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A geometric degree formula for A-discriminants and Euler obstructions of toric varieties

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Abstract

We give explicit formulas for the dimensions and the degrees of A-discriminant varieties introduced by Gelfand, Kapranov and Zelevinsky. Our formulas can be applied also to the case where the A-discriminant varieties are higher-codimensional and their degrees are described by the geometry of the configurations A. Moreover combinatorial formulas for the Euler obstructions of general (not necessarily normal) toric varieties will be also given.

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1. Introduction

The theory of discriminants is on the crossroad of various branches of mathematics, such as commutative algebra, combinatorics, algebraic geometry, singularity theory and topology. In [14], Gelfand, Kapranov and Zelevinsky generalized this classical theory to polynomials of several variables by introducing A-discriminant varieties and obtained many deep results. They thus laid the foundation of the modern theory of discriminants. The first aim of this paper is to give formulas for the dimensions and the degrees of A-discriminant varieties.

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Now let $M \simeq \mathbb{Z}^n$ be a $\mathbb{Z}$-lattice (free $\mathbb{Z}$-module) of rank n and $M_{\mathbb{R}} := \mathbb{R} \otimes_{\mathbb{Z}} M$ the real vector space associated with M. Let $A \subset M$ be a finite subset of M and denote by P its convex hull in $M_{\mathbb{R}}$. In this paper, we assume that $\dim P = n$. If $A = \{\alpha(1), \alpha(2), \dots, \alpha(m+1)\}$, we can define a morphism $\Phi_A : T \to \mathbb{P}^m$ ($m := \sharp A - 1$) from an algebraic torus $T := \operatorname{Spec}(\mathbb{C}[M]) = (\mathbb{C}^*)^n$ to a complex projective space $\mathbb{P}^m$ by

$$x = (x_1, x_2, \dots, x_n) \mapsto \left[x^{\alpha(1)} : x^{\alpha(2)} : \dots : x^{\alpha(m+1)}\right], \tag{1.1}$$

where for each $\alpha(i) \in A \subset M \simeq \mathbb{Z}^n$ and $x \in T$ we set $x^{\alpha(i)} = x_1^{\alpha(i)_1} x_2^{\alpha(i)_2} \cdots x_n^{\alpha(i)_n}$ as usual.

Definition 1.1. (See [14].) Let $X_A := \overline{\operatorname{im} \Phi_A}$ be the closure of the image of $\Phi_A : T \to \mathbb{P}^m$. Then the dual variety $X_A^* \subset (\mathbb{P}^m)^*$ of X_A is called the A-discriminant variety. If moreover X_A^* is a hypersurface in the dual projective space $(\mathbb{P}^m)^*$, then the defining homogeneous polynomial of X_A^* (which is defined up to non-zero constant multiples) is called the A-discriminant.

Note that the A-discriminant variety X_A^* is naturally identified with (the projectivization of) the closure of the set of Laurent polynomials $f : T = (\mathbb{C}^*)^n \to \mathbb{C}$ of the form $f(x) = \sum_{\alpha \in A} a_\alpha x^\alpha$ ($a_\alpha \in \mathbb{C}$) such that $\{x \in T \mid f(x) = 0\}$ is a singular hypersurface in T. In order to introduce the degree formula for A-discriminants proved by Gelfand et al. [14], we need the following.

Definition 1.2. (See [14].) For a subset $B \subset M \simeq \mathbb{Z}^n$, we define an affine $\mathbb{Z}$-sublattice $M(B)$ of M by

$$M(B) := \left\{ \sum_{v \in B} c_v \cdot v \;\middle|\; c_v \in \mathbb{Z},\ \sum_{v \in B} c_v = 1 \right\}. \tag{1.2}$$

Let $\Delta \prec P$ be a face of P and denote by $\mathbb{L}(\Delta)$ the smallest affine subspace of $M_{\mathbb{R}}$ containing Δ. Then $M(A \cap \Delta)$ is a $\mathbb{Z}$-lattice of rank $\dim \Delta = \dim \mathbb{L}(\Delta)$ in $\mathbb{L}(\Delta)$ and we have $(M(A \cap \Delta))_{\mathbb{R}} \simeq \mathbb{L}(\Delta)$. After fixing a $\mathbb{Z}$-basis of the lattice $M(A \cap \Delta)$, let vol be the Lebesgue measure of $(\mathbb{L}(\Delta), M(A \cap \Delta))$ by which the volume of the $(\dim \Delta)$-dimensional standard (smallest and integral) cube in it is measured to be 1. For a subset $K \subset \mathbb{L}(\Delta)$, we set

$$\operatorname{Vol}_{\mathbb{Z}}(K) := (\dim \Delta)! \cdot \operatorname{vol}(K). \tag{1.3}$$

We call it the normalized $(\dim \Delta)$-dimensional volume of K with respect to the lattice $M(A \cap \Delta)$. In other words, for the $(\dim \Delta)$-dimensional standard simplex S in $(\mathbb{L}(\Delta), M(A \cap \Delta))$ we set $\operatorname{Vol}_{\mathbb{Z}}(S) = 1$. Then the normalized volume of any integral polytope in $(\mathbb{L}(\Delta), M(A \cap \Delta))$ is an integer. Throughout this paper, we use this normalized volume $\operatorname{Vol}_{\mathbb{Z}}$ instead of the usual one vol.

The following formula is obtained by Gelfand et al. [14, Chapter 9, Theorem 2.8].

Theorem 1.3. *(See [14].) Assume that $X_A \subset \mathbb{P}^m$ is smooth and X_A^* is a hypersurface in $(\mathbb{P}^m)^*$. Then the degree of the A-discriminant is given by the formula:*

$$\deg X_A^* = \sum_{\Delta \prec P} (-1)^{\operatorname{codim} \Delta} (\dim \Delta + 1) \operatorname{Vol}_{\mathbb{Z}}(\Delta). \tag{1.4}$$

In order to state our generalization of Theorem 1.3 to the case where X_A^* may be higher-codimensional, recall that $T = \mathrm{Spec}(\mathbb{C}[M])$ acts naturally on X_A and we have a basic correspondence ($0 \leqslant k \leqslant n = \dim P$):

$$\{k\text{-dimensional faces of } P\} \overset{1:1}{\longleftrightarrow} \{k\text{-dimensional } T\text{-orbits in } X_A\} \tag{1.5}$$

proved by [14, Chapter 5, Proposition 1.9]. For a face $\Delta \prec P$ of P, we denote by T_Δ the corresponding T-orbit in X_A. We denote the value of the Euler obstruction $\mathrm{Eu}_{X_A} : X_A \to \mathbb{Z}$ of X_A on T_Δ by $\mathrm{Eu}(\Delta) \in \mathbb{Z}$. The precise definition of the Euler obstruction will be given later in Section 4. Here we simply recall that the Euler obstruction of X_A is constant along each T-orbit T_Δ and takes the value 1 on the smooth part of X_A. In particular, for $\Delta = P$ the T-orbit T_Δ is open dense in X_A and $\mathrm{Eu}(\Delta) = 1$. Then we have

Theorem 1.4. *For $1 \leqslant i \leqslant m$, set*

$$\delta_i := \sum_{\Delta \prec P} (-1)^{\mathrm{codim}\Delta} \left\{ \binom{\dim \Delta - 1}{i} + (-1)^{i-1}(i+1) \right\} \mathrm{Vol}_{\mathbb{Z}}(\Delta) \cdot \mathrm{Eu}(\Delta). \tag{1.6}$$

Then the codimension $r = \mathrm{codim} X_A^* = m - \dim X_A^*$ and the degree of the dual variety X_A^* are given by

$$r = \mathrm{codim} X_A^* = \min\{i \mid \delta_i \neq 0\}, \tag{1.7}$$

$$\deg X_A^* = \delta_r. \tag{1.8}$$

Remark 1.5.

(i) For $p \in \mathbb{Z}$ and $q \in \mathbb{Z}_{\geqslant 0}$, we used the generalized binomial coefficient

$$\binom{p}{q} = \frac{p(p-1)(p-2)\cdots(p-q+1)}{q!}. \tag{1.9}$$

For example, for a vertex $\Delta = \{v\} \prec P$, we have $\binom{\dim \Delta - 1}{i} = \binom{-1}{i} = (-1)^i$.

(ii) The number $\mathrm{codim} X_A^* - 1$ is called the dual defect of X_A.

Corollary 1.6. *Assume that X_A^* is a hypersurface in $(\mathbb{P}^m)^*$. Then the degree of the A-discriminant is given by*

$$\deg X_A^* = \sum_{\Delta \prec P} (-1)^{\mathrm{codim}\Delta} (\dim \Delta + 1) \mathrm{Vol}_{\mathbb{Z}}(\Delta) \cdot \mathrm{Eu}(\Delta). \tag{1.10}$$

The above theorem will be proved by using Ernström's degree formula for dual varieties in [8] and our result in [24]. Note that very recently by using tropical algebraic geometry, also Dickenstein et al. [5] obtained a totally different degree formula for the A-discriminant variety X_A^*. Their main application of the results in [5] is a generalization of the degree formula for X_A^* in [14] to the case where X_A^* is higher-codimensional. Indeed, if X_A^* is a hypersurface the degree formula for X_A^* is implicit in Gelfand–Kapranov–Zelevinsky's prime factorization theorem [14, Chapter 10, Theorem 1.2]. The formula in [5] is described by some combinatorics such as matroids related to P. On the other hand, our formula is described by the (normalized) volumes of the faces of P as the results in [14]. In particular, if X_A is a smooth hypersurface, our formula coincides with Gelfand–Kapranov–Zelevinsky's theorem (Theorem 1.3). In Section 4, we will give geometric formulas which express the Euler obstruction $\mathrm{Eu}_{X_A}: X_A \to \mathbb{Z}$ of X_A in terms of the normalized volumes of the faces of P. Note that a beautiful formula for the Euler obstructions of 2-dimensional normal toric varieties was proved by Gonzalez-Sprinberg [15]. Our formulas generalize it to arbitrary (not necessarily normal) toric varieties. Combining them with Theorem 1.4 above, we can describe the dimension and the degree of X_A^* by the geometry of P for any configuration $A \subset M = \mathbb{Z}^n$. Recently in [9] Esterov found some nice applications of our formulas.

Our functorial proof of the formula for the Euler obstruction $\mathrm{Eu}_{X_A}: X_A \to \mathbb{Z}$ leads us to other applications. In Section 5, we derive from it some formulas (Theorems 5.3 and 5.4) for the characteristic cycles of T-equivariant constructible sheaves on general (not necessarily normal) toric varieties. See [2] for another approach to this problem. In particular, combining it with the combinatorial description of the intersection cohomology complexes of toric varieties obtained by Bernstein, Khovanskii and MacPherson (unpublished), Denef and Loeser [4] and Fieseler [10], etc., we can derive combinatorial formulas for the characteristic cycle of the intersection cohomology complex of any normal toric variety. See Section 5 for the details. Note that in [14, Chapter 10, Theorem 2.11] also Gelfand et al. obtained a formula for the characteristic cycles in a special but important case, from which they could have obtained the same result by some generalization. However we included here a proof of Theorems 5.3 and 5.4, since we cannot find such an explicit presentation in the literature and the lack of it seems to be the source of many misunderstandings. For example, even in a special case, Schulze and Walther [27] corrected the formula in [13] for the characteristic cycles of the A-hypergeometric systems only very recently. We tried to show the power and the beauty of the sheaf-theoretical methods (see for example, [6,16,18]) by proving Theorems 4.7 and 5.4 functorially. Moreover, during the proof of these results, we obtained an explicit description (4.43) of the branches along torus orbits in non-normal toric varieties found in [14, Chapter 5, Theorem 3.1] and gave a rigorous justification to the argument on non-normal toric varieties in [14, Chapter 5, Theorem 3.1]. This explicit description seems to be new and useful in non-normal toric geometry. We believe that Theorems 1.4, 4.3 and 4.7 are new, although our proof of Theorem 4.7 heavily depends on some ingenious constructions in [14]. Finally, let us mention that combining our combinatorial description of the Euler obstructions of toric varieties with the result of Ehlers (unpublished) and Barthel et al. [1] we can now compute the Chern–Mather classes of complete toric varieties very easily.

2. Preliminary notions and results

In this section, we introduce basic notions and results which will be used in this paper. In this paper, we essentially follow the terminology of [6,16,18]. For example, for a topological space X we denote by $\mathbf{D}^b(X)$ the derived category whose objects are bounded complexes of sheaves of $\mathbb{C}_X$-modules on X.

Definition 2.1. Let X be an algebraic variety over $\mathbb{C}$. Then

(i) We say that a sheaf $\mathcal{F}$ on X is constructible if there exists a stratification $X = \bigsqcup_\alpha X_\alpha$ of X such that $\mathcal{F}|_{X_\alpha}$ is a locally constant sheaf of finite rank for any α.

