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On the Krein–Milman Property and the Bade Property

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ABSTRACT

Throughout this paper a study on the Krein–Milmam Property and the Bade Property is entailed reaching the following conclusions: If a real topological vector space satisfies the Krein–Milmam Property, then it is Hausdorff; if a real topological vector space satisfies the Krein–Milmam Property and is locally convex and metrizable, then all of its closed infinite dimensional vector subspaces have uncountable dimension; if a real pseudo-normed space has the Bade Property, then it is Hausdorff as well but could allow closed infinite dimensional vector subspaces with countable dimension. On other hand, we show the existence of infinite dimensional closed subspaces of ℓ_∞ with the Bade Property that are not the space of convergence associated to any series in a real topological vector space. Finally, we characterize unconditionally convergent series in real Banach spaces by means of a new concept called *uniform convergence of series*.

1. Introduction

This paper is a step forward on the ongoing search for the solutions to several open problems on the Krein–Milman Property and the Bade Property. The most interesting problem related to the Krein–Milmam Property is on determining whether a Banach space enjoying the Krein–Milman Property also has the Radon–Nikodym Property. The origin of this well-known and long-standing open problem goes back to the Summer of 1973 where Lindenstrauss (see, for instance, [7]) showed that every Banach space having the Radon–Nikodym Property also enjoys the Krein–Milman Property. In [10] the authors approach the above problem in the positive by proving that if a dual Banach space has the Krein–Milman Property, then it also verifies the Radon–Nikodym Property. We refer the reader to [15,

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Chapter 5, Section 5.4] for a proper discussion of the problem about the Radon–Nikodym Property being equivalent to the Krein–Milman Property. Another of the open problems mentioned at the very beginning has been recently treated in the early nineties and the early 2000's (see [1,2,14]) and deals with the relation between the Bade Property, the λ-Property, and the spaces of convergence associated to a series in a real Banach space. Other problems related to this topic in real topological vector spaces have been recently considered in [8] and in [9] and mostly deal with the structure of convex sets in real topological vector spaces as well as with the topological and algebraical impact of the Krein–Milman Property on real topological vector spaces. At this point, we remind the reader about the main concepts managed in this paper.

Definition 1.1 (*Krein and Milman [13]*)**.** Let X be a real topological vector space:

1. A closed bounded convex subset M of X is said to have the Krein–Milman Property exactly when $\operatorname{ext}(M) \neq \varnothing$.
2. X is said to have the Krein–Milmam Property exactly when every closed bounded and convex subset of X enjoys the Krein–Milman Property.

This definition finds its birth in the very well known Krein–Milmam Theorem (see [13]).

Theorem 1.1 (*Krein and Milman* [13])**.** *Let X be a Hausdorff locally convex real topological vector space. Let M be a compact convex subset of X. Then M has the Krein–Milman Property.*

The Krein–Milmam Property was originally defined for Hausdorff locally convex real topological vector spaces in the following way: A closed bounded convex subset M of a Hausdorff locally convex real topological vector space X is said to have the Krein–Milman Property exactly when $M = \overline{\text{co}}\,(\operatorname{ext}(M))$. Essentially, it can be shown that in a Hausdorff locally convex real topological vector space the fact that every closed bounded convex subset is the closed convex hull of its extreme points is equivalent to the fact that every closed bounded convex subset has an extreme point. However, this equivalence does not hold in general for real topological vector spaces. This is the reason for Definition 1.1. Examples of real topological vector spaces verifying the Krein–Milman Property include all Banach spaces with the Radon–Nikodym Property (in particular, all reflexive Banach spaces). In close connection with the Krein–Milman Property we find the Bade Property.

Definition 1.2 (*Bade [3]*)**.** A real pseudo-normed space X is said to have the Bade Property exactly when B_X satisfies the Krein–Milman Property in the original sense, that is, $\mathsf{B}_X = \overline{\text{co}}\,(\operatorname{ext}(\mathsf{B}_X))$.

The origins of this definition are to be found in Bade's Dissertation. Among a bunch of results, one can find its well known Bade's Theorem (see [3]):

Theorem 1.2 (*Bade [3]*)**.** *Let K be a Hausdorff compact topological space. Then $\mathcal{C}(K, \mathbb{R})$ has the Bade Property if and only if K is 0-dimensional.*

The next two sections of this paper are devoted to show that the Krein–Milman Property has a considerable exclusivity in the class of all real topological vector spaces. In concrete terms:

1. If a real topological vector space has the Krein–Milman Property, then it has to be Hausdorff.
2. If, in addition, the real topological vector space is locally convex and metrizable, then every closed infinite dimensional subspace must have uncountable dimension. Our technique also allowed us to find a nearly elementary proof of the fact that an infinite dimensional real Banach space must have uncountable dimension without involving the Baire Category Theorem.
3. The Bade Property implies the Hausdorff-ness of the space as well, but could allow some closed infinite dimensional subspaces to have countable dimension.

On the other hand, as mentioned at the beginning of the introduction, another well known problem (see [14]) is to determine when the convergence space associated to a given series satisfies the Bade Property.

Definition 1.3 (*Pérez-Fernández et al. [14]*). Let X be a real topological vector space. Let $\sum_{n=1}^{\infty} x_n$ be a series in X. The convergence space associated to $\sum_{n=1}^{\infty} x_n$ is defined as

$$\mathcal{S}\left(\sum_{n=1}^{\infty} x_n\right) := \left\{(a_n)_{n\in\mathbb{N}} \in \ell_\infty : \sum_{n=1}^{\infty} a_n x_n \text{ converges in } X\right\}.$$

In [14] it is proved that a series $\sum_{n=1}^{\infty} x_n$ in a real Banach space is weakly unconditionally Cauchy if and only if $\mathcal{S}\left(\sum_{n=1}^{\infty} x_n\right)$ is complete. On the other hand (see [2]) a series $\sum_{n=1}^{\infty} x_n$ in a real Banach space is conditionally convergent and weakly unconditionally Cauchy if and only if $c \subseteq \mathcal{S}\left(\sum_{n=1}^{\infty} x_n\right) \subsetneq \ell_\infty$. An interesting question is to determine for which series $\sum_{n=1}^{\infty} x_n$ does $\mathcal{S}\left(\sum_{n=1}^{\infty} x_n\right)$ have the Bade Property. It is well known [4,5] that a series $\sum_{n=1}^{\infty} x_n$ in a real Banach space is unconditionally convergent if and only if $\mathcal{S}\left(\sum_{n=1}^{\infty} x_n\right) = \ell_\infty$. Therefore, the space of convergence associated to unconditionally convergent series (in real Banach spaces) has the Bade Property. As far as we know, these are the only known examples of series whose associated convergence space has the Bade Property. Our contribution on this topic is in the last three sections of this paper, where we provide the following results among others:

1. c is not the space of convergence of any series in any real Banach space.
2. There are closed subspaces of ℓ_∞ with the Bade Property that are not the space of convergence associated to any series in any real topological vector space.
3. There exists a conditionally convergent series in a real Banach space whose associated convergence space does have the Bade Property.
4. Unconditionally convergent series in real Banach spaces can be characterized through their uniform convergence on the extreme points of B_{ℓ_∞}.

