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ON CRITICAL SUBGRAPHS OF COLOUR-CRITICAL GRAPHS

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Abstract. Some results on the distribution of critical subgraphs in colour-critical graphs are obtained. Characterizations of k-critical graphs in terms of their $(k-1)$-critical subgraphs are given. The special case $k = 4$ is considered, and it is proved that if a 4-critical graph Γ has a vertex x of large valency (compared to the number of vertices of Γ not adjacent to x), then Γ contains vertices of valency 3. Finally, a list of all 4-critical graphs with ≤ 9 vertices is exhibited.

1. Terminology and introduction

We consider *finite graphs without loops and multiple edges.* The set of *vertices* and the set of *edges* of a graph Γ are denoted $V(\Gamma)$ and $E(\Gamma)$, respectively. The *complete k-graph* is denoted $\langle k \rangle$. The terms *path* and *circuit* are used in the sense in which the corresponding terms Weg and Kreis are used in [6]. The *length* of a path or circuit is the number of edges contained in it. We allow a path to have length 0, but a circuit has length ≥ 3. A path or a circuit is odd or even according to whether its length is odd or even.

If Δ and Γ are graphs satisfying $V(\Delta) \subseteq V(\Gamma)$ and $E(\Delta) \subseteq E(\Gamma)$, then Δ is a *subgraph* of Γ, denoted $\Delta \subseteq \Gamma$. If $T \subseteq V(\Gamma)$, then $\Gamma(T)$ denotes the subgraph of Γ *spanned by T*, i.e., $V(\Gamma(T)) = T$ and $E(\Gamma(T))$ consists of all edges of $E(\Gamma)$ having both endvertices in T. If $E(\Gamma(T)) = \emptyset$, then T is an *independent* set of vertices of Γ, and T is *maximal independent* if it is not a proper subset of any other independent set of vertices of Γ. The *valency* of a vertex x of Γ, denoted $\mathrm{val}(x, \Gamma)$, is the number of edges of Γ incident with x. If $V' \subseteq V(\Gamma)$ and $E' \subseteq E(\Gamma)$, then $\Gamma - V' - E'$ denotes the subgraph of Γ obtained by deleting from Γ all vertices of V'

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and all edges of Γ incident with vertices of V', and then deleting all edges of E' that remain, without deleting any more vertices. If Γ is connected and $\Gamma - E'$ is disconnected, then E' is a *separating* set of edges of Γ.

A graph Γ is *k-colourable* if $V(\Gamma)$ can be partitioned into at most k mutually disjoint (colour) classes in such a way that each class is an independent set of vertices. If k is the least integer for which Γ is k-colourable, then k is the *chromatic number* of Γ and Γ is *k-chromatic*. An element $t \in V(\Gamma) \cup E(\Gamma)$ of a k-chromatic graph Γ is *critical* if $\Gamma - t$ is $(k-1)$-colourable. A connected graph Γ is *critical k-chromatic* (or simply *k-critical*) if it has chromatic number k and all edges – and consequently all vertices – of Γ are critical. Each vertex of a k-critical graph has valency $\geq k-1$.

The 1- and 2-critical graphs are the $\langle 1 \rangle$ and the $\langle 2 \rangle$, respectively. The 3-critical graphs are the odd circuits ([6, p. 151, Satz 12]). Hence each vertex of a k-critical graph with $k \leq 3$ has valency $k-1$. It seems hopeless to determine the structure of all 4-critical graphs. T. Gallai constructed an infinite class of regular 4-critical graphs of valency 4, thus proving that a 4-critical graph need not contain a vertex of valency 3 (see [4, p. 172, (2.3)] and [5]). M. Simonovits and the present author even proved that for any natural number h, there exist 4-critical graphs in which all vertices have valency $\geq h$ (see [8, Chapter 6] and [9]). However, it is still unknown whether any planar 4-critical graph necessarily contains a vertex of valency 3 (see [5]).

In Section 4 of this paper, we shall prove that if Γ is a 4-critical graph having a vertex adjacent to all except α vertices of Γ, then there is an upper bound $m(\alpha)$ depending only on α for the number of vertices of Γ of valency ≥ 4, i.e., if $|V(\Gamma)| > m(\alpha)$, then Γ contains at least $|V(\Gamma)| - m(\alpha)$ vertices of valency 3. The possible structures of Γ in the cases $\alpha = 2$ and $\alpha = 3$ are determined. This enables us to establish a catalogue of all 4-critical graphs with at most 8 vertices. Also the case of 9 vertices is mentioned. The proofs in Section 4 are based on a characterization in Section 3 of k-critical graphs in terms of their $(k-1)$-critical subgraphs. In Section 2, a more elementary result is obtained and also a new, simple proof of the result, that any separating set of edges in a k-critical graph contains $\geq k-1$ edges, is presented.

2. Critical subgraphs

By deleting any vertex or any edge from a k-critical graph, the remaining graph has chromatic number $k - 1$ and therefore contains at least one $(k - 1)$-critical subgraph. Theorem 2.1 gives extensions of this statement.