(ii) We say that an object $\mathcal{F}$ of $\mathbf{D}^b(X)$ is constructible if the cohomology sheaf $H^j(\mathcal{F})$ of $\mathcal{F}$ is constructible for any $j \in \mathbb{Z}$. We denote by $\mathbf{D}^b_c(X)$ the full subcategory of $\mathbf{D}^b(X)$ consisting of constructible objects $\mathcal{F}$.

Recall that for any morphism $f : X \to Y$ of algebraic varieties over $\mathbb{C}$ there exist functors

$$Rf_* : \mathbf{D}^b(X) \to \mathbf{D}^b(Y), \qquad Rf_* : \mathbf{D}^b_c(X) \to \mathbf{D}^b_c(Y) \tag{2.1}$$

of direct images. For other basic operations $Rf_!$, f^{-1}, $f^!$, etc. in derived categories, see [18] for the details.

Next we introduce the notion of constructible functions.

Definition 2.2. Let X be an algebraic variety over $\mathbb{C}$. Then we say a $\mathbb{Z}$-valued function $\rho : X \to \mathbb{Z}$ on X is constructible if there exists a stratification $X = \bigsqcup_\alpha X_\alpha$ of X such that $\rho|_{X_\alpha}$ is constant for any α. We denote by $\mathrm{CF}_{\mathbb{Z}}(X)$ the abelian group of constructible functions on X.

For a constructible function $\rho : X \to \mathbb{Z}$ take a stratification $X = \bigsqcup_\alpha X_\alpha$ of X such that $\rho|_{X_\alpha}$ is constant for any α as above. Denoting the Euler characteristic of X_α by $\chi(X_\alpha)$ we set

$$\int_X \rho := \sum_\alpha \chi(X_\alpha) \cdot \rho(x_\alpha) \in \mathbb{Z}, \tag{2.2}$$

where x_α is a reference point in X_α. Then we can easily show that $\int_X \rho \in \mathbb{Z}$ does not depend on the choice of the stratification $X = \bigsqcup_\alpha X_\alpha$ of X. We call $\int_X \rho \in \mathbb{Z}$ the topological (Euler) integral of ρ over X.

Among various operations in derived categories, the following nearby and vanishing cycle functors introduced by Deligne will be frequently used in this paper (see [6, Section 4.2] for an excellent survey of this subject).

Definition 2.3. Let $f : X \to \mathbb{C}$ be a non-constant regular function on an algebraic variety X over $\mathbb{C}$. Set $X_0 := \{x \in X \mid f(x) = 0\} \subset X$ and let $i_X : X_0 \hookrightarrow X$, $j_X : X \setminus X_0 \hookrightarrow X$ be inclusions. Let $p : \widetilde{\mathbb{C}^*} \to \mathbb{C}^*$ be the universal covering of $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ ($\widetilde{\mathbb{C}^*} \simeq \mathbb{C}$) and consider the Cartesian square

$$\begin{tikzcd} \widetilde{X \setminus X_0} \arrow[r] \arrow[d, "p_X"'] \arrow[dr, phantom, "\square"] & \widetilde{\mathbb{C}^*} \arrow[d, "p"] \\ X \setminus X_0 \arrow[r, "f"] & \mathbb{C}^*. \end{tikzcd} \tag{2.3}$$

Then for $\mathcal{F} \in \mathbf{D}^b(X)$ we set

$$\psi_f(\mathcal{F}) := i_X^{-1} R(j_X \circ p_X)_* (j_X \circ p_X)^{-1} \mathcal{F} \in \mathbf{D}^b(X_0) \tag{2.4}$$

and call it the nearby cycle of $\mathcal{F}$. We also define the vanishing cycle $\varphi_f(\mathcal{F}) \in \mathbf{D}^b(X_0)$ of $\mathcal{F}$ to be the third term of the distinguished triangle:

$$i_X^{-1}\mathcal{F} \longrightarrow \psi_f(\mathcal{F}) \longrightarrow \varphi_f(\mathcal{F}) \xrightarrow{+1} \tag{2.5}$$

in $\mathbf{D}^b(X_0)$, where $i_X^{-1}\mathcal{F} \to \psi_f(\mathcal{F})$ is the morphism induced by $\mathrm{id} \to R(j_X \circ p_X)_*(j_X \circ p_X)^{-1}$.

Since nearby and vanishing cycle functors preserve the constructibility, in the above situation we obtain functors

$$\psi_f, \varphi_f : \mathbf{D}_c^b(X) \to \mathbf{D}_c^b(X_0). \tag{2.6}$$

The following theorem will play a crucial role in this paper. For the proof, see for example, [6, Proposition 4.2.11].

Theorem 2.4. *(See [6].) Let $\pi : Y \to X$ be a proper morphism of algebraic varieties over $\mathbb{C}$ and $f : X \to \mathbb{C}$ a non-constant regular function on X. Set $g := f \circ \pi : Y \to \mathbb{C}$, $X_0 := \{x \in X \mid f(x) = 0\}$ and $Y_0 := \{y \in Y \mid g(y) = 0\} = \pi^{-1}(X_0)$. Then for any $\mathcal{G} \in \mathbf{D}_c^b(Y)$ we have*

$$R(\pi|_{Y_0})_*\psi_g(\mathcal{G}) = \psi_f(R\pi_*\mathcal{G}), \tag{2.7}$$

$$R(\pi|_{Y_0})_*\varphi_g(\mathcal{G}) = \varphi_f(R\pi_*\mathcal{G}) \tag{2.8}$$

in $\mathbf{D}_c^b(X_0)$, where $\pi|_{Y_0} : Y_0 \to X_0$ is the restriction of π.

Let us recall the definition of characteristic cycles of constructible sheaves. Let X be a smooth algebraic variety over $\mathbb{C}$ and $\mathcal{F} \in \mathbf{D}_c^b(X)$. Then there exists a Whitney stratification $X = \bigsqcup_\alpha X_\alpha$ of X consisting of connected strata X_α such that $H^j(\mathcal{F})|_{X_\alpha}$ is a locally constant sheaf for any $j \in \mathbb{Z}$ and α. For a point $x_\alpha \in X_\alpha$, take a holomorphic function $f : U_\alpha \to \mathbb{C}$ defined in a neighborhood U_α of x_α in X which satisfies the conditions

(i) $f(x_\alpha) = 0$,
(ii) $(x_\alpha; \mathrm{grad} f(x_\alpha)) \in T^*_{X_\alpha}X \setminus (\bigcup_{\beta \neq \alpha} \overline{T^*_{X_\beta}X})$,
(iii) $x_\alpha \in X_\alpha$ is a non-degenerate critical point of $f|_{X_\alpha}$,

and set

$$m_\alpha := -\chi\big(\varphi_f(\mathcal{F})_{x_\alpha}\big) \tag{2.9}$$

$$= -\sum_{j \in \mathbb{Z}} (-1)^j \dim_{\mathbb{C}}\big(H^j\big(\varphi_f(\mathcal{F})\big)_{x_\alpha}\big) \in \mathbb{Z}. \tag{2.10}$$

Then we can show that the integer m_α does not depend on the choice of the stratification $X = \bigsqcup_\alpha X_\alpha$, $x_\alpha \in X_\alpha$ and f.

Definition 2.5. By using the above integers $m_\alpha \in \mathbb{Z}$, we define a Lagrangian cycle $CC(\mathcal{F})$ in the cotangent bundle T^*X of X by

$$CC(\mathcal{F}) := \sum_\alpha m_\alpha \big[\overline{T^*_{X_\alpha}X}\big]. \tag{2.11}$$

We call $CC(\mathcal{F})$ the characteristic cycle of $\mathcal{F} \in \mathbf{D}_c^b(X)$. Its coefficient $m_\alpha \in \mathbb{Z}$ is called the multiplicity of $\mathcal{F}$ along the Lagrangian subvariety $\overline{T^*_{X_\alpha} X} \subset T^*X$.

Recall that in $\mathbf{D}_c^b(X)$ there exists a full abelian subcategory $\mathrm{Perv}(X)$ of perverse sheaves (see [16,18], etc. for the details of this subject). Although for the definition of perverse sheaves there are some different conventions of shifts in the literature, here we adopt the one in [16] by which the shifted constant sheaf $\mathbb{C}_X[\dim X] \in \mathbf{D}_c^b(X)$ on a smooth algebraic variety X is perverse. Then for any perverse sheaf $\mathcal{F} \in \mathrm{Perv}(X) \subset \mathbf{D}_c^b(X)$ on a smooth algebraic variety X we can easily show that the multiplicities in the characteristic cycle $CC(\mathcal{F})$ of $\mathcal{F}$ are non-negative.

Example 2.6. Let $X = \mathbb{C}^n_x$ and $Y = \{x_1 = \cdots = x_d = 0\} \subset X = \mathbb{C}^n_x$. Set $\mathcal{F} := \mathbb{C}_Y[n-d] \in \mathrm{Perv}(X)$. Then by an easy computation

$$m = -\chi\left(\varphi_f\left(\mathbb{C}_Y[n-d]\right)_0\right) = 1 \tag{2.12}$$

for $f(x) = x_1 + x_{d+1}^2 + \cdots + x_n^2$ at $0 \in Y \subset X = \mathbb{C}^n_x$ we obtain

$$CC(\mathcal{F}) = 1 \cdot \left[T^*_Y X\right]. \tag{2.13}$$

Finally, we recall a special case of Bernstein–Khovanskii–Kushnirenko's theorem [19,20].

Definition 2.7. Let $g(x) = \sum_{v \in \mathbb{Z}^n} a_v x^v$ be a Laurent polynomial on $(\mathbb{C}^*)^n$ ($a_v \in \mathbb{C}$). We call the convex hull of $\mathrm{supp}(g) := \{v \in \mathbb{Z}^n \mid a_v \neq 0\} \subset \mathbb{Z}^n \subset \mathbb{R}^n$ in $\mathbb{R}^n$ the Newton polytope of g and denote it by $NP(g)$.

Theorem 2.8. *(See [19,20].) Let Δ be an integral polytope in $\mathbb{R}^n$ and $g_1, \ldots, g_p$ generic Laurent polynomials on $(\mathbb{C}^*)^n$ satisfying $NP(g_i) = \Delta$. Then the Euler characteristic of the subvariety $Z^* = \{x \in (\mathbb{C}^*)^n \mid g_1(x) = \cdots = g_p(x) = 0\}$ of $(\mathbb{C}^*)^n$ is given by*

$$\chi\left(Z^*\right) = (-1)^{n-p} \binom{n-1}{p-1} \mathrm{Vol}_{\mathbb{Z}}(\Delta), \tag{2.14}$$

where $\mathrm{Vol}_{\mathbb{Z}}(\Delta) \in \mathbb{Z}$ is the normalized n-dimensional volume of Δ with respect to the lattice $\mathbb{Z}^n \subset \mathbb{R}^n$.

3. Proof of Theorem 1.4

In this section, we prove Theorem 1.4. First, by [14, Chapter 5, Proposition 1.2] we may assume that $M(A) = M$. Recall that for $\alpha(j) \in A$ ($1 \leqslant j \leqslant \sharp A = m+1$) the function

$$T = \left(\mathbb{C}^*\right)^n \ni x = (x_1, x_2, \ldots, x_n) \mapsto x^{\alpha(j)} \in \mathbb{C}^* \tag{3.1}$$

is defined by the canonical pairing

$$T \times M = \mathrm{Hom}_{\mathbb{Z}}\left(M, \mathbb{C}^*\right) \times M \to \mathbb{C}^*, \tag{3.2}$$

where we consider $\mathbb{C}^*$ as an abelian group (i.e. a $\mathbb{Z}$-module) and $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*)$ denotes the group of homomorphisms of $\mathbb{Z}$-modules from M to $\mathbb{C}^*$. Then by the condition $M(A) = M$ we can easily see that the morphism $\Phi_A : T \simeq (\mathbb{C}^*)^n \to \mathbb{P}^m$ induces an isomorphism $T \simeq \Phi_A(T) \simeq (\mathbb{C}^*)^n$. Note that $\Phi_A(T) \simeq (\mathbb{C}^*)^n$ is the largest T-orbit T_P in $X_A = \overline{\mathrm{im}\,\Phi_A} \subset \mathbb{P}^m$. We can construct such an isomorphism also for any T-orbit T_Δ ($\Delta \prec P$) in X_A as follows. For a face $\Delta \prec P$ of P, taking a point $\alpha(j) \in M(A \cap \Delta) \subset M \cap \mathbb{L}(\Delta)$ to be the origin of the lattices $M(A \cap \Delta)$ and $M \cap \mathbb{L}(\Delta)$, we consider $M(A \cap \Delta)$ as a sublattice ($\mathbb{Z}$-submodule) of $M \cap \mathbb{L}(\Delta)$. By this choice of the origin $0 = \alpha(j)$ of the lattice $M(A \cap \Delta) \simeq \mathbb{Z}^{\dim \Delta}$, we can construct a morphism $\Phi_{A\cap\Delta} : \mathrm{Hom}_{\mathbb{Z}}(M(A \cap \Delta), \mathbb{C}^*) \simeq (\mathbb{C}^*)^{\dim \Delta} \to \mathbb{P}^m$ as follows. First, for $x \in \mathrm{Hom}_{\mathbb{Z}}(M(A \cap \Delta), \mathbb{C}^*) \simeq (\mathbb{C}^*)^{\dim \Delta}$ and $\alpha \in M(A \cap \Delta)$ denote by $x^\alpha \in \mathbb{C}^*$ the image of the pair (x, α) by the canonical paring

$$\mathrm{Hom}_{\mathbb{Z}}\big(M(A \cap \Delta), \mathbb{C}^*\big) \times \big(M(A \cap \Delta)\big) \to \mathbb{C}^*. \tag{3.3}$$