2. The topological impact of the Krein–Milman Property

This section is aimed at showing that real topological vector spaces enjoying the Krein–Milman Property must be Hausdorff.

Theorem 2.1. *Let X be a real topological vector space. The set*

$$\{x \in X : x \text{ belongs to any neighborhood of } 0\}$$

is a closed bounded vector subspace of X whose induced topology is the trivial topology. Furthermore, it is always topologically complemented with every subspace with which is algebraically complemented.

Proof. Let us denote the above set by N. Then:

1. N is a vector subspace of X: Indeed, let U be any neighborhood of 0 and let $n, m \in N$ and $\alpha, \beta \in \mathbb{R}$. There exists a neighborhood V of 0 such that $V + V \subseteq U$. Now, there are W_1 and W_2 neighborhoods of 0 such that $\alpha W_1, \beta W_2 \subseteq V$. Observe that $n \in W_1$ and $m \in W_2$. Therefore

$$\alpha n + \beta m \in \alpha W_1 + \beta W_2 \subseteq V + V \subseteq U.$$

2. N is closed: Indeed, let $x \in X \setminus N$. There exists a neighborhood U of 0 such that $x \notin U$. There exists another neighborhood V of 0 such that $V + V \subseteq U$. Finally, $x + V$ is a neighborhood of x such that $(x + V) \cap N = \varnothing$.

3. N is bounded: Obvious since

$$N = \bigcap \{U \subseteq X : U \text{ is a neighborhood of } 0\} .$$

4. The relative topology of N is the trivial topology: Obvious from the above equality.
5. N is complemented in X: Indeed, let M be another vector subspace of X such that $N \cap M = \{0\}$ and $X = M + N$. Observe that the linear projection

$$\begin{aligned} X &\to N \\ m + n &\mapsto n \end{aligned}$$

is continuous since the induced topology on N is the trivial topology. Another way to show that the topology on X coincides with the product topology on $M \oplus N$ is by means of nets. If $(m_i + n_i)_{i \in I}$ is a net of $M \oplus N$ converging to $m + n \in M \oplus N$, then $(n_i)_{i \in I}$ converges to n (again, because the topology on N is trivial). Therefore, $(m_i)_{i \in I}$ converges to m. On the other hand, observe that M is not closed (unless $N = \{0\}$). Indeed, $0 \in M \subseteq \text{cl}\,(M)$, therefore $N \subseteq \text{cl}\,(M)$ and hence M is dense in X. $\square$

Corollary 2.1. *Let X be a real topological vector space. If X has the Krein–Milmam Property, then X is Hausdorff.*

The same situation occurs with the Bade Property.

Lemma 2.1. *Let X be a real pseudo-normed space. If $n \in X$ is so that $\|n\| = 0$, then $\|m + n\| = \|m\|$ for all $m \in X$.*

Proof. Observe that

$$\|m\| = |\|m\| - \|-n\|| \leqslant \|m + n\| \leqslant \|m\| + \|n\| = \|m\| . \quad \square$$

Theorem 2.2. *Let X be a real pseudo-normed space. If X is not Hausdorff, then B_X is free of extreme points.*

Proof. Define $N := \{x \in X : \|x\| = 0\}$. In accordance with Theorem 2.1, N is a bounded closed vector subspace of X whose induced topology is trivial and is topologically complemented with any subspace with which is algebraically complemented. Let M be an algebraical complement for N in X. We will show now that B_X is free of extreme points. Let $x \in \mathsf{B}_X$. There are $m \in M$ and $n \in N$ such that $x = m + n$. By Lemma 2.1, $\|x\| = \|m\| = \|m + 2n\|$, so $m, m + 2n \in \mathsf{B}_X$. Finally,

$$x = \frac{1}{2}m + \frac{1}{2}(m + 2n) ,$$

so $x \notin \text{ext}\,(\mathsf{B}_X)$. $\square$

Corollary 2.2. *Let X be a real pseudo-normed space. If X has the Bade Property, then X is Hausdorff.*

3. The algebraical impact of the Krein–Milman Property

This section is aimed at showing that the infinite dimensional closed vector subspaces of those metrizable locally convex real topological vector spaces enjoying the Krein–Milman Property must have uncountable dimension. In the meantime, we found an easier proof of the fact the infinite dimensional Banach spaces must have uncountable dimension without involving the Baire Category Theorem.

Theorem 3.1. *Let X be a Hausdorff locally convex real topological vector space. Assume that the dimension of X is countably infinite. There exists a biorthogonal system* $(e_n, e_n^*)_{n\in\mathbb{N}} \subseteq X \times X^*$ *such that* $X = \operatorname{span}\{e_n : n \in \mathbb{N}\}$ *and* $X^* = \overline{\operatorname{span}}^{\,\omega^*}\{e_n^* : n \in \mathbb{N}\}$.

Proof. Let $(u_n)_{n\in\mathbb{N}} \subset X$ be a Hamel basis for X. We will construct the biorthogonal system inductively:

Step 1. Choice of e_1:
- Step 1.1. Take $e_1 := u_1$. Obviously, $\operatorname{span}\{e_1\} = \operatorname{span}\{u_1\}$.
- Step 1.2. The Hahn–Banach Theorem allows us to find $e_1^* \in X^*$ such that $e_1^*(e_1) = 1$.

Step 2. Choice of e_2:
- Step 2.1. Take $e_2 := u_2 - e_1^*(u_2)\,u_1$. Note that $\operatorname{span}\{e_1, e_2\} = \operatorname{span}\{u_1, u_2\}$.
- Step 2.2. The Hahn–Banach Theorem allows us to find $e_2^* \in X^*$ such that $1 = e_2^*(e_2) > \sup e_2^*(\mathbb{R}e_1)$. Therefore $e_2^*(e_1) = 0$.