Theorem 2.1. *Let Γ be a k-critical graph ($k \geq 3$), and let α be an integer satisfying $1 \leq \alpha \leq k - 2$.*

(a) *By deleting at most α vertices from Γ, there exists for each vertex x of the remaining graph Δ, a $(k - \alpha)$-critical subgraph of Δ containing x.*

(b) *By deleting at most α edges from Γ, there exists for each edge e of the remaining graph Δ, a $(k - \alpha)$-critical subgraph of Δ containing e, but not containing both endvertices of any of the deleted edges.*

Theorem 2.1(a) follows easily by induction from the case $\alpha = 1$, and the case $\alpha = 1$ is an immediate corollary of a characterization of k-critical graphs given in Section 3. Theorem 2.1(a) was obtained earlier by Dirac [3, p. 45, (4)], and the following proof of Theorem 2.1(b) is similar to the proof in [3] of Theorem 2.1(a).

Proof of Theorem 2.1(b). Let $e_1, ..., e_\nu$ be a set of ν edges of Γ, where $\nu \leq \alpha$, and let e be an edge of $\Gamma - e_1 - ... - e_\nu (= \Delta)$. $\Gamma - e$ has a $(k - 1)$-colouring K with colours $1, 2, ..., k - 1$ such that the two endvertices of e both have the colour $k - 1$ and such that for $i = 1, ..., \nu$, at least one endvertex of e_i has a colour among $1, ..., \nu$. Delete from Γ all vertices having colours $1, ..., \alpha$ in K and call the remaining graph Δ'. $e_i \notin E(\Delta')$ for $i = 1, ..., \nu$, but since $\alpha \leq k - 2$, $e \in E(\Delta')$. Δ' has chromatic number $\geq k - \alpha$, because if Δ' were $(k - \alpha - 1)$-colourable, then Γ would be $(k - 1)$-colourable. However, K's restriction to $\Delta' - e$ shows that $\Delta' - e$ is $(k - \alpha - 1)$-colourable, hence e is contained in any $(k - \alpha)$-critical subgraph of Δ'. This proves Theorem 2.1(b).

For $\alpha = k - 2$, Theorem 2.1(a) is equivalent to the statement that each vertex of a k-critical graph has valency $\geq k - 1$, and (b) is trivial. The case $\alpha = k - 3$ of (b) gives a simple proof of the following well-known result [2, p. 45, Theorem 1]:

(1) *A separating set of edges of a k-critical graph ($k \geq 3$) contains at least $k - 1$ edges.*

Proof. For $k = 3$, the statement is true. Suppose that $k \geq 4$ and let E be a separating set of edges of the k-critical graph Γ. If we delete from Γ all edges of E except one, then the remaining edge of E is not contained in any circuit of the remaining graph. Then by Theorem 2.1(b) with $\alpha = k - 3$, at least $k - 2$ edges have been deleted, i.e., $|E| \geq k - 1$, and (1) has been proved. Another version of this proof based on the case $\alpha = 1$ of Theorem 2.1(b) and by induction over k can be given.

The negation of the following statement (2) would – if true – have generalized both parts of Theorem 2.1 in the case $\alpha = 1$.

(2) *An edge of a graph $\Gamma - x$, where Γ is k-critical ($k \geq 4$) and $x \in V(\Gamma)$, is not necessarily contained in a ($k - 1$)-critical subgraph of $\Gamma - x$.*

Proof. Let Γ be a graph obtained from two disjoint, k-critical graphs Δ_1 and Δ_2 ($k \geq 4$) by Hajós' construction, i.e., delete an edge (x_1, y_1) from Δ_1 and an edge (x_2, y_2) from Δ_2, identify x_1 and x_2 to a new vertex x and join y_1 and y_2 by a new edge e. Γ is k-critical and the edge e is not contained in any ($k - 1$)-critical subgraph of $\Gamma - x$. This proves (2).

The above considerations give rise to an unsolved problem: For Γ k-critical ($k \geq 4$) and $x \in V(\Gamma)$, characterize the set E_x of those edges of $\Gamma - x$ that are not contained in any ($k - 1$)-critical subgraph of $\Gamma - x$. By (1), any edge contained in a separating set of $\leq k - 3$ edges of $\Gamma - x$ belongs to E_x. A result of Dirac [3, p. 48, Corollary to Theorem 3] implies that for $k = 4$, also the converse of this statement is true, i.e., for $k = 4$, E_x consists of precisely the separating edges (also called the bridges) of $\Gamma - x$. This implies that the above examples showing (2) are the only such examples for $k = 4$. However, the situation changes for $k \geq 5$. Fig. 1 shows a 5-critical graph Γ (we leave it to the reader to check this) in which the edge e_1 belongs to E_x, but where it is not contained in any separating set of ≤ 2 edges of $\Gamma - x$. To see that $e_1 \in E_x$ assume on the contrary that there exists a 4-critical subgraph Δ of $\Gamma - x$

