} a_{j2} u_0 u_2 + a_{i1} a_{j1} u_1^2\}.$$

Since $\deg_A b_1 = \deg_A c_1$ and $x_1 z_1 \in P$, $W_2(c_1 x_1 z_1)$ must have the form

$$c_1 \sum_{(k,m) \in \mathscr{K}} \{a_{k0} a_{m2} u_0 u_2 + a_{k1} a_{m1} u_1^2\}.$$

By considering the "smallest" (i, j) in $\mathscr{J}$ and the "smallest" (k, m) in $\mathscr{K}$ we see that, in order for $W_2(b_1 y_1 z_1) = W_2(c_1 x_1 z_1)$, $\deg_A b_1$ and $\deg_A c_1$ would have to be greater than 2 unless $(i, j) = (k, l)$. Hence by an inductive argument, we have $\mathscr{J} = \mathscr{K}$, or equivalently, $b_1 = c_1$. Thus $b_0 = c_0$. A similar argument works for the weight 3 parts, so if $b_1 = c_1$ then $\mathbf{xz} = \boldsymbol{\alpha}$. Moreover, for any $w_0 \in \mathbf{R}_1 \mathbf{R} \cap \mathscr{C}(\mathbf{z}_1)$ we have $\mathbf{xw} = \mathbf{yw}$. A similar argument works for the other equivalence classes. Note that the above two paragraphs imply that if there exist $\mathbf{z}, \mathbf{z}'$ for two equivalent elements $\mathbf{x}, \mathbf{x}'$ such that $\mathbf{xz} = \mathbf{x'z'} = \boldsymbol{\alpha}$, then the lowest degree coefficients of the u_i's in x are the same as those in x' and $\operatorname{Ann} \mathbf{x} = \operatorname{Ann} \mathbf{x}'$. Given an equivalence class either there exists some y_1 in it such that $\mathbf{yz} = \boldsymbol{\alpha}$, or not. If not there is only one annihilator associated with that class. If there is such a y_1, let $\mathscr{C}_1$ be the set of elements in $\mathscr{C}$ with the lowest degree coefficients of the u_i's the same as those of y_1, and let $\mathscr{C}_2$, be the compliment of $\mathscr{C}_1$ in $\mathscr{C}$. We are done.

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