Then the morphism $\Phi_{A\cap\Delta} : \mathrm{Hom}_{\mathbb{Z}}(M(A \cap \Delta), \mathbb{C}^*) \to \mathbb{P}^m$ is defined by

$$\Phi_{A\cap\Delta}(x) = [\xi_1 : \xi_2 : \cdots : \xi_{m+1}] \tag{3.4}$$

for $x \in \mathrm{Hom}_{\mathbb{Z}}(M(A \cap \Delta), \mathbb{C}^*) \simeq (\mathbb{C}^*)^{\dim \Delta}$, where we set

$$\xi_k := \begin{cases} x^{\alpha(k)} & \text{if } \alpha(k) \in A \cap \mathbb{L}(\Delta), \\ 0 & \text{otherwise.} \end{cases} \tag{3.5}$$

In this situation, by [14, Proposition 1.2 and Proposition 1.9 in Chapter 5] the T-orbit T_Δ coincides with the image of $\Phi_{A\cap\Delta}$ and we can easily prove that the morphism

$$\Phi_{A\cap\Delta} : \mathrm{Hom}_{\mathbb{Z}}\big(M(A \cap \Delta), \mathbb{C}^*\big) \twoheadrightarrow \Phi_{A\cap\Delta}\big((\mathbb{C}^*)^{\dim \Delta}\big) = T_\Delta \tag{3.6}$$

is an isomorphism. By making use of this very simple description of $\Phi_{A\cap\Delta} : (\mathbb{C}^*)^{\dim \Delta} \xrightarrow{\sim} T_\Delta$ for the faces $\Delta \prec P$, we can now give a proof of our theorem. For $1 \leqslant i \leqslant m$, we take a generic linear subspace $H \simeq \mathbb{P}^{m-1}$ (resp. $H_{i+1} \simeq \mathbb{P}^{m-i-1}$) of $\mathbb{P}^m$ of codimension 1 (resp. $i+1$) and set

$$\delta_i := (-1)^{n+i-1} \left\{ i \int_{\mathbb{P}^m} \mathrm{Eu}_{X_A} - (i+1) \int_H \mathrm{Eu}_{X_A} + \int_{H_{i+1}} \mathrm{Eu}_{X_A} \right\}. \tag{3.7}$$

Here we set $H_{m+1} := \emptyset$. Then by [8, Theorem 1.1] and [24, Remark 3.3] (see also [22] and [23]) the codimension $r = \operatorname{codim} X_A^* = m - \dim X_A^*$ and the degree of the dual variety $X_A^* \subset (\mathbb{P}^m)^*$ of X_A are given by

$$r = \operatorname{codim} X_A^* = \min\{1 \leqslant i \leqslant m \mid \delta_i \neq 0\}, \tag{3.8}$$

$$\deg X_A^* = \delta_r. \tag{3.9}$$

Hence it remains for us to rewrite the above integers δ_i $(1 \leqslant i \leqslant m)$. First of all, since the Euler obstruction $\mathrm{Eu}_{X_A} : X_A \to \mathbb{Z}$ is constant on each T-orbit $T_\Delta \simeq (\mathbb{C}^*)^{\dim \Delta}$ for $\Delta \prec P$ and $\chi((\mathbb{C}^*)^d) = 0$ for $d \geqslant 1$, we have

$$\int_{\mathbb{P}^m} \mathrm{Eu}_{X_A} = \sum_{\substack{\Delta \prec P \\ \dim \Delta = 0}} \mathrm{Eu}(\Delta). \tag{3.10}$$

Next, by taking a generic hyperplane

$$H = \left\{ [\xi_1 : \xi_2 : \cdots : \xi_{m+1}] \in \mathbb{P}^m \;\middle|\; \sum_{j=1}^{m+1} a_j \xi_j = 0 \right\} \tag{3.11}$$

($a_j \in \mathbb{C}$) of $\mathbb{P}^m$, we can calculate the topological integral $\int_H \mathrm{Eu}_{X_A}$ as follows. Since $\Phi_{A\cap\Delta}$: $\mathrm{Hom}_{\mathbb{Z}}(M(A\cap\Delta), \mathbb{C}^*) \simeq (\mathbb{C}^*)^{\dim\Delta} \to T_\Delta$ is an isomorphism, for the Laurent polynomial

$$\begin{array}{cccc} L_\Delta : & \mathrm{Hom}_{\mathbb{Z}}(M(A\cap\Delta), \mathbb{C}^*) & \longrightarrow & \mathbb{C}^* \\ & \Psi & & \Psi \\ & x & \longmapsto & \displaystyle\sum_{\alpha(j)\in A\cap\Delta} a_j x^{\alpha(j)} \end{array} \tag{3.12}$$

on the torus $\mathrm{Hom}_{\mathbb{Z}}(M(A\cap\Delta), \mathbb{C}^*) \simeq (\mathbb{C}^*)^{\dim\Delta}$ we have

$$\chi(T_\Delta \cap H) = \chi\left(\left\{x \in \left(\mathbb{C}^*\right)^{\dim\Delta} \;\middle|\; L_\Delta(x) = 0\right\}\right). \tag{3.13}$$

Note that for a generic hyperplane $H \subset \mathbb{P}^m$ the hypersurface $\{x \in (\mathbb{C}^*)^{\dim\Delta} \mid L_\Delta(x) = 0\}$ in the torus $\mathrm{Hom}_{\mathbb{Z}}(M(A\cap\Delta), \mathbb{C}^*) \simeq (\mathbb{C}^*)^{\dim\Delta}$ cut out by H satisfies the assumption of Bernstein–Khovanskii–Kushnirenko's theorem (Theorem 2.8) for any $\Delta \prec P$. By Theorem 2.8, we thus obtain

$$\int_H \mathrm{Eu}_{X_A} = \sum_{\substack{\Delta \prec P \\ \dim \Delta \geqslant 1}} (-1)^{\dim\Delta - 1} \mathrm{Vol}_{\mathbb{Z}}(\Delta) \cdot \mathrm{Eu}(\Delta). \tag{3.14}$$

Similarly, by taking a generic linear subspace

$$H_{i+1} = \left\{ [\xi_1 : \xi_2 : \cdots : \xi_{m+1}] \in \mathbb{P}^m \;\middle|\; \sum_{j=1}^{m+1} a_j^{(k)} \xi_j = 0 \ (k = 1, 2, \ldots, i+1) \right\} \tag{3.15}$$

($a_j^{(k)} \in \mathbb{C}$) of $\mathbb{P}^m$ of codimension $i+1$ and using Theorem 2.8, we have

$$\int_{H_{i+1}} \mathrm{Eu}_{X_A} = \sum_{\substack{\Delta \prec P \\ \dim \Delta \geqslant i+1}} (-1)^{\dim\Delta - i - 1} \binom{\dim\Delta - 1}{i} \mathrm{Vol}_{\mathbb{Z}}(\Delta) \cdot \mathrm{Eu}(\Delta). \tag{3.16}$$

By (3.7), (3.10), (3.14) and (3.16), we finally obtain

$$\delta_i := \sum_{\Delta \prec P} (-1)^{\operatorname{codim}\Delta} \left\{ \binom{\dim\Delta - 1}{i} + (-1)^{i-1}(i+1) \right\} \mathrm{Vol}_{\mathbb{Z}}(\Delta) \cdot \mathrm{Eu}(\Delta). \tag{3.17}$$

This completes the proof.

4. Euler obstructions of toric varieties

In this section, we give some formulas for the Euler obstructions of toric varieties. A beautiful formula for the Euler obstructions of 2-dimensional normal toric varieties was proved by Gonzalez-Sprinberg [15]. Our result can be considered as a natural generalization of his formula.

First we recall the definition of Euler obstructions (for the details see [17], etc.). Let X be an algebraic variety over $\mathbb{C}$. Then the Euler obstruction Eu_X of X is a $\mathbb{Z}$-valued constructible function on X defined as follows. The value of Eu_X on the smooth part of X is defined to be 1. In order to define the value of Eu_X at a singular point $p \in X$, we take an affine open neighborhood U of p in X and a closed embedding $U \hookrightarrow \mathbb{C}^m$. Next we choose a Whitney stratification $U = \bigsqcup_{\alpha \in A} U_\alpha$ of U such that U_α are connected. Then the values $\mathrm{Eu}_X(U_\alpha)$ of Eu_X on the strata U_α are defined by induction on codimensions of U_α as follows:

(i) If U_α is contained in the smooth part of U, we set $\mathrm{Eu}_X(U_\alpha) = 1$.
(ii) Assume that for $k \geqslant 0$ the values of Eu_X on the strata U_α such that $\mathrm{codim} U_\alpha \leqslant k$ are already determined. Then for a stratum U_β such that $\mathrm{codim}\, U_\beta = k+1$ the value $\mathrm{Eu}_X(U_\beta)$ is defined by

$$\mathrm{Eu}_X(U_\beta) = \sum_{U_\beta \subsetneq \overline{U_\alpha}} \chi\big(U_\alpha \cap f^{-1}(\eta) \cap B(q;\varepsilon)\big) \cdot \mathrm{Eu}_X(U_\alpha) \tag{4.1}$$

for sufficiently small $\varepsilon > 0$ and $0 < \eta \ll \varepsilon$, where $q \in U_\beta$ and f is a holomorphic function defined on an open neighborhood W of q in $\mathbb{C}^m$ such that $U_\beta \cap W \subset f^{-1}(0)$ and $(q; \mathrm{grad} f(q)) \in T^*_{U_\beta}\mathbb{C}^m \setminus (\bigcup_{U_\beta \subsetneq \overline{U_\alpha}} \overline{T^*_{U_\alpha}\mathbb{C}^m})$.

The above integers $\chi(U_\alpha \cap f^{-1}(\eta) \cap B(q;\varepsilon))$ can be calculated by the nearby cycle functor ψ_f as

$$\chi\big(U_\alpha \cap f^{-1}(\eta) \cap B(q;\varepsilon)\big) = \chi\big(\psi_f(\mathbb{C}_{U_\alpha})_q\big). \tag{4.2}$$

Indeed by [6, Proposition 4.2.2], for the Milnor fiber $F_q := U \cap f^{-1}(\eta) \cap B(q;\varepsilon) = (f|_U)^{-1}(\eta) \cap B(q;\varepsilon)$ of $f|_U : U \to \mathbb{C}$ at $q \in U_\beta$ we have an isomorphism

$$R\Gamma(F_q; \mathbb{C}_{U_\alpha}) \simeq \psi_f(\mathbb{C}_{U_\alpha})_q. \tag{4.3}$$

Since $F_q = \bigsqcup_{\gamma \in A}(U_\gamma \cap F_q)$ is also a Whitney stratification of F_q ($f|_U : U \to \mathbb{C}$ has the isolated stratified critical value $0 \in \mathbb{C}$ by [21, Proposition 1.3]), we obtain $\chi(R\Gamma(F_q; \mathbb{C}_{U_\alpha})) = \chi(U_\alpha \cap F_q) = \chi(U_\alpha \cap f^{-1}(\eta) \cap B(q;\varepsilon))$ by [6, Theorem 4.1.22].