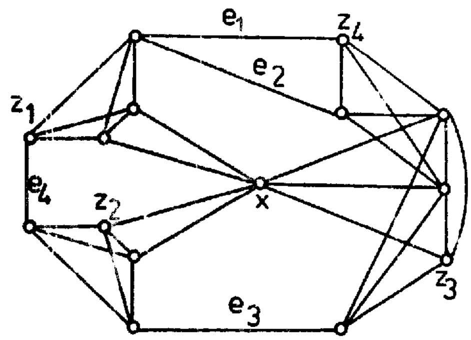


Fig. 1

containing e_1. Then by (1), also e_2 and e_3 are contained in Δ. However, $\{e_3, e_4\}$ is a separating set of two edges of $\Gamma - x$, and e_3 is therefore not contained in Δ. This is a contradiction. The example of Fig. 1 may be generalized to larger values of k by replacing each of the vertices z_1, z_2, z_3 and z_4 by a $\langle k - 4\rangle$, showing the existence of a k-critical graph ($k \geq 5$) with an edge $e \in E_x$, where e is not contained in any separating set of $\leq k - 3$ edges of $\Gamma - x$.

3. Characterizations of critical graphs

Let for a graph Γ and any vertex x of Γ, A_x denote the set of vertices of Γ not adjacent to x and let B_x denote the set of vertices of Γ adjacent to x. $V(\Gamma)$ is thus the disjoint union of $\{x\}$, A_x and B_x. Let Δ_x denote the subgraph $\Gamma - x - E(\Gamma(A_x))$.

Theorem 3.1. *Let Γ be a non-empty graph and let $k \geq 3$. The following four statements are equivalent.*

(a) *Γ is k-critical.*

(b) *For all $x \in V(\Gamma)$, the following statement holds:*

(*) *Let the maximal independent sets of vertices of $\Gamma(A_x)$ be $I_1^x, \ldots, I_\mu^x$. Then for $j = 1, \ldots, \mu$, $\Gamma - x - I_j^x$ contains a $(k-1)$-critical subgraph. Let for $j = 1, \ldots, \mu$, θ_j denote any such $(k-1)$-critical subgraph of $\Gamma - x - I_j^x$. Then $\Delta_x \subseteq \bigcup_{j=1}^{\mu} \theta_j$.*

(c) *Let $I_1, \ldots, I_\nu$ be all maximal independent sets of vertices of Γ. Then (**) holds.*

(∗∗) *For $j = 1, ..., \nu$, $\Gamma - I_j$ contains a $(k-1)$-critical subgraph. Let θ_j be any such $(k-1)$-critical subgraph of $\Gamma - I_j$. Then $\Gamma = \bigcup_{j=1}^{\nu} \theta_j$.*

(d) *There exist independent sets $I_1, ..., I_\nu$ of vertices of Γ such that (∗∗) holds and such that for at least one vertex x of Γ all maximal independent sets of vertices of Γ containing x are among $I_1, ..., I_\nu$.*

Remark 3.2. In Section 4, only the statement (a) ⇒ (b) will be used.

Remark 3.3. The θ_j's in (∗) and (∗∗) are not necessarily different and not necessarily uniquely determined.

Remark 3.4. The assertion of Theorem 2.1(a) in the case $\alpha = 1$ is an immediate consequence of (b).

Proof of Theorem 3.1. (a) ⇒ (b). Let Γ be k-critical and let $x \in V(\Gamma)$. We shall prove that (∗) holds. For each j, $1 \leq j \leq \mu$, $\{x\} \cup I_j^x$ is an independent set of vertices of Γ, hence $\Gamma - x - I_j^x$ is $(k-1)$-chromatic and therefore contains a $(k-1)$-critical subgraph θ_j. Let $t \in V(\Delta_x) \cup E(\Delta_x)$. In order to finish the proof of (∗) we shall prove that $t \in \theta_1 \cup ... \cup \theta_\mu$. Γ is k-critical, hence $\Gamma - t$ has a $(k-1)$-colouring K. Let I denote the set of vertices of A_x having the same colour as x in K. $\{x\} \cup I$ is one of the $k-1$ colour-classes of K, hence $\Gamma - x - I - t$ is $(k-2)$-colourable and since $t \in \Delta_x$, $\{x\} \cup I$ is an independent set of vertices of Γ. It follows that $\Gamma - x - I$ is $(k-1)$-chromatic and that t is contained in any $(k-1)$-critical subgraph of $\Gamma - x - I$. There exists a j such that $I \subseteq I_j^x$, hence either $t \in I_j^x$ or $t \in \theta_j$. If $t \in I_j^x$, then a $(k-1)$-colouring of Γ may be obtained from K by giving to t the colour of x in K. But Γ is k-chromatic, hence $t \in \theta_j$. This proves (a) ⇒ (b).