Step 3. Choice of e_3:
- Step 3.1. Take $e_3 := u_3 - e_1^*(u_3)\,e_1 - e_2^*(u_3)\,e_2$. Observe that $\operatorname{span}\{e_1, e_2, e_3\} = \operatorname{span}\{u_1, u_2, u_3\}$.
- Step 3.2. The Hahn–Banach Theorem allows us to find $e_3^* \in X^*$ such that $1 = e_3^*(e_3) > \sup e_3^*(\mathbb{R}e_1 \oplus \mathbb{R}e_2)$. Therefore $e_3^*(e_1) = e_3^*(e_2) = 0$.

We omit the rest of the steps. To see that $X^* = \overline{\operatorname{span}}^{\,\omega^*}\{e_n^* : n \in \mathbb{N}\}$ it suffices to realize that $\operatorname{span}\{e_n^* : n \in \mathbb{N}\}$ separates points of X. □

The first corollary we derive from the previous result is an easier proof of the fact that infinite dimensional Banach spaces must have uncountable dimension. Notice that we do not make use of the Baire Category Theorem.

Corollary 3.1. *Let X be a real normed space. Assume that the dimension of X is countably infinite. Then there exists an absolutely convergent series in X which is non-convergent, in other words, X is not complete.*

Proof. By Theorem 3.1, there exists a biorthogonal system $(e_n, e_n^*)_{n\in\mathbb{N}} \subseteq X \times X^*$ such that $X = \operatorname{span}\{e_n : n \in \mathbb{N}\}$ and $X^* = \overline{\operatorname{span}}^{\,\omega^*}\{e_n^* : n \in \mathbb{N}\}$. We may assume that $(e_n)_{n\in\mathbb{N}} \subset \mathsf{S}_X$. Note that the series $\sum_{n=1}^{\infty} \frac{1}{2^n} e_n$ is absolutely convergent. Assume that $\sum_{n=1}^{\infty} \frac{1}{2^n} e_n$ is convergent in X. There are $\lambda_1, \ldots, \lambda_p \in \mathbb{R}$ such that $\sum_{n=1}^{\infty} \frac{1}{2^n} e_n = \sum_{n=1}^{p} \lambda_n e_n$. Finally,

$$\frac{1}{2^{p+1}} = e_{p+1}^*\left(\sum_{n=1}^{\infty} \frac{1}{2^n} e_n\right) = e_{p+1}^*\left(\sum_{n=1}^{p} \lambda_n e_n\right) = 0,$$

which is impossible. □

It is time now to show that closed infinite dimensional subspaces of a metrizable locally convex real topological vector space with the Krein–Milmam Property must have uncountable dimension. We will rely again on Theorem 3.1.

Remark 3.1. In [9] it is shown that if X is a real topological vector space and $(e_n)_{n\in\mathbb{N}} \subset X$ is a linearly independent sequence, then the set

$$M := \left\{ \sum_{n=1}^{\infty} \lambda_n e_n : (\lambda_n)_{n\in\mathbb{N}} \in c_{00} \text{ and } |\lambda_n| \leqslant \frac{1}{2^n} \text{ for all } n \in \mathbb{N} \right\}$$

satisfies the following:

1. M is absolutely convex but free of extreme points.
2. If $(e_n)_{n\in\mathbb{N}}$ is bounded, then M is bounded.

3. If there exists a biorthogonal system $(e_i, e_i^*)_{i\in I} \subset X \times X^*$ such that $(e_n)_{n\in\mathbb{N}} \subset (e_i)_{i\in I}$, then M is closed in $\operatorname{span}\{e_i : i \in I\}$.

Corollary 3.2. *Let X be a metrizable locally convex real topological vector space. Assume that X has the Krein–Milmam Property. If Y is a closed infinite dimensional subspace of X, then the cardinality of a Hamel basis of Y is uncountable.*

Proof. Let Y be a closed infinite dimensional subspace of X. Assume that the dimension of Y is countable. In accordance with Theorem 3.1, there exists a biorthogonal system $(e_n, e_n^*)_{n\in\mathbb{N}} \subseteq Y \times Y^*$ such that $Y = \operatorname{span}\{e_n : n \in \mathbb{N}\}$. Since Y is first countable, we may assume that $(e_n)_{n\in\mathbb{N}}$ is bounded. Finally, by applying Remark 3.1 we deduce the contradiction that Y does not have the Krein–Milmam Property. $\square$

Remark 3.2. Let X be a real topological vector space and assume that $(e_n, e_n^*)_{n\in\mathbb{N}} \subseteq X \times X^*$ is a biorthogonal system such that $X = \operatorname{span}\{e_n : n \in \mathbb{N}\}$. The set M considered in Remark 3.1 is also a generator system of X, therefore it is absorbing (see [8, Lemma 2.4]) and hence a barrel.

The previous remark motivates the following question:

Question 3.1. *Let X be a Hausdorff locally convex real topological vector space. Assume that $(e_i, e_i^*)_{i\in I} \subseteq X \times X^*$ is a biorthogonal system in such a way that $X = \operatorname{span}\{e_i : i \in I\}$. Does there exists an extreme point-free closed absolutely convex subset M of $\operatorname{span}\{e_i : i \in I\}$ containing $\{e_i : i \in I\}$?*

Another interesting question would be determining whether a real Hausdorff locally convex topological vector space with a boundedly complete basis has the Krein–Milman Property. We remind the reader that a basic sequence $(b_j)_{j\in\mathbb{N}}$ in a real Banach space is boundedly complete provided whenever scalars $(c_j)_{j\in\mathbb{N}}$ satisfy

$$\sup_{n\in\mathbb{N}} \left\| \sum_{j=1}^{n} c_j b_j \right\| < \infty,$$

then $\sum_{j=1}^{n} c_j b_j$ converges. The notion of boundedly complete basis in the setting of real Hausdorff locally convex topological vector spaces has been defined and studied in [11,12]. To finish this section we will show that the Bade Property does not have such a strong impact on the dimension of the infinite dimensional closed vector subspaces.

Example 3.1. Let X be any countably infinite dimensional real normed space. Since X is separable, it is well known that X admits an equivalent rotund renorming (see for instance [6]). As a consequence, X enjoys the Bade Property endowed with this equivalent norm. However, no infinite dimensional closed vector subspace of X has uncountable dimension.

4. A simplified reformulation of the Krein–Milman Property

Our reformulation of the Krein–Milmam Property relies on the fact that checking whether or not a certain real topological vector space has the Krein–Milmam Property depends *only* on the bounded closed absolutely convex subsets.

Theorem 4.1. *Let X be a real topological vector space. The following conditions are equivalent:*

1. *X has the Krein–Milmam Property.*
2. *If $M \subseteq X$ is closed, bounded and absolutely convex, then* $\operatorname{ext}(M) \neq \varnothing$.