4.1. The case of affine toric varieties

From now on, we shall consider the toric case. Let $N \simeq \mathbb{Z}^n$ be a $\mathbb{Z}$-lattice of rank n and σ a strongly convex rational polyhedral cone in $N_\mathbb{R} = \mathbb{R} \otimes_\mathbb{Z} N$. We denote by M the dual lattice of N and define the polar cone $\sigma^\vee$ of σ in $M_\mathbb{R} = \mathbb{R} \otimes_\mathbb{Z} M$ by

$$\sigma^\vee = \big\{ v \in M_\mathbb{R} \;\big|\; \langle u, v\rangle \geqslant 0 \text{ for any } u \in \sigma \big\}. \tag{4.4}$$

Then the dimension of $\sigma^\vee$ is n and we obtain a semigroup $\mathcal{S}_\sigma := \sigma^\vee \cap M$ and an n-dimensional affine toric variety $X := U_\sigma = \mathrm{Spec}(\mathbb{C}[\mathcal{S}_\sigma])$ (see [11,26], etc.). Recall also that the algebraic torus $T = \mathrm{Spec}(\mathbb{C}[M]) \simeq (\mathbb{C}^*)^n$ acts naturally on $X = U_\sigma$ and the T-orbits in X are indexed by the faces $\Delta_\alpha \prec \sigma^\vee$ of $\sigma^\vee$. We denote by $\mathbb{L}(\Delta_\alpha)$ the smallest linear subspace of $M_\mathbb{R}$ containing Δ_α. For a face Δ_α of $\sigma^\vee$, denote by T_α the T-orbit $\mathrm{Spec}(\mathbb{C}[M \cap \mathbb{L}(\Delta_\alpha)])$ which corresponds to Δ_α. Then we obtain a decomposition $X = \bigsqcup_{\Delta_\alpha \prec \sigma^\vee} T_\alpha$ of $X = U_\sigma$ into T-orbits. By the above recursive definition (ii) of Eu_X, in order to compute the Euler obstruction $\mathrm{Eu}_X : X \to \mathbb{Z}$ it suffices to determine the following numbers.

Definition 4.1. For two faces Δ_α, Δ_β of $\sigma^\vee$ such that $\Delta_\beta \precneqq \Delta_\alpha$ ($\Leftrightarrow T_\beta \subsetneq \overline{T_\alpha}$), we define the linking number $l_{\alpha,\beta} \in \mathbb{Z}$ of T_α along T_β as follows. For a point $q \in T_\beta$ and a closed embedding $\iota : X = U_\sigma \hookrightarrow \mathbb{C}^m$, we set

$$l_{\alpha,\beta} := \chi\big(\psi_f(\mathbb{C}_{T_\alpha})_q\big), \tag{4.5}$$

where f is a holomorphic function defined on an open neighborhood W of q in $\mathbb{C}^m$ such that $T_\beta \cap W \subset f^{-1}(0)$ and $(q; \mathrm{grad}\, f(q)) \in T^*_{T_\beta}\mathbb{C}^m \setminus (\bigcup_{\Delta_\beta \precneqq \Delta_\gamma} \overline{T^*_{T_\gamma}\mathbb{C}^m})$.

Note that the above definition of $l_{\alpha,\beta}$ does not depend on the choice of $q \in T_\beta$, ι and f, etc. We will show that $l_{\alpha,\beta}$ can be described by the geometry of the cones Δ_α and Δ_β. First let us consider the $\mathbb{Z}$-lattice $M_\beta := M \cap \mathbb{L}(\Delta_\beta)$ of rank $\dim \Delta_\beta$. Next set $\mathbb{L}(\Delta_\beta)' := M_\mathbb{R}/\mathbb{L}(\Delta_\beta)$ and let $p_\beta : M_\mathbb{R} \to \mathbb{L}(\Delta_\beta)'$ be the natural projection. Then $M'_\beta := p_\beta(M) \subset \mathbb{L}(\Delta_\beta)'$ is a $\mathbb{Z}$-lattice of rank $(n - \dim \Delta_\beta)$ and $K_{\alpha,\beta} := p_\beta(\Delta_\alpha) \subset \mathbb{L}(\Delta_\beta)'$ is a proper convex cone with apex $0 \in \mathbb{L}(\Delta_\beta)'$.

Definition 4.2. For two faces Δ_α and Δ_β of $\sigma^\vee$ such that $\Delta_\beta \precneqq \Delta_\alpha$, we define the normalized relative subdiagram volume $\mathrm{RSV}_\mathbb{Z}(\Delta_\alpha, \Delta_\beta)$ of Δ_α along Δ_β by

$$\mathrm{RSV}_\mathbb{Z}(\Delta_\alpha, \Delta_\beta) := \mathrm{Vol}_\mathbb{Z}(K_{\alpha,\beta} \setminus \Theta_{\alpha,\beta}), \tag{4.6}$$

where $\Theta_{\alpha,\beta}$ is the convex hull of $K_{\alpha,\beta} \cap (M'_\beta \setminus \{0\})$ in $\mathbb{L}(\Delta_\beta)' \simeq \mathbb{R}^{n-\dim \Delta_\beta}$ and $\mathrm{Vol}_\mathbb{Z}(K_{\alpha,\beta} \setminus \Theta_{\alpha,\beta})$ is the normalized $(\dim \Delta_\alpha - \dim \Delta_\beta)$-dimensional volume of $K_{\alpha,\beta} \setminus \Theta_{\alpha,\beta}$ with respect to the lattice $M'_\beta \cap \mathbb{L}(K_{\alpha,\beta})$. If $\Delta_\alpha = \Delta_\beta$, we set $\mathrm{RSV}_\mathbb{Z}(\Delta_\alpha, \Delta_\alpha) := 1$.

Theorem 4.3. *For two faces Δ_α and Δ_β of $\sigma^\vee$ such that $\Delta_\beta \precneqq \Delta_\alpha$, the linking number $l_{\alpha,\beta}$ of T_α along T_β is given by*

$$l_{\alpha,\beta} = (-1)^{\dim \Delta_\alpha - \dim \Delta_\beta - 1}\mathrm{RSV}_\mathbb{Z}(\Delta_\alpha, \Delta_\beta). \tag{4.7}$$

Proof. First recall that we have $T_\beta = \mathrm{Spec}(\mathbb{C}[M_\beta]) \simeq (\mathbb{C}^*)^{\dim \Delta_\beta}$. For each face Δ_α of $\sigma^\vee$ such that $\Delta_\beta \precneqq \Delta_\alpha$, consider the semigroups $\mathcal{S}_\alpha := M \cap \Delta_\alpha$ and $\mathcal{S}_{\alpha,\beta} := M'_\beta \cap K_{\alpha,\beta}$. In the special case when $\Delta_\alpha = \sigma^\vee$, we set also $\mathcal{S}_{\sigma,\beta} := M'_\beta \cap p_\beta(\sigma^\vee)$. Then for any face $\Delta_\alpha \prec \sigma^\vee$ such that $\Delta_\beta \precneqq \Delta_\alpha$ it is easy to see that

$$\mathcal{S}_\alpha + M_\beta = \mathcal{S}_{\alpha,\beta} \oplus M_\beta \tag{4.8}$$

and in a neighborhood of T_β in X we have

$$\overline{T_\alpha} = \operatorname{Spec}\big(\mathbb{C}[\mathcal{S}_\alpha + M_\beta]\big) \tag{4.9}$$

$$= \operatorname{Spec}\big(\mathbb{C}[\mathcal{S}_{\alpha,\beta}]\big) \times T_\beta \tag{4.10}$$

(see the proof of [14, Chapter 5, Theorem 3.1]). In particular, for $\Delta_\alpha = \sigma^\vee$ we have

$$X = \operatorname{Spec}\big(\mathbb{C}[\mathcal{S}_{\sigma,\beta}]\big) \times T_\beta \tag{4.11}$$

in a neighborhood of T_β. More precisely, there exists a unique point $q \in X_{\sigma,\beta} := \operatorname{Spec}(\mathbb{C}[\mathcal{S}_{\sigma,\beta}])$ such that $\{q\} \times T_\beta = T_\beta$. Now let us take a face $\Delta_\alpha \prec \sigma^\vee$ such that $\Delta_\beta \precneqq \Delta_\alpha$ and set $X_{\alpha,\beta} := \operatorname{Spec}(\mathbb{C}[\mathcal{S}_{\alpha,\beta}])$. Then by the inclusion $\mathcal{S}_{\alpha,\beta} \hookrightarrow \mathcal{S}_{\sigma,\beta}$ we obtain a surjective homomorphism

$$\mathbb{C}[\mathcal{S}_{\sigma,\beta}] \twoheadrightarrow \mathbb{C}[\mathcal{S}_{\alpha,\beta}] \tag{4.12}$$

of $\mathbb{C}$-algebras and hence a closed embedding $X_{\alpha,\beta} \hookrightarrow X_{\sigma,\beta}$. Denote by $T_{\alpha,\beta}$ the open dense torus $\operatorname{Spec}(\mathbb{C}[M'_\beta \cap \mathbb{L}(K_{\alpha,\beta})]) \simeq (\mathbb{C}^*)^{\dim \Delta_\alpha - \dim \Delta_\beta}$ of the toric variety $X_{\alpha,\beta}$. Note that we have $T_\alpha \simeq T_{\alpha,\beta} \times T_\beta$. Now let $v_1, v_2, \dots, v_m$ be generators of the semigroup $\mathcal{S}_{\sigma,\beta}$ and consider a surjective morphism

$$\mathbb{C}[t_1, t_2, \dots, t_m] \to \mathbb{C}[\mathcal{S}_{\sigma,\beta}] \tag{4.13}$$

of $\mathbb{C}$-algebras defined by $t_i \mapsto [v_i]$. Then it induces a closed embedding $X_{\sigma,\beta} \hookrightarrow \mathbb{C}^m$ by which the point $q \in X_{\sigma,\beta}$ is sent to $0 \in \mathbb{C}^m$. If we consider $T_{\alpha,\beta}$ as a locally closed subset of $\mathbb{C}^m$ by this embedding, then the linking number $l_{\alpha,\beta}$ of T_α along T_β is given by

$$l_{\alpha,\beta} = \chi\big(\psi_f(\mathbb{C}_{T_{\alpha,\beta}})_0\big), \tag{4.14}$$

where $f : \mathbb{C}^m \to \mathbb{C}$ is a generic linear form. By applying Theorem 2.4 to the closed embedding $X_{\alpha,\beta} \hookrightarrow \mathbb{C}^m$, we obtain

$$l_{\alpha,\beta} = \chi\big(\psi_g(\mathbb{C}_{T_{\alpha,\beta}})_0\big), \tag{4.15}$$

where we set $g := f|_{X_{\alpha,\beta}}$. Finally it follows from [25, Corollary 3.6] (whose special case used here can be deduced also from the proof of [14, Chapter 10, Theorem 2.12]) that

$$l_{\alpha,\beta} = (-1)^{\dim \Delta_\alpha - \dim \Delta_\beta - 1} \operatorname{Vol}_{\mathbb{Z}}(K_{\alpha,\beta} \setminus \Theta_{\alpha,\beta}). \tag{4.16}$$

This completes the proof. □

Since the Euler obstruction $\operatorname{Eu}_X : X \to \mathbb{Z}$ of X is constant on each T-orbit T_α ($\Delta_\alpha \prec \sigma^\vee$), we denote by $\operatorname{Eu}(\Delta_\alpha)$ the value of Eu_X on T_α. Then we have

Corollary 4.4. *All the values* $\operatorname{Eu}(\Delta_\alpha)$ *of* $\operatorname{Eu}_X : X \to \mathbb{Z}$ *are determined by induction on codimensions of faces of* $\sigma^\vee$ *as follows:*

(i) $\mathrm{Eu}(\sigma^\vee) := \mathrm{Eu}_X(T) = 1$,
(ii) $\mathrm{Eu}(\Delta_\beta) = \sum_{\Delta_\beta \precneqq \Delta_\alpha} (-1)^{\dim \Delta_\alpha - \dim \Delta_\beta - 1} \mathrm{RSV}_{\mathbb{Z}}(\Delta_\alpha, \Delta_\beta) \cdot \mathrm{Eu}(\Delta_\alpha)$.

4.2. The case of toric varieties associated with lattice points

We consider the situation considered in Sections 1 and 3 and inherit the notations there. From now on, we shall give a combinatorial description of Eu_{X_A} for the variety X_A. By our assumption $\dim P = n$, the rank of the affine $\mathbb{Z}$-lattice $M(A)$ generated by A is n. For each face Δ_α of P, consider the smallest affine subspace $\mathbb{L}(\Delta_\alpha)$ of $M_\mathbb{R}$ containing Δ_α and the affine $\mathbb{Z}$-lattice $M_\alpha := M(A \cap \Delta_\alpha)$ generated by $A \cap \Delta_\alpha$ in $\mathbb{L}(\Delta_\alpha)$. Now let us fix two faces Δ_α, Δ_β of P such that $\Delta_\beta \prec \Delta_\alpha$. By taking a suitable affine transformation of the lattice $M(A)$, we may assume that the origin 0 of $M(A)$ is a vertex of the smaller face Δ_β. By this choice of the origin $0 \in \Delta_\beta \cap M(A)$, we define the subsemigroup $\mathcal{S}_\alpha$ of M_α generated by $A \cap \Delta_\alpha$. Although $\mathcal{S}_\alpha$ depends also on Δ_β, etc., we denote it by $\mathcal{S}_\alpha$ to simplify the notation. Denote by M_α/Δ_β the quotient lattice $M_\alpha/(M_\alpha \cap \mathbb{L}(\Delta_\beta))$ of rank $(\dim \Delta_\alpha - \dim \Delta_\beta)$. Then the following definitions are essentially due to [14, Chapter 5, p. 178].

Definition 4.5. (See [14].)