(b) ⇒ (c). Let Γ satisfy (b). For each integer j, $1 \leq j \leq \nu$, and $x \in I_j$ there exists an integer h such that $I_j = \{x\} \cup I_h^x$, hence by (∗), $\Gamma - I_j = \Gamma - x - I_h^x \supseteq \theta_j$, where θ_j is $(k-1)$-critical. This proves the first part of (∗∗) and implies $|V(\Gamma)| \geq k$. By Remark 3.4, we have that for any vertex x of Γ, each vertex of $\Gamma - x$ is contained in a $(k-1)$-critical subgraph of $\Gamma - x$, hence no connected component of Γ is a $\langle 1 \rangle$ or a $\langle 2 \rangle$, which implies that $\bigcup_{x \in V(\Gamma)} \Delta_x = \Gamma$. For each $x \in V(\Gamma)$ and each I_h^x, there exists a j such that $\{x\} \cup I_h^x = I_j$, hence by (∗),

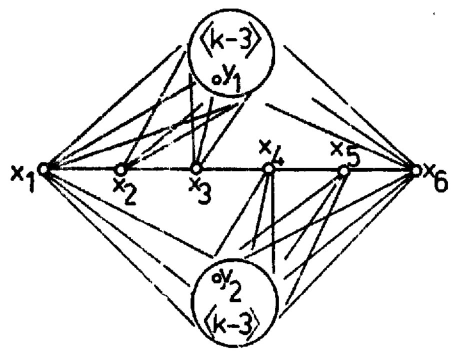


Fig. 2.

$$\Gamma = \bigcup_{x \in V(\Gamma)} \Delta_x \subseteq \bigcup_{j=1}^{\nu} \theta_j \subseteq \Gamma,$$

which proves (b) ⇒ (c).

(c) ⇒ (d). Trivial.

(d) ⇒ (a). Let Γ satisfy (d). Suppose that Γ has a $(k-1)$-colouring K and let C be the colour-class of K containing x. Then $\Gamma - C$ is $(k-2)$-colourable and since $x \in C$, there exists a j such that $C \subseteq I_j$, hence also $\Gamma - I_j$ is $(k-2)$-colourable. This contradicts (**). Hence Γ has chromatic number $\geq k$. In order to prove that Γ is k-critical it is therefore sufficient to prove that for any element $t \in V(\Gamma) \cup E(\Gamma)$, $\Gamma - t$ is $(k-1)$-colourable. Let $t \in V(\Gamma) \cup E(\Gamma)$. By (**), there exists a j such that t is necessarily contained in any $(k-1)$-critical subgraph of $\Gamma - I_j$, because otherwise each θ_j could be chosen such that $\theta_j \subseteq \Gamma - t$, contradicting $\Gamma = \bigcup_{j=1}^{\nu} \theta_j$. It follows that there exists a j such that $\Gamma - I_j - t$ is $(k-2)$-colourable, but then $\Gamma - t$ is $(k-1)$-colourable, because I_j is an independent set of vertices. This proves (d) ⇒ (a).

Theorem 3.1 has then been proved.

Let us remark that the proof of (a) ⇒ (b) may be applied also in the case where $t \in E(\Gamma(A_x))$ if $\Gamma - t$ has a $(k-1)$-colouring in which the two endvertices of t and x have not all three the same colour. This implies that if $t \in E_x$ (defined in Section 2), then in any $(k-1)$-colouring of $\Gamma - t$ the two endvertices of t and x have necessarily all three the same colour.

The condition of (d), that there exists a vertex x such that all maximal independent sets containing x are among $I_1, \ldots, I_\nu$, cannot be omitted, for $k \geq 4$ not even if $I_1, \ldots, I_\nu$ are required all to be maximal independent. This is shown, for $k \geq 4$, by the graph Γ of Fig. 2. For this graph Γ, the maximal independent sets $I_1 = \{x_2, y_2\}$, $I_2 = \{x_3, y_2\}$, $I_3 = \{x_4, y_1\}$ and $I_4 = \{x_5, y_1\}$ satisfy (**), but Γ is $(k-1)$-colourable. It can, however, be proved that a non-empty graph Γ is 3-critical, i.e., an odd circuit, if and only if there exist maximal independent sets $I_1, ..., I_\nu$ of vertices of Γ such that (**) with $k = 3$ is satisfied.

The following well-known results on k-critical graphs are easy consequences of Theorem 3.1(b).

(3) *If $A_x \neq \emptyset$, then each vertex of $\Gamma(A_x)$ has valency ≥ 1 in $\Gamma(A_x)$, especially $|A_x| \geq 2$.*

Proof. If $y \in A_x$, then $y \in V(\Delta_x)$, hence by (*), y is not contained in all maximal independent sets of vertices of $\Gamma(A_x)$. This proves (3).

(4) *If $A_x \neq \emptyset$, then $\Gamma(B_x)$ does not contain a $(k-1)$-critical subgraph.*

Proof. If θ is a $(k-1)$-critical subgraph of $\Gamma(B_x)$, then we may choose $\theta_j = \theta$ for all j, contradicting $\Delta_x \subseteq \bigcup_{j=1}^{\mu} \theta_j = \theta$.