Proof. Assume that X does not have the Krein–Milmam Property. Then there exists a closed, bounded and convex subset N of X free of extreme points. Take M to be the closed absolutely convex hull of N. It is easy to see that M is bounded and that $M = \{\lambda n : n \in N, |\lambda| \leqslant 1\}$. Let $m \in M \setminus \{0\}$. There are $0 < |\lambda| \leqslant 1$ and $n \in N$ such that $m = \lambda n$. Since $\operatorname{ext}(N) = \varnothing$, there must exist $t \in (0, 1)$ and $n_1 \neq n_2 \in N$ such that $n = tn_1 + (1-t)\, n_2$. Note then that $m = t\,(\lambda n_1) + (1-t)\,(\lambda n_2)$. As a consequence, $\operatorname{ext}(M) = \varnothing$. □

The previous result simplifies the rule to follow in order to check whether a certain real topological vector space enjoys or not the Krein–Milmam Property. The previous result also serves as motivation for the following question:

Question 4.1. *Let X be a real topological vector space such that every barrel of every closed subspace of X has extreme points. Does then X have the Krein–Milman Property?*

The reader may notice that a positive answer to the previous question would provide an even simpler procedure to check whether or not a certain real topological vector space has the Krein–Milman Property. The previous question also motivates the following one, with which we finalize this section:

Question 4.2. *Let X be a real Banach space such that every closed subspace of X enjoys the Bade Property under any equivalent norm. Does then X have the Krein–Milman Property?*

5. Some preliminary results on series in real topological vector spaces

At this point, we would like to recall the reader about several fundamental facts that we will rely on throughout the next sections. In the first place,

$$\operatorname{ext}\left(\mathsf{B}_{\ell_\infty}\right) = \left\{(\varepsilon_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : |\varepsilon_n| = 1 \text{ for all } n \in \mathbb{N}\right\}.$$

On the other hand, $\operatorname{ext}\left(\mathsf{B}_c\right) = \operatorname{ext}\left(\mathsf{B}_{\ell_\infty}\right) \cap c$, and $\operatorname{ext}\left(\mathsf{B}_{c_0}\right) = \operatorname{ext}\left(\mathsf{B}_{c_{00}}\right) = \varnothing$. Next, we will point out a series of results which will be of helpful use for our purposes. The first lemma is crucial to achieve our goals since it completely describes the extreme points of the convergence space of associated to a given series.

Lemma 5.1. *Let X be a real topological vector space. Let $\sum_{n=1}^{\infty} x_n$ be a series in X and denote $\mathcal{S} := \mathcal{S}\left(\sum_{n=1}^{\infty}\right)$. Then* $\operatorname{ext}\left(\mathsf{B}_{\mathcal{S}}\right) = \mathsf{B}_{\mathcal{S}} \cap \operatorname{ext}\left(\mathsf{B}_{\ell_\infty}\right)$.

Proof. Obviously, $\operatorname{ext}\left(\mathsf{B}_{\mathcal{S}}\right) \supseteq \mathsf{B}_{\mathcal{S}} \cap \operatorname{ext}\left(\mathsf{B}_{\ell_\infty}\right)$. Let $(\varepsilon_n)_{n\in\mathbb{N}} \in \operatorname{ext}\left(\mathsf{B}_{\mathcal{S}}\right)$ and assume that there exists $n_0 \in \mathbb{N}$ such that $\left|\varepsilon_{n_0}\right| < 1$. Take $\delta := \frac{1-\varepsilon_{n_0}}{2} > 0$ and consider the sequences $(a_n)_{n\in\mathbb{N}}$ y $(b_n)_{n\in\mathbb{N}}$ defined by:

$$a_n := \begin{cases} \varepsilon_n & \text{if } n \neq n_0, \\ \varepsilon_{n_0} + \delta & \text{if } n = n_0, \end{cases}$$

and

$$b_n := \begin{cases} \varepsilon_n & \text{if } n \neq n_0, \\ \varepsilon_{n_0} - \delta & \text{if } n = n_0. \end{cases}$$

Finally, observe that $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}} \in \mathsf{B}_{\mathcal{S}}$ and

$$(\varepsilon_n)_{n\in\mathbb{N}} = \frac{1}{2}(a_n)_{n\in\mathbb{N}} + \frac{1}{2}(b_n)_{n\in\mathbb{N}}. \quad \square$$

The next theorem is a characterization of convergent series. The reader may observe that the Bade Property (for c) already comes into play.

Theorem 5.1. *Let X be a real topological vector space. Let $\sum_{n=1}^{\infty} x_n$ be a series in X and denote $\mathcal{S} := \mathcal{S}\left(\sum_{n=1}^{\infty}\right)$. The following conditions are equivalent:*

1. *$\sum_{n=1}^{\infty} x_n$ is convergent.*
2. *$\operatorname{ext}(\mathsf{B}_c) \subseteq \operatorname{ext}(\mathsf{B}_{\mathcal{S}})$.*
3. *$\operatorname{ext}(\mathsf{B}_c) \cap \operatorname{ext}(\mathsf{B}_{\mathcal{S}}) \neq \varnothing$.*

Observe that in this situation we have that $c \subseteq \operatorname{cl}(\mathcal{S})$. Furthermore, if, in addition, $\sum_{n=1}^{\infty} x_n$ has a non-trivial convergent subseries, then $\operatorname{ext}(\mathsf{B}_c) \subsetneq \operatorname{ext}(\mathsf{B}_{\mathcal{S}})$.

Proof. We will divide the proof in three parts:

1. In the first place, we will show the equivalence of the three assertions above. Assume first that $\sum_{n=1}^{\infty} x_n$ is convergent. Observe that $\sum_{n=1}^{\infty} -x_n$ is also convergent. If $(\varepsilon_n)_{n\in\mathbb{N}} \in \operatorname{ext}(\mathsf{B}_c)$, then one of the two sets $\{n \in \mathbb{N} : \varepsilon_n = 1\}$ and $\{n \in \mathbb{N} : \varepsilon_n = -1\}$ must be finite. In either case we have that $(\varepsilon_n)_{n\in\mathbb{N}} \in \operatorname{ext}(\mathsf{B}_{\mathcal{S}})$. Assume next that $\operatorname{ext}(\mathsf{B}_c) \cap \operatorname{ext}(\mathsf{B}_{\mathcal{S}}) \neq \varnothing$. Let $(\varepsilon_n)_{n\in\mathbb{N}} \in \operatorname{ext}(\mathsf{B}_c) \cap \operatorname{ext}(\mathsf{B}_{\mathcal{S}})$. Again one the two sets $\{n \in \mathbb{N} : \varepsilon_n = 1\}$ and $\{n \in \mathbb{N} : \varepsilon_n = -1\}$ must be finite. So we deduce that either $\sum_{n=1}^{\infty} x_n$ or $\sum_{n=1}^{\infty} -x_n$ is convergent.
2. In the second place, observe that if $\operatorname{ext}(\mathsf{B}_c) \subseteq \operatorname{ext}(\mathsf{B}_{\mathcal{S}})$, then we have that
$$\mathsf{B}_c = \overline{\operatorname{co}}(\operatorname{ext}(\mathsf{B}_c)) \subseteq \overline{\operatorname{co}}(\operatorname{ext}(\mathsf{B}_{\mathcal{S}})) \subseteq \operatorname{cl}(\mathcal{S}),$$
in virtue of the fact that c has the Bade Property (cf. [3]).
3. Finally, suppose in addition that $\sum_{n=1}^{\infty} x_n$ has a non-trivial convergent subseries. There exists $N \subset \mathbb{N}$ infinite such that $M := \mathbb{N} \setminus N$ is also infinite and $\sum_{n\in N} x_n$ is convergent. Observe that in this situation $\sum_{n\in M} x_n$ is also convergent. Consider the sequences $(\chi_N(n))_{n\in\mathbb{N}}$ and $(\chi_M(n))_{n\in\mathbb{N}}$ where χ_N and χ_M denote the characteristic functions of N and M, respectively. Note that $(\chi_N(n) - \chi_M(n))_{n\in\mathbb{N}} \in \operatorname{ext}(\mathsf{B}_{\mathcal{S}}) \setminus \operatorname{ext}(\mathsf{B}_c)$. $\square$

Corollary 5.1. *Let X be a real Banach space. Let $\sum_{n=1}^{\infty} x_n$ be a series in X and denote $\mathcal{S} := \mathcal{S}\left(\sum_{n=1}^{\infty}\right)$. Then $c \neq \mathcal{S}$.*

Proof. Assume $c = \mathcal{S}$. In lieu of the previous theorem, $\sum_{n=1}^{\infty} x_n$ is convergent and thus $(x_n)_{n\in\mathbb{N}}$ must tend to 0, therefore a non-trivial subsequence $\left(x_{n_k}\right)_{k\in\mathbb{N}}$ can be found in such a way that $\sum_{k=1}^{\infty} x_{n_k}$ is absolutely convergent, and therefore unconditionally convergent. We apply now the last part of the previous theorem to reach a contradiction. $\square$

Observe that the previous corollary shows the existence of infinite dimensional closed subspaces of ℓ_∞ with the Bade Property that are not the convergence space associated to any series in any real Banach space. In the next section we will show the existence of infinite dimensional closed subspaces of ℓ_∞ with the Bade Property that are not the convergence space associated to any series in any real topological vector space. Next, it is time for a characterization of subseries convergent series.

Theorem 5.2. *Let X be a real topological vector space. Let $\sum_{n=1}^{\infty} x_n$ be a series in X and denote $\mathcal{S} := \mathcal{S}\left(\sum_{n=1}^{\infty}\right)$. The following conditions are equivalent:*

1. *$\sum_{n=1}^{\infty} x_n$ is subseries convergent.*
2. *$\text{ext}\left(\mathsf{B}_{\ell_\infty}\right) \subseteq \text{ext}\left(\mathsf{B}_{\mathcal{S}}\right)$.*
3. *$\frac{1}{2}(\varepsilon_n)_{n\in\mathbb{N}} + \frac{1}{2}(\delta_n)_{n\in\mathbb{N}} \in \mathcal{S}$ for every $(\varepsilon_n)_{n\in\mathbb{N}}, (\delta_n)_{n\in\mathbb{N}} \in \text{ext}\left(\mathsf{B}_{\ell_\infty}\right)$.*

In this situation, $\ell_\infty \subseteq \text{cl}(\mathcal{S})$.

Proof. We will divide the proof in two parts:

1. We will first show the equivalence of the three assertions above. Assume that $\sum_{n=1}^{\infty} x_n$ is subseries convergent. Let $(\varepsilon_n)_{n\in\mathbb{N}} \in \text{ext}\left(\mathsf{B}_{\ell_\infty}\right)$. Denote $M_+ := \{n \in \mathbb{N} : \varepsilon_n = 1\}$ and $M_- := \{n \in \mathbb{N} : \varepsilon_n = -1\}$. Observe that

$$\sum_{n=1}^{\infty} \varepsilon_n x_n = \sum_{n\in M_+} x_n - \sum_{n\in M_-} x_n.$$

Conversely, assume that $\frac{1}{2}(\varepsilon_n)_{n\in\mathbb{N}} + \frac{1}{2}(\delta_n)_{n\in\mathbb{N}} \in \mathcal{S}$ for every $(\varepsilon_n)_{n\in\mathbb{N}}, (\delta_n)_{n\in\mathbb{N}} \in \text{ext}\left(\mathsf{B}_{\ell_\infty}\right)$. Let M be a subset of $\mathbb{N}$. It suffices to consider $(\varepsilon_n)_{n\in\mathbb{N}}$ and $(\delta_n)_{n\in\mathbb{N}}$ given by

$$\varepsilon_n := \begin{cases} 1 & \text{if } n \in M \\ -1 & \text{if } n \in \mathbb{N} \setminus M \end{cases}$$

and $\delta_n := 1$ for all $n \in \mathbb{N}$. Observe that

$$\sum_{n\in M} x_n = \sum_{n=1}^{\infty} \left(\frac{1}{2}\varepsilon_n + \frac{1}{2}\delta_n\right) x_n.$$

2. Finally, observe that if $\text{ext}\left(\mathsf{B}_{\ell_\infty}\right) \subseteq \text{ext}\left(\mathsf{B}_{\mathcal{S}}\right)$, then we have that

$$\mathsf{B}_{\ell_\infty} = \overline{\text{co}}\left(\text{ext}\left(\mathsf{B}_{\ell_\infty}\right)\right) \subseteq \overline{\text{co}}\left(\text{ext}\left(\mathsf{B}_{\mathcal{S}}\right)\right) \subseteq \text{cl}(\mathcal{S}),$$

in virtue of the fact that ℓ_∞ has the Bade Property (cf. [3]). □

The next theorem is important in order to find out what series are to be dismissed.