(i) We denote by $\mathcal{S}_\alpha/\Delta_\beta$ the image of $\mathcal{S}_\alpha \subset M_\alpha$ in the quotient $\mathbb{Z}$-lattice M_α/Δ_β.
(ii) We denote by $K(\mathcal{S}_\alpha/\Delta_\beta)$ (resp. $K_+(\mathcal{S}_\alpha/\Delta_\beta)$) the convex hull of $\mathcal{S}_\alpha/\Delta_\beta$ (resp. $(\mathcal{S}_\alpha/\Delta_\beta) \setminus \{0\}$) in $(M_\alpha/\Delta_\beta)_\mathbb{R}$ and set

$$K_-(\mathcal{S}_\alpha/\Delta_\beta) := \overline{K(\mathcal{S}_\alpha/\Delta_\beta) \setminus K_+(\mathcal{S}_\alpha/\Delta_\beta)}. \tag{4.17}$$

We call $K_-(\mathcal{S}_\alpha/\Delta_\beta)$ the subdiagram part of the semigroup $\mathcal{S}_\alpha/\Delta_\beta$ and denote by $u(\mathcal{S}_\alpha/\Delta_\beta)$ its normalized $(\dim \Delta_\alpha - \dim \Delta_\beta)$-dimensional volume with respect to the $\mathbb{Z}$-lattice $M_\alpha/\Delta_\beta \subset (M_\alpha/\Delta_\beta)_\mathbb{R}$. If $\Delta_\alpha = \Delta_\beta$, we set $u(\mathcal{S}_\alpha/\Delta_\alpha) := 1$.

Finally, recall the definition of the index $i(\Delta_\alpha, \Delta_\beta) \in \mathbb{Z}_{>0}$ in [14, Chapter 5, (3.1)].

Definition 4.6. (See [14].) For two faces Δ_α, Δ_β of P such that $\Delta_\beta \prec \Delta_\alpha$, we define $i(\Delta_\alpha, \Delta_\beta)$ to be the index

$$i(\Delta_\alpha, \Delta_\beta) := \bigl[M_\alpha \cap \mathbb{L}(\Delta_\beta) : M_\beta\bigr]. \tag{4.18}$$

Now recall that by [14, Chapter 5, Proposition 1.9] we have the basic correspondence:

$$\{\text{faces of } P\} \overset{1:1}{\longleftrightarrow} \{T\text{-orbits in } X_A\}. \tag{4.19}$$

For a face $\Delta_\alpha \prec P$ of P, we denote by T_α the corresponding T-orbit in X_A. We also denote by $\mathrm{Eu}(\Delta_\alpha)$ the value of the Euler obstruction $\mathrm{Eu}_{X_A} : X_A \to \mathbb{Z}$ on T_α.

Theorem 4.7. *The values* $\mathrm{Eu}(\Delta_\alpha)$ *are determined by:*

(i) $\mathrm{Eu}(P) = 1$,
(ii) $\mathrm{Eu}(\Delta_\beta) = \sum_{\Delta_\beta \precneqq \Delta_\alpha} (-1)^{\dim \Delta_\alpha - \dim \Delta_\beta - 1} i(\Delta_\alpha, \Delta_\beta) \cdot u(\mathcal{S}_\alpha/\Delta_\beta) \cdot \mathrm{Eu}(\Delta_\alpha)$.

Proof. Let $\Delta_\alpha \prec P$ be a face of P. Then by [14, Chapter 5, Proposition 1.9] the closure $\overline{T_\alpha}$ of T_α in X_A is isomorphic to the projective toric variety $X_{A\cap\Delta_\alpha} \subset \mathbb{P}^{\sharp(A\cap\Delta_\alpha)-1}$ defined by the finite subset $A\cap\Delta_\alpha$ in the lattice $M_\alpha = M(A\cap\Delta_\alpha) \simeq \mathbb{Z}^{\dim\Delta_\alpha}$. Moreover the cone $\mathrm{Cone}(\overline{T_\alpha}) \subset \mathbb{C}^{\sharp(A\cap\Delta_\alpha)}$ over $\overline{T_\alpha} \subset \mathbb{P}^{\sharp(A\cap\Delta_\alpha)-1}$ is an affine variety as follows. Let

$$i_\alpha : M_\alpha \hookrightarrow \Xi_\alpha := M_\alpha \oplus \mathbb{Z} \simeq \mathbb{Z}^{\dim\Delta_\alpha+1} \tag{4.20}$$

be the embedding defined by $v \mapsto (v,1)$ and $\widetilde{\mathcal{S}}_\alpha$ the subsemigroup of the lattice Ξ_α generated by $i_\alpha(A\cap\Delta_\alpha)$ and $0\in\Xi_\alpha$. Then by [14, Chapter 5, Proposition 2.3] the cone $\mathrm{Cone}(\overline{T_\alpha}) \subset \mathbb{C}^{\sharp(A\cap\Delta_\alpha)}$ is isomorphic to the affine toric variety $\mathrm{Spec}(\mathbb{C}[\widetilde{\mathcal{S}}_\alpha])$. In the special case when $\Delta_\alpha = P$, we set $\Xi := \Xi_\alpha$ $(= M(A)\oplus\mathbb{Z})$ and $\widetilde{\mathcal{S}} := \widetilde{\mathcal{S}}_\alpha$ for short. Since $\widetilde{\mathcal{S}}_\alpha$ is a subsemigroup of $\widetilde{\mathcal{S}}$ via the inclusions $M_\alpha \subset M(A)$ and $\Xi_\alpha \subset \Xi$, there exists a natural surjection

$$\mathbb{C}[\widetilde{\mathcal{S}}] \twoheadrightarrow \mathbb{C}[\widetilde{\mathcal{S}}_\alpha]. \tag{4.21}$$

This corresponds to the closed embedding

$$\mathrm{Cone}(\overline{T_\alpha}) \simeq \mathrm{Spec}\bigl(\mathbb{C}[\widetilde{\mathcal{S}}_\alpha]\bigr) \hookrightarrow \mathrm{Cone}(X_A) \simeq \mathrm{Spec}\bigl(\mathbb{C}[\widetilde{\mathcal{S}}]\bigr). \tag{4.22}$$

Now let Δ_α and Δ_β be two faces of P such that $\Delta_\beta \precneqq \Delta_\alpha$ ($\Leftrightarrow T_\beta \subsetneqq \overline{T_\alpha}$). We have to determine the linking number $l_{\alpha,\beta}$ of T_α along T_β (defined as in Definition 4.1). Since the singularity of $\overline{T_\alpha}$ along T_β is the same as that of $\mathrm{Cone}(\overline{T_\alpha}) \simeq \mathrm{Spec}(\mathbb{C}[\widetilde{\mathcal{S}}_\alpha])$ along $\mathrm{Cone}(T_\beta) \simeq \mathrm{Spec}(\mathbb{C}[\Xi_\beta]) \simeq (\mathbb{C}^*)^{\dim\Delta_\beta+1}$, it suffices to study the pair $\mathrm{Cone}(T_\beta) \subset \mathrm{Cone}(\overline{T_\alpha})$. Moreover, by the proof of [14, Chapter 5, Theorem 3.1], in a neighborhood of $\mathrm{Cone}(T_\beta)$ in $\mathrm{Cone}(X_A) \subset \mathbb{C}^{\sharp A}$, we have

$$\mathrm{Cone}(\overline{T_\alpha}) = \mathrm{Spec}\bigl(\mathbb{C}[\widetilde{\mathcal{S}}_\alpha + \Xi_\beta]\bigr), \tag{4.23}$$

$$\mathrm{Cone}(X_A) = \mathrm{Spec}\bigl(\mathbb{C}[\widetilde{\mathcal{S}} + \Xi_\beta]\bigr) \tag{4.24}$$

and the fibers of the morphisms

$$\mathrm{Cone}(\overline{T_\alpha}) \simeq \mathrm{Spec}\bigl(\mathbb{C}[\widetilde{\mathcal{S}}_\alpha + \Xi_\beta]\bigr) \to \mathrm{Cone}(T_\beta) \simeq \mathrm{Spec}\bigl(\mathbb{C}[\Xi_\beta]\bigr), \tag{4.25}$$

$$\mathrm{Cone}(X_A) \simeq \mathrm{Spec}\bigl(\mathbb{C}[\widetilde{\mathcal{S}} + \Xi_\beta]\bigr) \to \mathrm{Cone}(T_\beta) \simeq \mathrm{Spec}\bigl(\mathbb{C}[\Xi_\beta]\bigr) \tag{4.26}$$

induced by $\Xi_\beta \subset \widetilde{\mathcal{S}}_\alpha + \Xi_\beta$ and $\Xi_\beta \subset \widetilde{\mathcal{S}} + \Xi_\beta$ are $\mathrm{Spec}(\mathbb{C}[(\widetilde{\mathcal{S}}_\alpha + \Xi_\beta)/\Xi_\beta])$ and $\mathrm{Spec}(\mathbb{C}[(\widetilde{\mathcal{S}} + \Xi_\beta)/\Xi_\beta])$ respectively. Let us set

$$Y_\alpha := \mathrm{Spec}\bigl(\mathbb{C}\bigl[(\widetilde{\mathcal{S}}_\alpha + \Xi_\beta)/\Xi_\beta\bigr]\bigr), \tag{4.27}$$

$$Y := \mathrm{Spec}\bigl(\mathbb{C}\bigl[(\widetilde{\mathcal{S}} + \Xi_\beta)/\Xi_\beta\bigr]\bigr). \tag{4.28}$$

Since the natural morphism

$$(\widetilde{\mathcal{S}}_\alpha + \Xi_\beta)/\Xi_\beta \to (\widetilde{\mathcal{S}} + \Xi_\beta)/\Xi_\beta \tag{4.29}$$

is injective, we obtain a surjection

$$\mathbb{C}\big[(\widetilde{\mathcal{S}}+\Xi_\beta)/\Xi_\beta\big] \twoheadrightarrow \mathbb{C}\big[(\widetilde{\mathcal{S}}_\alpha+\Xi_\beta)/\Xi_\beta\big] \tag{4.30}$$

and hence a closed embedding $Y_\alpha \hookrightarrow Y$. Note that $Y \cap \mathrm{Cone}(T_\beta) = Y_\alpha \cap \mathrm{Cone}(T_\beta)$ consists of a single point. We denote this point by q. Now let us consider the open subset $W_\alpha := \mathrm{Spec}(\mathbb{C}[\Xi_\alpha/\Xi_\beta])$ of $Y_\alpha = \mathrm{Spec}(\mathbb{C}[(\widetilde{\mathcal{S}}_\alpha+\Xi_\beta)/\Xi_\beta])$. It is easy to see that W_α is the intersection of $\mathrm{Cone}(T_\alpha) \simeq (\mathbb{C}^*)^{\dim \Delta_\alpha+1}$ and Y_α. Let $v_1, v_2, \dots, v_m$ be generators of the semigroup $(\widetilde{\mathcal{S}}+\Xi_\beta)/\Xi_\beta$ and consider a surjective morphism

$$\mathbb{C}[t_1, t_2, \dots, t_m] \to \mathbb{C}\big[(\widetilde{\mathcal{S}}+\Xi_\beta)/\Xi_\beta\big] \tag{4.31}$$

of $\mathbb{C}$-algebras defined by $t_i \mapsto [v_i]$. Then it induces a closed embedding $Y \hookrightarrow \mathbb{C}^m$ by which the point $q \in Y$ is sent to $0 \in \mathbb{C}^m$. If we consider W_α as a locally closed subset of $\mathbb{C}^m$ by this embedding, then the linking number $l_{\alpha,\beta}$ is given by

$$l_{\alpha,\beta} = \chi\big(\psi_f(\mathbb{C}_{W_\alpha})_0\big), \tag{4.32}$$

where $f : \mathbb{C}^m \to \mathbb{C}$ is a generic linear form. By applying Theorem 2.4 to the closed embedding $Y_\alpha \hookrightarrow \mathbb{C}^m$, we obtain also

$$l_{\alpha,\beta} = \chi\big(\psi_g(\mathbb{C}_{W_\alpha})_0\big), \tag{4.33}$$

where we set $g := f|_{Y_\alpha}$. In order to calculate this last term $\chi(\psi_g(\mathbb{C}_{W_\alpha})_0)$, we shall investigate the structure of W_α more precisely. By the inclusion $(\Xi_\beta)_\mathbb{R} \subset (\Xi_\alpha)_\mathbb{R}$, we set $\Xi'_\alpha := \Xi_\alpha \cap (\Xi_\beta)_\mathbb{R}$. Since we assumed that the origin of the lattice M_α is a vertex of Δ_β, the two lattices Ξ'_α and Ξ_β contain the subgroup $\{(0,t) \in \Xi_\alpha \mid t \in \mathbb{Z}\} \simeq \mathbb{Z}$ of Ξ_α. Hence we obtain an isomorphism

$$\Xi'_\alpha/\Xi_\beta \simeq \big(M_\alpha \cap \mathbb{L}(\Delta_\beta)\big)/M_\beta. \tag{4.34}$$