4. 4-critical graphs

Let now Γ be a 4-critical graph. By Theorem 3.1, Γ satisfies Theorem 3.1(b) in the case $k = 4$ in which the graphs θ_j are odd circuits. M. Simonovits made me aware of this extension (for $k = 4$) of (4):

(5) *If $A_x \neq \emptyset$, then $\Gamma(B_x)$ does not contain any circuits.*

Proof. Suppose that θ is a circuit of $\Gamma(B_x)$. By (4), θ is even and therefore the two endvertices of an edge $e \in E(C)$ have different colours in a 3-colouring of $\Gamma - e$. But then Γ is 3-colourable, which is a contradiction. This proves (5).

By (5), $\Gamma(B_x) \cap \theta_j$ is either empty or it consists of a set of mutually disjoint paths. If $\Gamma(B_x) \cap \theta_j \neq \emptyset$, then let these paths be $P_1^j, P_2^j, ..., P_{f(j)}^j$, where $f(j) \geq 1$, otherwise let $f(j) = 0$.

(6) *Let $1 \leq j < m \leq \mu$ and let $1 \leq i \leq f(j)$ and $1 \leq h \leq f(m)$. Then $P_i^j \cap P_h^m$ is either empty or a path.*

Proof. If not, then $\Gamma(B_x)$ would contain a circuit, contradicting (5). This proves (6).

On the basis of Theorem 3.1(b) and (5) we get:

Theorem 4.1. *Let α be an integer ≥ 2. There exist integers $n(\alpha)$ and $m(\alpha)$ depending only on α such that if Γ is a 4-critical graph containing a vertex x of valency $|V(\Gamma)| - \alpha - 1$, then:*

(a) $|E(\Gamma)| \leq 2 \cdot |V(\Gamma)| + n(\alpha)$;

(b) *Γ contains at most $m(\alpha)$ vertices of valency ≥ 4, i.e., if $|V(\Gamma)| > m(\alpha)$, then Γ contains at least $|V(\Gamma)| - m(\alpha)$ vertices of valency 3.*

Remark 4.2. The statement for 5-critical graphs corresponding to (b) of Theorem 4.1 (i.e., the statement obtained from (b) by replacing 3 and 4 by 4, 5, respectively) is not true for any value of α. Counterexamples of infinitely many 5-critical graphs Γ containing vertices x of valencies $|V(\Gamma)| - \alpha - 1$ in which all vertices have valencies $\geq h$, where h is a given integer ≥ 5, may be constructed from the 4-critical graphs having large minimal valency (they may be constructed such that the resulting graphs have separating sets of two vertices of which x is one).

Proof of Theorem 4.1. Let Γ be 4-critical and let $x \in V(\Gamma)$ have valency $|V(\Gamma)| - \alpha - 1$ in Γ, i.e., $|A_x| = \alpha$. The number of edges of $\Gamma(B_x)$ is at most $|B_x| - 1$ by (5). The number of edges of $\Gamma(A_x)$ is at most $\frac{1}{2}\alpha(\alpha - 1)$. The number of edges having one endvertex in A_x and the other in B_x is $\leq 2 \cdot \Sigma_{j=1}^{\mu} f(j)$ by Theorem 3.1(b). μ is bounded by the number of different non-empty subsets of A_x, i.e., $\mu \leq 2^\alpha - 1$, in fact, $\mu \leq 2^\alpha - 2$ since the vertices of A_x are not independent by (3). $f(j) \leq \alpha - 1$ since the circuit θ_j is "passing through" $\Gamma(A_x)$ at most $\alpha - 1$ times. It follows that

$$\begin{aligned} |E(\Gamma)| &\leq |V(\Gamma)| - \alpha - 1 + |B_x| - 1 + \tfrac{1}{2}\alpha(\alpha - 1) + 2(2^\alpha - 2)(\alpha - 1) \\ &= 2 \cdot |V(\Gamma)| - 2\alpha - 3 + \tfrac{1}{2}\alpha(\alpha - 1) + 2(2^\alpha - 2)(\alpha - 1) \\ &= 2 \cdot |V(\Gamma)| + n(\alpha) . \end{aligned}$$

This proves (a).

By the formula $\Sigma_{y \in V(\Gamma)} \mathrm{val}(y, \Gamma) = 2|E(\Gamma)|$, we get

$$|V(\Gamma)| - \alpha - 1 + 4(|V(\Gamma)| - 1 - r) + 3r \leq 2|E(\Gamma)| ,$$

where r denotes the number of vertices of $A_x \cup B_x$ having valency 3 in Γ. By (a), this implies

$$r \geq |V(\Gamma)| - 2n(\alpha) - \alpha - 5 ,$$

hence Γ contains at most $2n(\alpha) + \alpha + 5$ vertices of valency ≥ 4. This proves (b), hence Theorem 4.1 has been proved.