Theorem 5.3. *Let X be a real topological vector space. Let $\sum_{n=1}^{\infty} x_n$ be a series in X and denote $\mathcal{S} := \mathcal{S}\left(\sum_{n=1}^{\infty}\right)$. Then:*

1. *If $\text{ext}\left(\mathsf{B}_{\mathcal{S}}\right) \neq \varnothing$, then $(x_n)_{n\in\mathbb{N}}$ converges to 0.*
2. *If $(x_n)_{n\in\mathbb{N}}$ has no subsequences converging to 0, then $\mathcal{S} \subseteq c_0$.*
3. *If $\sum_{n=1}^{\infty} x_n$ has a subseries $\sum_{k=1}^{\infty} x_{n_k}$ which is subseries convergent, then $\mathcal{S}$ is not separable.*

Proof

1. Let $(\varepsilon_n)_{n\in\mathbb{N}} \in \text{ext}\left(\mathsf{B}_{\mathcal{S}}\right)$. Then $\sum_{n=1}^{\infty} \varepsilon_n x_n$ is convergent, so $\lim_{n\to\infty} \varepsilon_n x_n = 0$. Since there exists a local basis of balanced and absorbing neighborhoods of 0, we deduce that $\lim_{n\to\infty} x_n = 0$.
2. Consider $(a_n)_{n\in\mathbb{N}} \in \mathcal{S}\backslash c_0$. There exists a subsequence $\left(a_{n_k}\right)_{k\in\mathbb{N}}$ such that $a := \inf\left\{\left|a_{n_k}\right| : k \in \mathbb{N}\right\} > 0$. Since $\sum_{n=1}^{\infty} a_n x_n$ is convergent, we deduce that $(a_n x_n)_{n\in\mathbb{N}}$ converges to 0 and so does $\left(a_{n_k} x_{n_k}\right)_{k\in\mathbb{N}}$. Let U be a balanced and absorbing neighborhood of 0. There exists $k_0 \in \mathbb{N}$ such that if $k \geqslant k_0$, then $a_{n_k} x_{n_k} \in aU$. Since U is balanced, we conclude that $x_{n_k} \in U$ for every $k \geqslant k_0$. This is a contradiction.

3. It suffices to notice that

$$\left\{(\chi_N(n))_{n\in\mathbb{N}} : N \subseteq \{n_k : k \in \mathbb{N}\}\right\}$$

is an uncountable family of elements of $\mathcal{S}$ the distance between every two elements of which is 1. □

As an immediate corollary we obtain the following:

Corollary 5.2. *Let X be a real Banach space. Let $\sum_{n=1}^{\infty} x_n$ be a series in X and denote $\mathcal{S} := \mathcal{S}\left(\sum_{n=1}^{\infty}\right)$. Then:*

1. *If $(x_n)_{n\in\mathbb{N}}$ ha a subsequence converging to 0, then $\mathcal{S}$ has a complemented subspace linearly isometric to ℓ_∞.*
2. *$(x_n)_{n\in\mathbb{N}}$ has no subsequences convergent to 0 if and only if $\mathcal{S}$ is separable.*

Proof

1. We may assume without any loss that $(x_n)_{n\in\mathbb{N}}$ has a subsequence $\left(x_{n_k}\right)_{k\in\mathbb{N}}$ whose associated series is absolutely convergent and thus unconditionally convergent. Then

$$\left\{(\alpha_n)_{n\in\mathbb{N}} \in \ell_\infty : \alpha_n = 0 \text{ for all } n \in \mathbb{N} \setminus \{n_k : k \in \mathbb{N}\}\right\}$$

is a complemented subspace of $\mathcal{S}$ linearly isometric to ℓ_∞.
2. It is a direct consequence of 1. and the previous theorem. □

6. Closed subspaces of ℓ_∞ with the Bade Property not associated to series

In the previous section we have seen that no series $\sum_{n=1}^{\infty} x_n$ in a real Banach space X satisfies that $\mathcal{S}\left(\sum_{n=1}^{\infty} x_n\right) = c$. In this section we will construct another closed subspace of ℓ_∞ with the Bade Property which is not the space of convergence associated to any series in any real topological vector space.

Remark 6.1

1. A subset C of ℓ_∞ is said to satisfy the First Terms Property exactly when

$$\left\{(\beta_n)_{n\in\mathbb{N}} \in \ell_\infty : \text{ exist } n_0 \in \mathbb{N} \text{ and } (\alpha_n)_{n\in\mathbb{N}} \in C \text{ such that } \beta_n = \alpha_n \text{ for } n \geqslant n_0\right\} \subseteq C.$$

2. Any vector subspace of ℓ_∞ verifying the First Terms Property must contain c_{00}.
3. Let X be a real topological vector space. Let $\sum_{n=1}^{\infty} x_n$ be a series in X and denote $\mathcal{S} := \mathcal{S}\left(\sum_{n=1}^{\infty}\right)$. It is obvious that $\mathcal{S}$ verifies the First Terms Property.

Theorem 6.1. *Let*

$$A := \left\{(\varepsilon_n)_{n\in\mathbb{N}} \in \operatorname{ext}\left(\mathsf{B}_{\ell_\infty}\right) : \varepsilon_{2n} = 1 \textit{ for all } n \in \mathbb{N}\right\}.$$

Then span (A) *verifies the following:*

1. *It is a closed subspace of ℓ_∞ enjoying the Bade Property. Even more,* span (A) *is linearly isometric to $\mathbb{R} \oplus_\infty \ell_\infty$.*
2. *It does not satisfy the First Terms Property and thus it is not the space of convergence associated to any series whatsoever.*

Proof. In the first place, notice that the convex hull of A and the absolutely convex hull of A are

$$\operatorname{co}(A) = \left\{(\varepsilon_n)_{n\in\mathbb{N}} \in \mathsf{B}_{\ell_\infty} : \varepsilon_{2n} = 1 \text{ for all } n \in \mathbb{N}\right\}$$

and

$$\operatorname{aco}(A) = \left\{(\varepsilon_n)_{n\in\mathbb{N}} \in \mathsf{B}_{\ell_\infty} : \text{ exists } \lambda \in [-1, 1] \text{ such that } \varepsilon_{2n} = \lambda \text{ for all } n \in \mathbb{N}\right\},$$

respectively. Therefore, $\operatorname{aco}(A) = \operatorname{co}(A \cup -A)$ and both $\operatorname{co}(A)$ and $\operatorname{aco}(A)$ are closed. On the other hand,

$$\operatorname{span}(A) = \left\{(\varepsilon_n)_{n\in\mathbb{N}} \in \ell_\infty : (\varepsilon_{2n})_{n\in\mathbb{N}} \text{ is constant}\right\}$$

is also closed and its unit ball is $\operatorname{aco}(A)$. Furthermore,

$$\operatorname{ext}(\operatorname{aco}(A)) = A \cup -A,$$

therefore $\operatorname{span}(A)$ has the Bade Property. Finally, in order to see that $\operatorname{span}(A)$ is not the space of convergence of any series we refer the reader to Remark 6.1. □