Namely Ξ_β is a sublattice of Ξ'_α with index $l := i(\Delta_\alpha, \Delta_\beta)$. By the fundamental theorem of finitely generated abelian groups, we may assume that $G := \Xi'_\alpha/\Xi_\beta$ is a cyclic group $\mathbb{Z}/l\mathbb{Z}$ of order $l = i(\Delta_\alpha, \Delta_\beta)$. Now let us take a sublattice Ξ''_α of Ξ_α such that $\Xi_\alpha = \Xi'_\alpha \oplus \Xi''_\alpha$. Then we have

$$\Xi_\alpha/\Xi_\beta \simeq G \oplus \Xi''_\alpha \tag{4.35}$$

and

$$W_\alpha \simeq \mathrm{Spec}\big(\mathbb{C}[G]\big) \times \mathrm{Spec}\big(\mathbb{C}\big[\Xi''_\alpha\big]\big) \tag{4.36}$$

$$\simeq \big\{z \in \mathbb{C} \bigm| z^l = 1\big\} \times \big(\mathbb{C}^*\big)^{\dim \Delta_\alpha - \dim \Delta_\beta}. \tag{4.37}$$

Let $\Psi : \Xi_\alpha \twoheadrightarrow G = \mathbb{Z}/l\mathbb{Z}$ be the composite of

$$\Xi_\alpha \twoheadrightarrow \Xi'_\alpha \twoheadrightarrow G = \Xi'_\alpha/\Xi_\beta. \tag{4.38}$$

For $s \in \Xi_\alpha$, we define an integer $e(s) \in \{0, 1, 2, \dots, l-1\}$ by $\Psi(s) = [e(s)] \in G \simeq \mathbb{Z}/l\mathbb{Z}$. Then for $k = 0, 1, \dots, l-1$ there exist surjective homomorphisms

$$I_k : \mathbb{C}\big[(\widetilde{\mathcal{S}}_\alpha + \Xi_\beta)/\Xi_\beta\big] \twoheadrightarrow \mathbb{C}\big[\big(\widetilde{\mathcal{S}}_\alpha + \Xi'_\alpha\big)/\Xi'_\alpha\big] \tag{4.39}$$

of $\mathbb{C}$-algebras defined by

$$\sum_{s_i \in \widetilde{\mathcal{S}}_\alpha} a_i \cdot [s_i + \Xi_\beta] \mapsto \sum_{s_i \in \widetilde{\mathcal{S}}_\alpha} a_i \cdot \mu_l^{ke(s_i)} \cdot \big[s_i + \Xi'_\alpha\big], \tag{4.40}$$

where $\mu_l = \exp(\frac{2\pi\sqrt{-1}}{l})$ is the primitive l-th root of unity. On the other hand, since $\{(0, t) \in \Xi_\alpha \mid t \in \mathbb{Z}\} \simeq \mathbb{Z}$ is a subgroup of Ξ'_α, we have isomorphisms

$$\Xi_\alpha/\Xi'_\alpha \simeq M_\alpha/\Delta_\beta = M_\alpha/\big(M_\alpha \cap \mathbb{L}(\Delta_\beta)\big), \tag{4.41}$$

$$\big(\widetilde{\mathcal{S}}_\alpha + \Xi'_\alpha\big)/\Xi'_\alpha \simeq \mathcal{S}_\alpha/\Delta_\beta. \tag{4.42}$$

Let us set $Z_\alpha := \mathrm{Spec}(\mathbb{C}[\mathcal{S}_\alpha/\Delta_\beta])$. Then by the above surjective homomorphisms I_k ($k = 0, 1, 2, \dots, l-1$) we obtain closed embeddings

$$\iota_k : Z_\alpha \hookrightarrow Y_\alpha \quad (k = 0, 1, 2, \dots, l-1). \tag{4.43}$$

We can see that the images of these embeddings $\iota_k : Z_\alpha \hookrightarrow Y_\alpha$ are the explicit realizations of the branches along T-orbits found in [14, Chapter 5, Theorem 3.1]. Indeed, denote by T_0 the open dense torus $\mathrm{Spec}(\mathbb{C}[\Xi''_\alpha]) \simeq (\mathbb{C}^*)^{\dim \Delta_\alpha - \dim \Delta_\beta}$ of Z_α. Then the open dense subset $W_\alpha \subset Y_\alpha$ is a direct sum $T_0 \sqcup T_0 \sqcup \cdots \sqcup T_0$ of l copies of T_0. For $k = 0, 1, \dots, l-1$, consider also surjective homomorphisms

$$I'_k : \mathbb{C}[\Xi_\alpha/\Xi_\beta] \twoheadrightarrow \mathbb{C}\big[\Xi''_\alpha\big] \simeq \mathbb{C}\big[\Xi_\alpha/\Xi'_\alpha\big] \tag{4.44}$$

of $\mathbb{C}$-algebras defined by

$$\sum_{s_i \in \Xi_\alpha} a_i \cdot [s_i + \Xi_\beta] \mapsto \sum_{s_i \in \Xi_\alpha} a_i \cdot \mu_l^{ke(s_i)} \cdot \big[s_i + \Xi'_\alpha\big]. \tag{4.45}$$

Then by this homomorphism I'_k we obtain a closed embedding

$$\iota'_k : T_0 \hookrightarrow W_\alpha \tag{4.46}$$

which induces an isomorphism from T_0 to the $(k+1)$-th component of W_α. Moreover ι'_k fits into the commutative diagram

$$\begin{tikzcd} Z_\alpha \arrow[r, hook, "\iota_k"] & Y_\alpha \\ T_0 \arrow[u, hook] \arrow[r, hook, "\iota'_k"] & W_\alpha. \arrow[u, hook] \end{tikzcd} \tag{4.47}$$

Then we have an isomorphism

$$\bigoplus_{k=0}^{l-1}(\iota_k)_*(\mathbb{C}_{T_0}) \simeq \mathbb{C}_{W_\alpha} \tag{4.48}$$

in $\mathbf{D}_c^b(Y_\alpha)$. Therefore, applying Theorem 2.4 to ι_k ($k = 0, 1, 2, \dots, l-1$), we obtain

$$l_{\alpha,\beta} = \sum_{k=0}^{l-1} \chi\big(\psi_{g_k}(\mathbb{C}_{T_0})_0\big), \tag{4.49}$$

where we set $g_k := g \circ \iota_k \in \mathbb{C}[\mathcal{S}_\alpha/\Delta_\beta]$ ($k = 0, 1, 2, \dots, l-1$). Finally by [25, Corollary 3.6] (see also the proof of [14, Chapter 10, Theorem 2.12]) we get

$$l_{\alpha,\beta} = (-1)^{\dim\Delta_\alpha - \dim\Delta_\beta - 1} i(\Delta_\alpha, \Delta_\beta) \cdot u(\mathcal{S}_\alpha/\Delta_\beta). \tag{4.50}$$

This completes the proof. □

Example 4.8. We give some examples of integral polytopes for which the degrees of the A-discriminant varieties are easily computed by our method. We fix a $\mathbb{Z}$-basis $\{m_1, m_2, \dots, m_n\}$ of the lattice $M = \mathbb{Z}^n$.

(i) For natural numbers $a_1, a_2, \dots, a_n \geqslant 1$, consider the finite subset $A = \{0, m_1, m_1 + m_n, \dots, m_1 + a_1 m_n, m_2, m_2 + m_n, \dots, m_2 + a_2 m_n, \dots, m_{n-1}, m_{n-1} + m_n, \dots, m_{n-1} + a_{n-1} m_n, m_n, 2m_n, \dots, a_n m_n\}$ of $M = \mathbb{Z}^n$. Let P be the convex hull of A in $M_{\mathbb{R}} = \mathbb{R}^n$. Then P is an integral polytope such that $P \cap M = A$. Let Σ_P be the normal fan of P in $(M_{\mathbb{R}})^* = \mathbb{R}^n$ and X_{Σ_P} the (normal) toric variety associated with Σ_P. Since P satisfies the condition of [26, Theorem 2.13], the line bundle on X_{Σ_P} associated with P is very ample. This implies that we have an isomorphism $X_{\Sigma_P} \simeq X_A \subset \mathbb{P}^{a_1+\cdots+a_n+n-1}$. Moreover we can easily see that X_{Σ_P} and hence X_A are smooth. However, according to [3, Section 5] and [7], the dual variety X_A^* of X_A is not a hypersurface in general. By our method, we can compute not only the codimension but also the degree of X_A^*. Since X_A is smooth, the Euler obstruction Eu_{X_A} of X_A is the constant function 1. By (1.6), if $n = 3$ we have

$$\delta_1 = \sum_{\Delta \prec P} (-1)^{\operatorname{codim}\Delta}(1 + \dim\Delta)\mathrm{Vol}_{\mathbb{Z}}(\Delta) \tag{4.51}$$

$$= -1 \cdot 1 \cdot 6 + 1 \cdot 2 \cdot (6 + a_1 + a_2 + a_3)$$
$$- 1 \cdot 3 \cdot (2 + 2a_1 + 2a_2 + 2a_3) + 1 \cdot 4 \cdot (a_1 + a_2 + a_3) \tag{4.52}$$

$$= 0, \tag{4.53}$$

$$\delta_2 = \sum_{\Delta \prec P} (-1)^{\operatorname{codim}\Delta} \left\{ \binom{\dim\Delta - 1}{2} - 3 \right\} \mathrm{Vol}_{\mathbb{Z}}(\Delta) \tag{4.54}$$

$$= -1 \cdot (-2) \cdot 6 + 1 \cdot (-3) \cdot (6 + a_1 + a_2 + a_3)$$
$$- 1 \cdot (-3) \cdot (2 + 2a_1 + 2a_2 + 2a_3) + 1 \cdot (-2) \cdot (a_1 + a_2 + a_3) \tag{4.55}$$

$$= a_1 + a_2 + a_3. \tag{4.56}$$

For $n = 3$, by Theorem 1.4 we thus obtain

$$\operatorname{codim} X_A^* = 2, \qquad \deg X_A^* = a_1 + a_2 + a_3. \tag{4.57}$$

More generally, for any $n \geqslant 2$ we can easily show that

$$\delta_1 = \cdots = \delta_{n-2} = 0, \qquad \delta_{n-1} = a_1 + \cdots + a_n. \tag{4.58}$$

Then by Theorem 1.4 we obtain

$$\operatorname{codim} X_A^* = n - 1, \qquad \deg X_A^* = a_1 + \cdots + a_n. \tag{4.59}$$

(ii) For $a \geqslant 2$, let $A = \{0, m_1, \ldots, m_{n-1}, m_n, 2m_n, \ldots, am_n\}$ be a finite subset of M and denote by P its convex hull in $M_{\mathbb{R}}$. Then P is an n-dimensional simplex whose vertices are $v_0 = 0$, $v_1 = m_1, \ldots, v_{n-1} = m_{n-1}$, $v_n = am_n$. Also in this case, we have an isomorphism $X_{\Sigma_P} \simeq X_A \subset \mathbb{P}^{a+n-1}$. But X_A is a singular variety this time. Nevertheless, we can compute the Euler obstruction Eu_{X_A} of X_A by our algorithm. For a subset $\alpha \subset \{0, 1, 2, \ldots, n\}$, we denote by Δ_α the face of P whose vertices are $\{v_i \mid i \in \alpha\}$. We can easily determine the values of Eu_{X_A} on the n- and $(n-1)$-dimensional T-orbits in X_A as

$$\mathrm{Eu}(P) = 1, \qquad \mathrm{Eu}(\Delta_\alpha) = 1 \quad (\sharp\alpha = n). \tag{4.60}$$

Starting from the values (4.60), by Theorem 4.7 we can calculate the values of Eu_{X_A} on the lower-dimensional T-orbits in X_A inductively:

$$\mathrm{Eu}(\Delta_\alpha) = \begin{cases} 2 - a & (0, n \notin \alpha), \\ 1 & (\text{otherwise}). \end{cases} \tag{4.61}$$

By the isomorphism $X_{\Sigma_P} \simeq X_A$, we can use also Corollary 4.4 to calculate these numbers. For example, if $n = 3$, $\mathrm{Eu}(\Delta_{12})$ is computed as

$$\begin{aligned} \mathrm{Eu}(\Delta_{12}) &= -u(\mathcal{S}_{0234}/\Delta_{12}) \cdot \mathrm{Eu}(P) + u(\mathcal{S}_{123}/\Delta_{12}) \cdot \mathrm{Eu}(\Delta_{123}) \qquad (4.62)\\ &\quad + u(\mathcal{S}_{012}/\Delta_{12}) \cdot \mathrm{Eu}(\Delta_{012}) \\ &= -a \cdot 1 + 1 \cdot 1 + 1 \cdot 1 = 2 - a. \qquad (4.63) \end{aligned}$$

Now let us compute the codimension and the degree of the dual variety X_A^* of X_A. By (1.6), we have

$$\delta_1 = \cdots = \delta_{n-1} = 0, \qquad \delta_n = 2a - 2. \tag{4.64}$$

Then by Theorem 1.4 we obtain

$$\operatorname{codim} X_A^* = n, \qquad \deg X_A^* = 2a - 2. \tag{4.65}$$

5. Characteristic cycles of constructible sheaves

In this section, we give a formula for the characteristic cycles of T-invariant constructible sheaves (see Definition 5.1 below) on toric varieties and apply it to GKZ hypergeometric systems and intersection cohomology complexes.