Unfortunately, I know next to nothing on exact values and orders of magnitudes of the best possible $n(\alpha)$ and $m(\alpha)$ that may be used in Theorem 4.1. The estimation in the proof is not best possible. Thus the number of edges of $\Gamma(A_x)$ is $\leq \frac{1}{4}\alpha^2$ by Turán's theorem [11], and $\mu \leq 3^{\alpha/3}$ by a theorem of Moon and Moser [7]. However, even this gives an estimation which is probably far from the correct order of magnitude. The estimation of the proof gives for $\alpha = 2$, $n(2) = -2$ and $m(2) = 3$, which is best possible as we shall see in the following, where the special cases $\alpha = 2$ and $\alpha = 3$ are treated. Dirac and Gallai conjectured [5, p. 44, Conjecture] that if Γ is assumed also to be planar, then $n(\alpha) = -2$ may be used for all values of α.

4.1. $\alpha = 2$

Let $A_x = \{y, z\}$. By (3), $\Gamma(A_x) = \langle 2 \rangle$ and $\mu = 2$. We may assume that $I_1^x = \{y\}$ and $I_2^x = \{z\}$. By (5), the odd circuit θ_1 contains z, but not y, and the odd circuit θ_2 contains y, but not z. By Theorem 3.1(b),

$$\Delta_x = \Gamma - x - (y, z) = \theta_1 \cup \theta_2 ,$$

and by (c) $\theta_1 \cap \theta_2$ is either empty or a path. If $\theta_1 \cap \theta_2$ is a path with ≥ 2 vertices, then none of the endvertices of that path are adjacent to both vertices of A_x since otherwise the edge from x to such an endvertex would not be critical. (The case $\alpha = 2$ was also considered in [10, §6].)

4.2. $\alpha = 3$

Let $A_x = \{p, y, z\}$. By (3), $\Gamma(A_x)$ is either

(i) a path, or
(ii) complete.

Let us consider the two cases in turn.

(i) $E(\Gamma(A_x)) = \{(p, y), (y, z)\}$.

$\mu = 2$ and we may assume that $I_1^x = \{p, z\}$ and $I_2^x = \{y\}$. By (5), the odd circuit θ_1 contains y, but not p nor z. By Theorem 3.1(b), the odd circuit θ_2 contains p and z, but not y, and it consists therefore of an even path P_1 of length ≥ 2 and an odd path P_2 of length ≥ 3 both joining p and z and with no interior vertices in common. By Theorem 3.1(b),

$$\Delta_x = \Gamma - x - (p, y) - (y, z) = \theta_1 \cup P_1 \cup P_2 .$$

If $P_i \cap \theta_1 \neq \emptyset$, then $P_i \cap \theta_1$ is a path by (6) and $P_j \cap \theta_1 = \emptyset$ ($j = 1$ if $i = 2$ and conversely) since otherwise the edges (p, y) and (y, z) would not both be critical. If P_1 has length 2 and if the interior vertex v of P_1 is on θ_1, then v is not adjacent to y since otherwise (x, v) would not be critical.

(ii) $\Gamma(A_x) = \langle 3 \rangle$.

$\mu = 3$ and we may assume that $I_1 = \{p\}$, $I_2 = \{y\}$ and $I_3 = \{z\}$. By (5), each of θ_1, θ_2 and θ_3 contains at least one vertex of A_x. We shall consider two cases.

(ii.1) Suppose that one of θ_1, θ_2 and θ_3 may be chosen such that it contains only one vertex of A_x, say θ_1 contains y, but not p nor z. Then we may choose $\theta_3 = \theta_1$. By Theorem 3.1(b), θ_2 contains necessarily both p and z, but not y, hence there is in Δ_x an even path P of length ≥ 2 joining p and z and not containing y. θ_2 may therefore be chosen as P together with the edge (p, z). By Theorem 3.1(b),

$$\Delta_x = \theta_1 \cup P ,$$

where $\theta_1 \cap P = \emptyset$ since otherwise one of the two edges of θ_1 incident with y would not be critical.

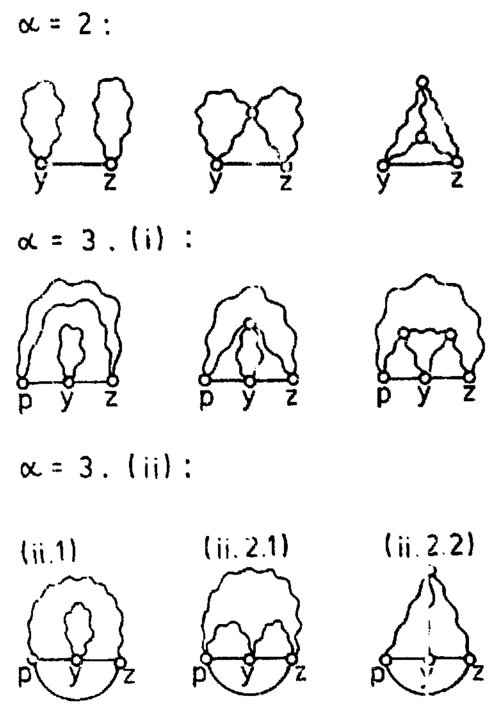


Fig. 3.