Observe that another way to see that the space constructed in the previous theorem is not the space of convergence associated to any series is by noting that such space does not contain c_{00}. A slight modification of the previous space will give us the following:

Theorem 6.2. *Let*

$$B := \left\{(\varepsilon_n)_{n\in\mathbb{N}} \in \operatorname{ext}\left(\mathsf{B}_{\ell_\infty}\right) : \textit{ exists } n_0 \in \mathbb{N} \textit{ such that } \varepsilon_{2n} = 1 \textit{ for all } n \geqslant n_0\right\}.$$

Then $\overline{\operatorname{span}}(B)$ *verifies the following:*

1. *It contains* c*, satisfies the First Terms Property, and enjoys the Bade Property. Even more,* $\overline{\operatorname{span}}(B)$ *is linearly isometric to* $c \oplus_\infty \ell_\infty$*.*
2. *It is not the space of convergence associated to any series in a real Banach space.*

Proof. In the first place, observe that

$$\begin{aligned}
\operatorname{co}(B) &= \left\{(\varepsilon_n)_{n\in\mathbb{N}} \in \mathsf{B}_{\ell_\infty} : \text{ exists } n_0 \in \mathbb{N} \text{ such that } \varepsilon_{2n} = 1 \text{ for all } n \geqslant n_0\right\},\\
\operatorname{aco}(B) &= \left\{(\varepsilon_n)_{n\in\mathbb{N}} \in \mathsf{B}_{\ell_\infty} : \text{ exist } n_0 \in \mathbb{N} \text{ and } \lambda \in [-1, 1] \text{ such that } \varepsilon_{2n} = \lambda \right.\\
&\qquad \left. \text{for all } n \geqslant n_0\right\},\\
\operatorname{span}(B) &= \left\{(\varepsilon_n)_{n\in\mathbb{N}} \in \ell_\infty : \text{ exists } n_0 \in \mathbb{N} \text{ such that } (\varepsilon_{2n})_{n\geqslant n_0} \text{ is constant}\right\}.
\end{aligned}$$

Now take into account that

$$\begin{aligned}
\overline{\operatorname{co}}(B) &= \left\{(\varepsilon_n)_{n\in\mathbb{N}} \in \mathsf{B}_{\ell_\infty} : (\varepsilon_{2n})_{n\in\mathbb{N}} \text{ converges to } 1\right\},\\
\overline{\operatorname{aco}}(B) &= \left\{(\varepsilon_n)_{n\in\mathbb{N}} \in \mathsf{B}_{\ell_\infty} : (\varepsilon_{2n})_{n\in\mathbb{N}} \text{ is convergent}\right\},\\
\overline{\operatorname{span}}(B) &= \left\{(\varepsilon_n)_{n\in\mathbb{N}} \in \ell_\infty : (\varepsilon_{2n})_{n\in\mathbb{N}} \text{ is convergent}\right\}.
\end{aligned}$$

Finally, assume that $\sum_{n=1}^\infty x_n$ is a series in a real Banach space X verifying that $\mathcal{S}\left(\sum_{n=1}^\infty x_n\right) = \overline{\operatorname{span}}(B)$. Let $M := \{2n : n \in \mathbb{N}\}$. Note that $(\chi_M(n))_{n\in\mathbb{N}} \in \overline{\operatorname{span}}(B) = \mathcal{S}\left(\sum_{n=1}^\infty x_n\right)$. Therefore $\sum_{n=1}^\infty x_{2n}$ is convergent and there exists a subsequence $(x_{2n_k})_{k\in\mathbb{N}}$ such that $\sum_{k=1}^{\infty} x_{2n_k}$ is absolutely convergent and hence unconditionally convergent. As a consequence,

$$\{(\alpha_n)_{n\in\mathbb{N}} \in \ell_\infty : \alpha_n = 0 \text{ for all } n \in \mathbb{N} \setminus \{2n_k : k \in \mathbb{N}\}\}$$

is a closed subspace of $\mathcal{S}\left(\sum_{n=1}^{\infty} x_n\right) = \overline{\text{span}}\,(B)$ linearly isometric to ℓ_∞. This is a contradiction with the construction of $\overline{\text{span}}\,(B)$. □

The end of this section is devoted to state a conjecture in whose truthfulness we tend to believe.

Conjecture 6.1. *Let X be a real Banach space. Let $\sum_{n=1}^{\infty} x_n$ be a series in X. Then $\mathcal{S} := \mathcal{S}\left(\sum_{n=1}^{\infty} x_n\right)$ has the Bade Property if and only if $\sum_{n=1}^{\infty} x_n$ is unconditionally convergent.*

7. A characterization of unconditionally convergent series in real Banach spaces

The final part of this manuscript is devoted to characterize unconditionally convergent series in real Banach spaces.

Definition 7.1. Let X be a real Banach space. We will say that a series $\sum_{n=1}^{\infty} x_n$ is uniformly convergent in $\mathcal{M} \subseteq \mathcal{S} := \mathcal{S}\left(\sum_{n=1}^{\infty} x_n\right)$ if for every $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that for every $k \geqslant k_0$ and every $(a_n)_{n\in\mathbb{N}} \in \mathcal{M}$ we have that $\left\|\sum_{n=k}^{\infty} a_n x_n\right\| < \varepsilon$.

It is immediate to see that every absolutely convergent series in a real Banach space is uniformly convergent in $\text{ext}\left(\mathsf{B}_{\ell_\infty}\right)$.

Proposition 7.1. *Let X be a real Banach space. If $\sum_{n=1}^{\infty} x_n$ is an unconditionally convergent series in X, then it is uniformly convergent in* $\text{ext}\left(\mathsf{B}_{\ell_\infty}\right)$.

Proof. Suppose to the contrary that the series $\sum_{n=1}^{\infty} x_n$ is not uniformly convergent in $\text{ext}\left(\mathsf{B}_{\ell_\infty}\right)$. There exists $\delta > 0$ such that for every $i \geqslant 1$ there are $j > i$ and $\left(\varepsilon_n^{(j)}\right)_{n\in\mathbb{N}} \in \text{ext}\left(\mathsf{B}_{\ell_\infty}\right)$ verifying that $\left\|\sum_{n=j}^{\infty} \varepsilon_n^{(j)} x_n\right\| \geqslant \delta$. We can consider a strictly increasing sequence of indices

$$i_1 < k_1 < i_2 < k_2 < \cdots < i_j < k_j < i_{j+1} < \cdots$$

satisfying that

$$\left\|\sum_{n=i_j}^{\infty} \varepsilon_n^{(j)} x_n\right\| > \delta$$

and

$$\left\|\sum_{n=i_j}^{k_j} \varepsilon_n^{(j)} x_n\right\| > \frac{\delta}{2}, \tag{1}$$

for every $j \in \mathbb{N}$. Next, we will construct a sequence $(\overline{\varepsilon}_n)_{n\in\mathbb{N}} \in \text{ext}\left(\mathsf{B}_{\ell_\infty}\right)$ as follows:

- If $i \in [i_j, k_j]$ for some $j \in \mathbb{N}$, then $\overline{\varepsilon}_i = \varepsilon_i^{(j)}$.
- The rest of the $\overline{\varepsilon}_i$'s can be either 1 or -1.