First, let X be a (not necessarily normal) toric variety over $\mathbb{C}$ and $T \subset X$ the open dense torus which acts on X itself. Let $X = \bigsqcup_\alpha X_\alpha$ be the decomposition of X into T-orbits.

Definition 5.1.

(i) We say that a constructible sheaf $\mathcal{F}$ on X is T-invariant if $\mathcal{F}|_{X_\alpha}$ is a locally constant sheaf of finite rank for any α.
(ii) We say that a constructible object $\mathcal{F} \in \mathbf{D}_c^b(X)$ is T-invariant if the cohomology sheaf $H^j(\mathcal{F})$ of $\mathcal{F}$ is T-invariant for any $j \in \mathbb{Z}$.

Note that the so-called T-equivariant constructible sheaves on X are T-invariant in the above sense. Recall also that to any object $\mathcal{F}$ of $\mathbf{D}_c^b(X)$ we can associate a $\mathbb{Z}$-valued constructible function $\rho(\mathcal{F}) \in \mathrm{CF}_{\mathbb{Z}}(X)$ defined by

$$\rho(\mathcal{F})(x) = \sum_{j \in \mathbb{Z}} (-1)^j \dim_{\mathbb{C}} H^j(\mathcal{F})_x \quad (x \in X). \tag{5.1}$$

If moreover $\mathcal{F}$ is T-invariant, clearly $\rho(\mathcal{F})$ is constant on each T-orbit X_α. In this case, we denote the value of $\rho(\mathcal{F})$ on X_α by $\rho(\mathcal{F})_\alpha \in \mathbb{Z}$. By using the fact that vanishing and nearby cycle functors send distinguished triangles to distinguished triangles, we can easily prove the following.

Proposition 5.2. *Let $f : X \to \mathbb{C}$ be a non-constant regular function on the toric variety X and set $X_0 = \{x \in X \mid f(x) = 0\} \subset X$. Then for any T-invariant object $\mathcal{F} \in \mathbf{D}_c^b(X)$ and $x \in X_0$ we have*

$$\chi\big(\psi_f(\mathcal{F})_x\big) = \sum_\alpha \rho(\mathcal{F})_\alpha \cdot \chi\big(\psi_f(\mathbb{C}_{X_\alpha})_x\big), \tag{5.2}$$

$$\chi\big(\varphi_f(\mathcal{F})_x\big) = \sum_\alpha \rho(\mathcal{F})_\alpha \cdot \chi\big(\varphi_f(\mathbb{C}_{X_\alpha})_x\big). \tag{5.3}$$

Now let $X \hookrightarrow Z$ be a closed embedding of the toric variety X into a smooth algebraic variety Z and $\mathcal{F} \in \mathbf{D}_c^b(X)$ a T-invariant object. We consider $\mathcal{F}$ as an object in $\mathbf{D}_c^b(Z)$ by this embedding and denote by $CC(\mathcal{F})$ its characteristic cycle in the cotangent bundle T^*Z. Then there exist some integers $m_\alpha \in \mathbb{Z}$ such that

$$CC(\mathcal{F}) = \sum_\alpha m_\alpha \big[\overline{T^*_{X_\alpha} Z}\big] \tag{5.4}$$

in T^*Z. It is well known that the coefficients m_α satisfy the formula

$$\rho(\mathcal{F}) = \sum_\alpha (-1)^{\dim X_\alpha} m_\alpha \cdot \mathrm{Eu}_{\overline{X_\alpha}}. \tag{5.5}$$

Moreover m_α are uniquely determined by this formula. Since the calculation of the Euler obstructions $\mathrm{Eu}_{\overline{X_\alpha}}$ does not depend on the choice of the embedding $X \hookrightarrow Z$ (see [17]), the coefficients m_α do not depend on the choice of the smooth ambient space Z.

Now let $N \simeq \mathbb{Z}^n$ be a $\mathbb{Z}$-lattice of rank n and σ a strongly convex rational polyhedral cone in $N_{\mathbb{R}}$. We take the dual $\mathbb{Z}$-lattice M of N and consider the polar cone $\sigma^\vee$ of σ in $M_{\mathbb{R}}$ as before. Then $X = \mathrm{Spec}(\mathbb{C}[\sigma^\vee \cap M])$ is a normal toric variety and its open dense torus T is $\mathrm{Spec}(\mathbb{C}[M])$. We denote by X_α the T-orbit which corresponds to a face Δ_α of $\sigma^\vee$ and consider the decomposition $X = \bigsqcup_{\Delta_\alpha \prec \sigma^\vee} X_\alpha$ of X into T-orbits. In this situation, we have the following result.

Theorem 5.3. *Let $X \hookrightarrow Z$ be a closed embedding of X into a smooth algebraic variety Z and $\mathcal{F} \in \mathbf{D}_c^b(X)$ a T-invariant object. Then the coefficients $m_\beta \in \mathbb{Z}$ in the characteristic cycle*

$$CC(\mathcal{F}) = \sum_{\Delta_\beta \prec \sigma^\vee} m_\beta \big[\overline{T^*_{X_\beta} Z}\big] \tag{5.6}$$

are given by the formula

$$m_\beta = \sum_{\Delta_\beta \prec \Delta_\alpha \prec \sigma^\vee} (-1)^{\dim \Delta_\alpha} \rho(\mathcal{F})_\alpha \cdot \mathrm{RSV}_{\mathbb{Z}}(\Delta_\alpha, \Delta_\beta) \tag{5.7}$$

(for the definition of the normalized relative subdiagram volume $\mathrm{RSV}_{\mathbb{Z}}(\Delta_\alpha, \Delta_\beta)$ *see Definition* 4.2*).*

Proof. Since the coefficients of the characteristic cycle $CC(\mathcal{F})$ are calculated by vanishing cycles as we explained in Section 2, by Proposition 5.2 we have

$$CC(\mathcal{F}) = \sum_{\Delta_\alpha \prec \sigma^\vee} \rho(\mathcal{F})_\alpha \cdot CC(\mathbb{C}_{X_\alpha}) \tag{5.8}$$

in T^*Z. For a face $\Delta_\beta \prec \sigma^\vee$ of $\sigma^\vee$, we will show (5.7). It is enough to prove that for any face $\Delta_\alpha \prec \sigma^\vee$ of $\sigma^\vee$ such that $\Delta_\beta \prec \Delta_\alpha$ the coefficient $m_{\alpha,\beta} \in \mathbb{Z}$ of $[\overline{T^*_{X_\beta} Z}]$ in the characteristic cycle $CC(\mathbb{C}_{X_\alpha})$ of $\mathbb{C}_{X_\alpha} \in \mathbf{D}_c^b(Z)$ is given by $m_{\alpha,\beta} = (-1)^{\dim \Delta_\alpha} \mathrm{RSV}_{\mathbb{Z}}(\Delta_\alpha, \Delta_\beta)$. Since in the case $\Delta_\beta = \Delta_\alpha$ we obtain it easily, it is enough to consider the case $\Delta_\beta \neq \Delta_\alpha$. From now on, we shall inherit and freely use the notations in the proof of Theorem 4.3. In particular, in a neighborhood of $X_\beta = T_\beta$ in X we have

$$\overline{X_\alpha} = X_{\alpha,\beta} \times T_\beta, \tag{5.9}$$

$$X = X_{\sigma,\beta} \times T_\beta \tag{5.10}$$

and there exists a unique point $q \in X_{\sigma,\beta}$ such that $\{q\} \times T_\beta = T_\beta$. Let us take a closed embedding $X_{\sigma,\beta} \hookrightarrow \mathbb{C}^m$ by which the point $q \in X_{\sigma,\beta}$ is sent to $0 \in \mathbb{C}^m$. Since the coefficient $m_{\alpha,\beta}$ in the characteristic cycle $CC(\mathbb{C}_{X_\alpha})$ is independent of the choice of the ambient manifold Z, we may replace Z by $Z' := \mathbb{C}^m \times T_\beta$ and compute it in Z'. Since $X_\alpha = T_\alpha \simeq T_{\alpha,\beta} \times X_\beta$, we obtain an isomorphism

$$\mathbb{C}_{X_\alpha} \simeq \big(\mathbb{C}_{T_{\alpha,\beta}}[-\dim \Delta_\beta]\big) \boxtimes \big(\mathbb{C}_{X_\beta}[\dim \Delta_\beta]\big) \tag{5.11}$$

in $\mathbf{D}_c^b(Z')$. Hence we get

$$CC(\mathbb{C}_{X_\alpha}) = CC\big(\mathbb{C}_{T_{\alpha,\beta}}[-\dim \Delta_\beta]\big) \times CC\big(\mathbb{C}_{X_\beta}[\dim \Delta_\beta]\big) \tag{5.12}$$

$$= CC\big(\mathbb{C}_{T_{\alpha,\beta}}[-\dim \Delta_\beta]\big) \times \big[T^*_{X_\beta} X_\beta\big] \tag{5.13}$$

in $T^*Z' = T^*(\mathbb{C}^m) \times T^*X_\beta$. Since we have $\overline{T^*_{X_\beta} Z'} = T^*_{\{0\}}(\mathbb{C}^m) \times T^*_{X_\beta} X_\beta$, $m_{\alpha,\beta}$ is equal to the coefficient of $[T^*_{\{0\}}(\mathbb{C}^m)]$ in the characteristic cycle $CC(\mathbb{C}_{T_{\alpha,\beta}}[-\dim \Delta_\beta])$ of $\mathbb{C}_{T_{\alpha,\beta}}[-\dim \Delta_\beta] \in \mathbf{D}_c^b(\mathbb{C}^m)$. Hence by taking a generic linear form $f : \mathbb{C}^m \to \mathbb{C}$ we have

$$m_{\alpha,\beta} = -\chi\big(\varphi_f\big(\mathbb{C}_{T_{\alpha,\beta}}[-\dim \Delta_\beta]\big)_0\big) \tag{5.14}$$

$$= (-1)^{\dim \Delta_\beta + 1} \chi\big(\varphi_f(\mathbb{C}_{T_{\alpha,\beta}})_0\big). \tag{5.15}$$

By applying Theorem 2.4 to the closed embedding $X_{\alpha,\beta} \hookrightarrow \mathbb{C}^m$ we obtain

$$m_{\alpha,\beta} = (-1)^{\dim \Delta_\beta + 1} \chi\big(\varphi_g(\mathbb{C}_{T_{\alpha,\beta}})_0\big), \tag{5.16}$$

where we set $g := f|_{X_{\alpha,\beta}}$. Note that if $\Delta_\beta \subsetneq \Delta_\alpha$ the stalk of $\mathbb{C}_{T_{\alpha,\beta}}$ at $0 \in X_{\alpha,\beta}$ is zero and

$$\chi\big(\varphi_g(\mathbb{C}_{T_{\alpha,\beta}})_0\big) = \chi\big(\psi_g(\mathbb{C}_{T_{\alpha,\beta}})_0\big). \tag{5.17}$$

Finally by [25, Corollary 3.6] (see also the proof of [14, Chapter 10, Theorem 2.12]) we obtain the desired formula

$$m_{\alpha,\beta} = (-1)^{\dim \Delta_\alpha} \mathrm{RSV}_{\mathbb{Z}}(\Delta_\alpha, \Delta_\beta). \tag{5.18}$$

This completes the proof. □

By the proof of Theorem 4.7, we can prove also a similar result for projective toric varieties associated with lattice points. Let A be a finite subset of $M \simeq \mathbb{Z}^n$ such that the convex hull P of A in $M_{\mathbb{R}}$ is n-dimensional. We inherit the notations in Sections 3 and 4. Let us consider the projective toric variety $X_A \subset Z = \mathbb{P}^{\sharp A-1}$ associated with A. For a face Δ_α of P, denote by X_α $(= T_\alpha)$ the T-orbit which corresponds to Δ_α. Then we obtain a decomposition $X_A = \bigsqcup_{\Delta_\alpha \prec P} X_\alpha$ of X_A into T-orbits.