(ii.2) The alternative to consider is the case where each of θ_1, θ_2 and θ_3 necessarily contains precisely two vertices of A_x. In this case, θ_1, θ_2 and θ_3 may be chosen as even paths P_1, P_2 and P_3 of length ≥ 2 joining the vertices y and z, p and z, p and y respectively, together with the edges (y, z), (p, z) and (p, y), respectively. By Theorem 3.1(b),

$$\Delta_x = P_1 \cup P_2 \cup P_3 \,.$$

One possibility is that P_1, P_2 and P_3 are mutually disjoint outside A_x. The alternative is that, say, $V(P_1) \cap V(P_2) \cap B_x \neq \emptyset$. Let q be the first vertex of P_1 (going from y towards z) belonging also to P_2. Let R_1, R_2 and R_3 be three paths joining q with p, y and z respectively, where R_1 and R_3 are parts of P_2 and R_2 is a part of P_1. R_1, R_2 and R_3 are mutually disjoint except for the vertex q, and since P_2 is even, R_1 and R_3 have the same parity. By (6), $P_1 \cap P_2$ is a path and since (ii.1) is not the case, also R_2 and R_3 have the same parity, i.e., R_1, R_2 and R_3 have all the same parity. Then θ_1, θ_2 and θ_3 may be chosen as $R_2 \cup R_3 \cup (y, z)$, $R_1 \cup R_3 \cup (p, z)$ and $R_1 \cup R_2 \cup (p, y)$, respectively. Hence by Theorem 3.1(b),

$$\Delta_x = R_1 \cup R_2 \cup R_3 .$$

Since $\Gamma - x$ is 3-colourable, the lengths of R_1, R_2 and R_3 are not all 1.

This completes the treatment of the cases $\alpha = 2$ and $\alpha = 3$. Fig. 3 shows the various possibilities for $\Gamma - x$. It can be proved that the conditions on the structure of Γ that we have established are not only necessary, but also sufficient conditions for Γ to be 4-critical. The analysis gives as a corollary:

Theorem 4.3. *Let* Γ *be a 4-critical graph with a vertex* x *of valency* $|V(\Gamma)| - \alpha - 1$.

(a) *If* $\alpha = 2$, *then all vertices of* $\Gamma - x$ *have valency* 3 *in* Γ *except possibly* either *one vertex of valency* 5 or *two vertices each of valency* 4. *All three cases occur.*

(b) *If* $\alpha = 3$, *then all vertices of* $\Gamma - x$ *have valency* 3 *in* Γ *except* either *one vertex of valency* 4 or *two vertices of valencies* 4 *and* 5 *respectively*, or *three vertices each of valency* 4. *All three cases occur.*

It follows from Theorem 4.3 that the best possible values of $n(\alpha)$ and $m(\alpha)$ in the cases $\alpha = 2$ and $\alpha = 3$ are $n(2) = n(3) = -2$, $m(2) = 3$ and $m(3) = 4$.

The case $\alpha = 4$ has also been considered, and by the above method it is in this case possible to prove:

(7) *There exists precisely one 4-critical graph with* 9 *vertices in which each vertex has valency* 3 *or* 4, *namely the last graph of Fig. 4.*

I shall leave out my rather cumbersome proof of (7).

If Γ is a 4-critical graph with at most 8 vertices, then by the theorem of Brooks [1] it contains a vertex x of valency $|V(\Gamma)| - \alpha - 1$ for either $\alpha = 0$, $\alpha = 2$ or $\alpha = 3$. By (7), this holds also if $|V(\Gamma)| = 9$ and $\Gamma \neq$ the graph of (7). If $\alpha = 0$, then it is well known that Δ_x is an odd circuit, hence the above analysis provides us with a complete list of all 4-critical graphs with ≤ 9 vertices. There are 30 such graphs and they are exhibited in Fig. 4. One of the graphs has 4 vertices, one has 6 vertices, 2 have 7 vertices, 5 have 8 vertices and 21 have 9 vertices.