Clearly $\sum_{n=1}^{\infty} \overline{\varepsilon}_n x_n$ is not a Cauchy series, so $\sum_{n=1}^{\infty} x_n$ is not unconditionally convergent. □

Corollary 7.1. *Let X be a real Banach space. Let* $\sum_{n=1}^{\infty} x_n$ *be a series in X. The following conditions are equivalent:*

1. $\sum_{n=1}^{\infty} x_n$ *is unconditionally convergent.*
2. $\sum_{n=1}^{\infty} x_n$ *is a convergent series and is uniformly convergent on* ext $(\mathsf{B}_{\mathcal{S}})$.

Proof. Assume that $\sum_{n=1}^{\infty} x_n$ is convergent and uniformly convergent on ext $(\mathsf{B}_{\mathcal{S}})$ but not unconditionally convergent. There exists $(\varepsilon_n)_{n\in\mathbb{N}} \in$ ext $(\mathsf{B}_{\mathcal{S}})$ such that $\sum_{n=1}^{\infty} \varepsilon_n x_n$ does not converge. Then there exists $\delta > 0$ such that for every $q \in \mathbb{N}$ we can find $p > q$ verifying that $\left\|\sum_{i=q}^{p} \varepsilon_i x_i\right\| > \delta$. Following an inductive process we can construct a strictly increasing sequence of naturals

$$p_1 < p_2 < \cdots < p_n < \cdots$$

such that for every $n \in \mathbb{N}$ we have that $\left\|\sum_{i=p_n+1}^{p_{n+1}} \varepsilon_i x_i\right\| > \delta$. Since $\sum_{n=1}^{\infty} x_n$ is convergent, there exists $n_0 \in \mathbb{N}$ such that for every $n \geqslant n_0$ we have that $\left\|\sum_{i=n}^{\infty} x_i\right\| < \frac{\delta}{2}$. Next, for every $k \in \mathbb{N}$ we choose $p_{n_k} > \max\{n_0, k\}$ and define a sequence $\left(\alpha_n^{(k)}\right)_{n\in\mathbb{N}}$ as follows:

$$\alpha_n^{(k)} = \begin{cases} \varepsilon_n, & \text{if } n \in \{p_{n_k}+1, \ldots, p_{n_k+1}\} \\ 1, & \text{if } n \in \mathbb{N} \setminus \{p_{n_k}+1, \ldots, p_{n_k+1}\}. \end{cases}$$

Observe that $\left(\alpha_n^{(k)}\right)_{n\in\mathbb{N}} \in$ ext $(\mathsf{B}_{\mathcal{S}})$ for every $k \in \mathbb{N}$. Now, if $j = p_{n_k} + 1$, then $j > k$ and

$$\begin{aligned} \left\|\sum_{i=j}^{\infty} \alpha_i^{(k)} x_i\right\| &= \left\|\sum_{i=p_{n_k}+1}^{p_{n_k+1}} \varepsilon_i x_i + \sum_{i=p_{n_k+1}}^{\infty} x_i\right\| \\ &\geqslant \left\|\sum_{i=p_{n_k}+1}^{p_{n_k+1}} \varepsilon_i x_i\right\| - \left\|\sum_{i=p_{n_k+1}}^{\infty} x_i\right\| \\ &> \delta - \frac{\delta}{2} \\ &= \frac{\delta}{2}. \end{aligned}$$

This is a contradiction. □

References

[1] A. Aizpuru-Tomás, F. Benítez-Trujillo, The Bade property and the λ-property in spaces of convergent sequences, Collect. Math. 42 (3) (1991) 245–251.
[2] A. Aizpuru, F.J. Pérez-Fernández, Characterization of series in Banach spaces, Acta Math. Univ. Comenian. 68 (1999) 337–344.
[3] W.G. Bade, The Banach Space $\mathcal{C}(S)$, Lecture Notes Series, vol. 26, Matematisk Institut, Aarhus Universitet, Aarhus, 1971.
[4] J. Diestel, Sequences and Series in Banach Spaces, Springer-Verlag, New York, 1984.
[5] J. Diestel, H. Jarchov, A. Tonge, Absolutely Summing Operators, Cambridge University Press, Cambridge, 1995.
[6] R. Deville, G. Godefroy, V. Zizler, Smoothness and Renormings in Banach Spaces, Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 64, Longman Scientific & Technical, Harlow, 1993. (co-published in the United States with John Wiley & Sons Inc., New York).
[7] V.P. Fonf, J. Lindenstrauss, R.R. Phelps, Infinite Dimensional Convexity, Handbook of the Geometry of Banach Spaces, vol. I, North-Holland, Amsterdam, 2001., pp. 599–670.
[8] F.J. García-Pacheco, Non-continuous linear functionals on topological vector spaces, Banach J. Math. Anal. 2 (1) (2008) 11–15.
[9] F.J. García-Pacheco, Z. Bentuo, Geometric properties on non-complete spaces, Quaest. Math., accepted for publication.
[10] R.E. Huff, P.D. Morris, Dual spaces with the Krein–Milman property have the Radon–Nikodym property, Proc. Amer. Math. Soc. 49 (1975) 104–108.

[11] N.J. Kalton, Schauder decompositions in locally convex spaces, Proc. Cambridge Philos. Soc. 68 (1970) 377–392.
[12] N.J. Kalton, Schauder bases and reflexivity, Studia Math. 38 (1970) 255–266.
[13] M. Krein, D. Milman, On extreme points of regular convex sets, Studia Math. 9 (1940) 133–138.
[14] F.J. Pérez-Fernández, F. Benítez-Trujillo, A. Aizpuru, Characterizations of completeness of normed spaces through weakly unconditionally Cauchy series, Czechoslovak Math. J. 50 (4) (2000) 889–896.
[15] A. Pietsch, History of Banach Spaces and Linear Operators, Birkhauser Boston Inc., Boston, MA, 2007.