Theorem 5.4. *Let $X_A \hookrightarrow Z = \mathbb{P}^{\sharp A-1}$ be the projective embedding of X_A and $\mathcal{F} \in \mathbf{D}_c^b(X_A)$ a T-invariant object. Then the coefficients $m_\beta \in \mathbb{Z}$ in the characteristic cycle*

$$CC(\mathcal{F}) = \sum_{\Delta_\beta \prec P} m_\beta \big[\overline{T^*_{X_\beta} Z}\big] \tag{5.19}$$

are given by the formula

$$m_\beta = \sum_{\Delta_\beta \prec \Delta_\alpha \prec P} (-1)^{\dim \Delta_\alpha} \rho(\mathcal{F})_\alpha \cdot i(\Delta_\alpha, \Delta_\beta) \cdot u(\mathcal{S}_\alpha/\Delta_\beta) \tag{5.20}$$

(for the definitions of $i(\Delta_\alpha, \Delta_\beta)$ and $u(\mathcal{S}_\alpha/\Delta_\beta)$ see Definitions 4.5 *and* 4.6*).*

Since the proof of this theorem is similar to that of Theorem 5.3, we omit it.

Example 5.5. Assume that the finite set $A = \{\alpha(1), \alpha(2), \dots, \alpha(m+1)\} \subset \mathbb{Z}^n$ generates $M = \mathbb{Z}^n$. For $j = 1, 2, \dots, m+1$, set $\widetilde{\alpha(j)} := (\alpha(j), 1) \in \mathbb{Z}^{n+1}$ and consider the $(n+1) \times (m+1)$ integer matrix

$$\widetilde{A} := \begin{pmatrix} {}^t\widetilde{\alpha(1)} & {}^t\widetilde{\alpha(2)} & \cdots & {}^t\widetilde{\alpha(m+1)} \end{pmatrix} = (a_{ij}) \in M(n+1, m+1; \mathbb{Z}) \tag{5.21}$$

whose j-th column is ${}^t\widetilde{\alpha(j)}$. For $\gamma \in \mathbb{C}^{n+1}$, we set

$$P_i := \sum_{j=1}^{m+1} a_{ij} x_j \frac{\partial}{\partial x_j} - \gamma_i \quad (1 \leqslant i \leqslant n+1), \tag{5.22}$$

$$\square_b := \prod_{b_j > 0} \left(\frac{\partial}{\partial x_j}\right)^{b_j} - \prod_{b_j < 0} \left(\frac{\partial}{\partial x_j}\right)^{-b_j} \quad \bigl(b \in \operatorname{Ker} \widetilde{A} \cap \mathbb{Z}^{m+1}\bigr). \tag{5.23}$$

Then the GKZ hypergeometric system on $\mathbb{C}_x^{m+1}$ associated with $\widetilde{A}$ and the parameter $\gamma \in \mathbb{C}^{n+1}$ is

$$\begin{cases} P_i f(x) = 0 & (1 \leqslant i \leqslant n+1), \\ \square_b f(x) = 0 & \bigl(b \in \operatorname{Ker} \widetilde{A} \cap \mathbb{Z}^{m+1}\bigr) \end{cases} \tag{5.24}$$

(see [12,13,28]). Let $\mathcal{D}_{\mathbb{C}_x^{m+1}}$ be the sheaf of differential operators with holomorphic coefficients on $\mathbb{C}_x^{m+1}$. Then the coherent $\mathcal{D}_{\mathbb{C}_x^{m+1}}$-module

$$\mathcal{M}_{A,\gamma} := \mathcal{D}_{\mathbb{C}_x^{m+1}} \Big/ \Biggl(\sum_{1 \leqslant i \leqslant n+1} \mathcal{D}_{\mathbb{C}_x^{m+1}} P_i + \sum_{b \in \operatorname{Ker} \widetilde{A} \cap \mathbb{Z}^{m+1}} \mathcal{D}_{\mathbb{C}_x^{m+1}} \square_b \Biggr) \tag{5.25}$$

which corresponds to the above GKZ system is holonomic. Let $\mathbb{C}_\xi^{m+1}$ be the dual vector space of $\mathbb{C}_x^{m+1}$ and $\mathcal{M}_{A,\gamma}^\wedge$ the Fourier transform of $\mathcal{M}_{A,\gamma}$ on $\mathbb{C}_\xi^{m+1}$ (see [16], etc.). We denote by S_0 the image of the map $\Psi_{\widetilde{A}} : (\mathbb{C}^*)^{n+1} \to \mathbb{C}_\xi^{m+1}$ defined by $\Psi_{\widetilde{A}}(y) = (y_1^{\widetilde{\alpha(1)}}, y_2^{\widetilde{\alpha(2)}}, \dots, y_{m+1}^{\widetilde{\alpha(m+1)}})$ and let $j : S_0 \hookrightarrow \mathbb{C}_\xi^{m+1}$ be the inclusion. Then $S_0 \subset \mathbb{C}_\xi^{m+1}$ is the cone over the open dense T-orbit in $X_A \subset \mathbb{P}^m$. In [12], for a local system $\mathcal{L}$ of rank one on S_0 Gelfand et al. constructed a morphism

$$j_! \mathcal{L}[n+1] \to R\mathcal{H}om_{\mathcal{D}_{\mathbb{C}_\xi^{m+1}}}\bigl(\mathcal{M}_{A,\gamma}^\wedge, \mathcal{O}_{\mathbb{C}_\xi^{m+1}}\bigr)[m+1] \tag{5.26}$$

in $\mathrm{Perv}(\mathbb{C}_\xi^{m+1})$. In [12, Theorem 4.6], by calculating the characteristic cycles of both sides of (5.26), they proved that (5.26) is an isomorphism if $\gamma \in \mathbb{C}^{n+1}$ is generic (non-resonant in the sense of [12, Theorem 4.6]). For each face $\Delta \prec P$ let $V_0(\Delta) \subset (\mathbb{P}^m)^*$ be the dual variety of the closure $\overline{T_\Delta} \subset \mathbb{P}^m$ of the T-orbit in X_A which corresponds to Δ and denote by $V(\Delta) \subset \mathbb{C}_x^{m+1}$ the cone over $V_0(\Delta) \subset (\mathbb{P}^m)^*$. Then by Theorem 5.4 we have

$$CC\big(j_!\mathcal{L}[n+1]\big) = \mathrm{Vol}_{\mathbb{Z}}(P)\big[T^*_{\mathbb{C}_x^{m+1}}\mathbb{C}_x^{m+1}\big] + \sum_{\Delta \prec P} i(P,\Delta)\cdot u(\Delta)\big[\overline{T^*_{V(\Delta)_{\mathrm{reg}}}\mathbb{C}_x^{m+1}}\big] \tag{5.27}$$

in $T^*\mathbb{C}_\xi^{m+1} \simeq T^*\mathbb{C}_x^{m+1}$, where for $\Delta_\beta = \Delta \prec \Delta_\alpha = P$ we set $u(\mathcal{S}_\alpha/\Delta_\beta) =: u(\Delta) \in \mathbb{Z}_{\geqslant 1}$. Moreover if $\gamma \in \mathbb{C}^{n+1}$ is non-resonant, by the proof of [13, Theorem 5] and (4.43) we can show that the characteristic cycle $CC(\mathcal{M}_{A,\gamma}) = CC(\mathcal{M}^\wedge_{A,\gamma}) = CC(R\mathcal{H}om_{\mathcal{D}_{\mathbb{C}_\xi^{m+1}}}(\mathcal{M}^\wedge_{A,\gamma}, \mathcal{O}_{\mathbb{C}_\xi^{m+1}})[m+1])$ has the same expression. It seems that the integers $i(P,\Delta)$ are forgotten in [13, Theorem 5]. Recently in [27, Theorem 4.21] Schulze and Walther proved it in the wider case where γ is not rank-jumping.

From now on, we shall apply Theorem 5.3 to the intersection cohomology complexes on projective toric varieties. Let $M \simeq \mathbb{Z}^n$ be a $\mathbb{Z}$-lattice of rank n and N its dual lattice. Let P be an integral polytope in $M_{\mathbb{R}}$ such that $\dim P = n = \dim M_{\mathbb{R}}$ and Σ_P its normal fan in $N_{\mathbb{R}}$. Denote by X_{Σ_P} the (normal) toric variety associated with Σ_P. Then by [26, Theorem 2.13], if P is sufficiently large and $A = P \cap M$, the natural morphism $\Phi_A : X_{\Sigma_P} \to \mathbb{P}^{\sharp A-1}$ induces an isomorphism $X_{\Sigma_P} \xrightarrow{\sim} X_A$. Let us consider the intersection cohomology complex $IC_{X_A} \in \mathbf{D}_c^b(X_A)$ of such a projective toric variety $X_A \simeq X_{\Sigma_P} \subset \mathbb{P}^{\sharp A-1}$. For simplicity, we set $X := X_A$ and $Z := \mathbb{P}^{\sharp A-1}$. For a face $\Delta_\alpha \prec P$ of P, denote by X_α the T-orbit in X which corresponds to Δ_α. Then $\mathcal{F} = IC_X[n] \in \mathbf{D}_c^b(X)$ is a T-equivariant perverse sheaf on X. Considering $\mathcal{F}$ as a perverse sheaf on $Z = \mathbb{P}^{\sharp A-1}$ via the embedding $X \hookrightarrow Z$, we obtain the following results. For $\Delta_\beta \prec \Delta_\alpha \prec P$, we set $V_{\alpha,\beta} := \mathrm{RSV}_{\mathbb{Z}}(\Delta_\alpha, \Delta_\beta)$ and $V_{P,\beta} := \mathrm{RSV}_{\mathbb{Z}}(P, \Delta_\beta)$ for short.

Example 5.6. For $n = 2, 3, 4$ the characteristic cycle of $\mathcal{F} = IC_X[n] \in \mathrm{Perv}(Z)$ in T^*Z is given by

(i) $n = 2$:

$$CC(\mathcal{F}) = \big[\overline{T^*_T Z}\big] + \sum_{\substack{\Delta_\beta \prec P \\ \dim \Delta_\beta = 0}} (V_{P,\beta} - 1)\big[\overline{T^*_{X_\beta} Z}\big], \tag{5.28}$$

(ii) $n = 3$:

$$\begin{aligned} CC(\mathcal{F}) = \big[\overline{T^*_T Z}\big] &+ \sum_{\substack{\Delta_\beta \prec P \\ \dim \Delta_\beta = 1}} (V_{P,\beta} - 1)\big[\overline{T^*_{X_\beta} Z}\big] \\ &+ \sum_{\substack{\Delta_\beta \prec P \\ \dim \Delta_\beta = 0}} \Bigg\{ V_{P,\beta} - \sum_{\substack{\Delta_\beta \prec \Delta_\alpha \prec P \\ \dim \Delta_\alpha = 2}} V_{\alpha,\beta} + 2 \Bigg\}\big[\overline{T^*_{X_\beta} Z}\big], \end{aligned} \tag{5.29}$$

(iii) $n = 4$:

$$
\begin{aligned}
CC(\mathcal{F}) = \big[\overline{T_T^* Z}\big] &+ \sum_{\substack{\Delta_\beta \prec P \\ \dim \Delta_\beta = 2}} (V_{P,\beta} - 1)\big[\overline{T_{X_\beta}^* Z}\big] \\
&+ \sum_{\substack{\Delta_\beta \prec P \\ \dim \Delta_\beta = 1}} \Bigg\{ V_{P,\beta} - \sum_{\substack{\Delta_\beta \prec \Delta_\alpha \prec P \\ \dim \Delta_\alpha = 3}} V_{\alpha,\beta} + 2 \Bigg\} \big[\overline{T_{X_\beta}^* Z}\big] \\
&+ \sum_{\substack{\Delta_\beta \prec P \\ \dim \Delta_\beta = 0}} \Bigg\{ V_{P,\beta} - \sum_{\substack{\Delta_\beta \prec \Delta_\alpha \prec P \\ \dim \Delta_\alpha = 3}} (V_{\alpha,\beta} + 1) + \sum_{\substack{\Delta_\beta \prec \Delta_\alpha \prec P \\ \dim \Delta_\alpha = 2}} V_{\alpha,\beta} + 1 \Bigg\} \big[\overline{T_{X_\beta}^* Z}\big].
\end{aligned}
\tag{5.30}
$$

Note that (i) is a consequence of the main result of Gonzalez-Sprinberg [15]. Also (i) and (ii) can be deduced from Theorem 5.3 and the combinatorial formula for the intersection cohomology complex $IC_X \in \mathbf{D}_c^b(X)$ proved by Fieseler [10], etc. We leave the proof to the reader. By (i) and a result of Gonzalez-Sprinberg [15], we obtain the following.

Corollary 5.7. *If $n = 2$, then the following three conditions are equivalent:*

(i) $X = X_A \simeq X_{\Sigma_P}$ *is smooth.*
(ii) $\mathrm{Eu}_X \equiv 1$ *on* X.
(iii) *The characteristic cycle* $CC(\mathcal{F})$ *of* $\mathcal{F} = IC_X[n]$ *is irreducible.*

Motivated by some calculations in the dimensions $n = 2$, 3 and 4, we conjecture that the above equivalence of (i) and (ii) holds also for $n \geqslant 3$.

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