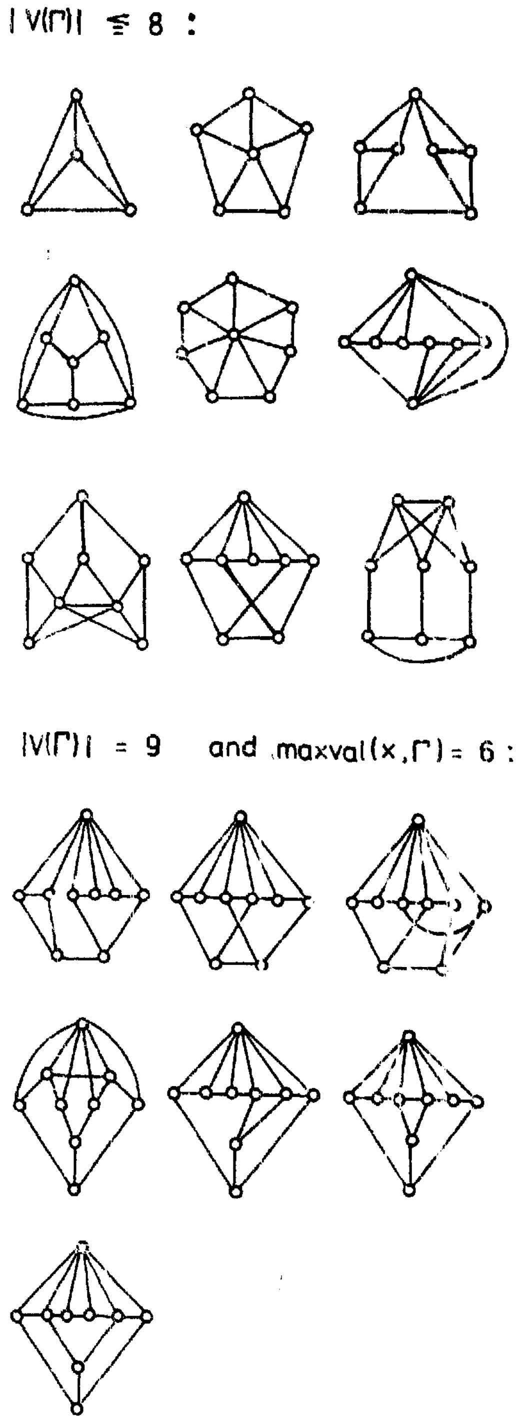
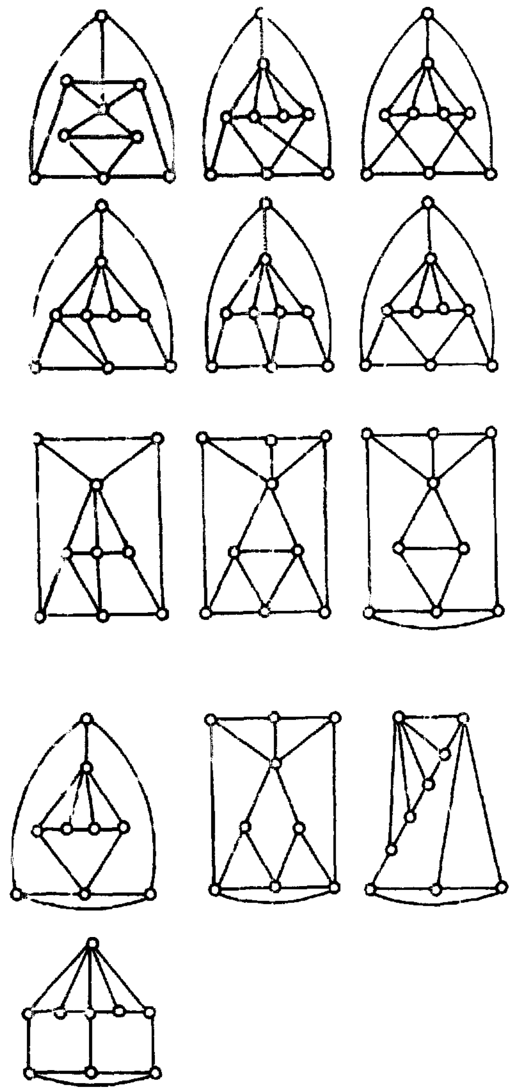


Fig. 4.

$|V(\Gamma)| = 9$ and $\mathrm{maxval}(x,\Gamma) = 5$:



$|V(\Gamma)| = 9$ and $\mathrm{maxval}(x,\Gamma) = 4$:

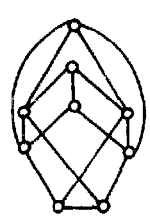


Fig. 4 (continued).

References

[1] R.L. Brooks. On colouring the nodes of a network, Proc. Cambridge Philos. Soc. 37 (1941) 194–197.
[2] G.A. Dirac, The structure of k-chromatic graphs, Fund. Math. 40 (1953) 42–55.
[3] G.A. Dirac, On the structure of 5- and 6-chromatic abstract graphs, J. Reine Angew. Math. 214/215 (1964) 43–52.
[4] T. Gallai, Kritische Graphen I, Publ. Math. Inst. Hungar. Acad. Sci. 8 (1963) 165–192.
[5] T. Gallai, Critical graphs, in: Theory of Graphs and its Applications, Proc. of the Symposium held in Smolenice in June 1963 (Czechoslovak Akad. Sciences, Prague, 1964) 43–45.
[6] D. König, Theorie der endlichen und unendlichen Graphen (Leipzig, 1936; Chelsea, New York, 1950).
[7] J.W. Moon and L. Moser, On cliques in graphs, Israel J. Math. 3 (1965) 23–28.
[8] B. Toft, Some contributions to the theory of colour-critical graphs, Ph.D. Thesis, University of London, 1970, published as No. 14 in Various Publications Series, Matematisk Institut, Aarhus Universitet.
[9] B. Toft, Two theorems on critical 4-chromatic graphs, Studia Sci. Math. Hungar. 7 (1972) 83–89.
[10] B. Toft. Colour-critical graphs with complements of low connectivity, submitted to Math. Scand.
[11] P. Turán, On the theory of graphs, Colloq. Math. 3 (1954) 19–30.