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The Road to the Discrete Analogue of the Painlevé Property: Nevanlinna Meets Singularity Confinement

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Abstract—The question of integrability of discrete systems is analysed in the light of the recent findings of Ablowitz *et al.*, who have conjectured that a fast growth of the solutions of a difference equation is an indication of nonintegrability. The study of the behaviour of the solutions of a mapping is based on the theory of Nevanlinna. In this paper, we show how this approach can be implemented in the case of second-order mappings which include the discrete Painlevé equations. Since the Nevanlinna approach does offer only a necessary condition which is not restrictive enough, we complement it by the singularity confinement requirement, first in an autonomous setting and then for deautonomisation. We believe that this three-tiered approach is the closest one can get to a discrete analogue of the Painlevé property.

Keywords—Painlevé equations, Discrete systems, Discrete integrability, Singularity confinement, Meromorphic functions, Order and Nevanlinna theory.

1. THE FALL AND RISE OF SINGULARITY CONFINEMENT

What is this special property of integrable systems that sets them apart from their nonintegrable brethren? Integrability is associated to a host of special properties [1]. Most of them, however, are of the constructive type. For example, the existence of a Lax pair is intimately associated with integrability. However, the derivation of a Lax pair when we are given some dynamical system cannot be seriously undertaken unless we have some substantial indication that the system is indeed integrable. Fortunately, among the miraculous properties of integrable systems, some are of predictive type. One can easily test a given system for such a property and use the answer of the test in order to predict the possible integrability of the system.

The property of this type, for differential systems, which is best known (at least to the authors of the present paper) is the Painlevé property. When faced with the problem of constructing new

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functions out of the solutions of nonlinear ordinary differential equations, Painlevé [2] decided that the way to deal with movable critical singularities (which prevented uniformisation) was to require their absence. This property, which was subsequently named after Painlevé, turned out to be indeed related to integrability. As a matter of fact, one can easily convince oneself that this is true in an almost tautological way. The usefulness of the Painlevé approach was established in an indisputable way when it led to the discovery of new functions, the Painlevé transcendents, which can only be defined as the solutions of the respective differential equations. The actual integration of these equations had to wait for more than 70 years [3], and it was made possible only after the introduction of inverse scattering transform (IST) techniques.

The discovery of integrable evolution, "soliton" [4], equations led to a revival of the Painlevé approach. It is worthwhile at this point to recall the ARS [5] conjecture: "a differential equation which is integrable through IST methods has the Painlevé property". While the formulation of the conjecture is not quite the original one, it follows closely the spirit of its proponents.

With the discovery of integrable discrete systems, it soon became clear that something like the Painlevé property was needed in the discrete case. In this domain, the first important step was made by the authors of the present paper [6] and independently by Joshi [7]. It was the discovery of the property dubbed "singularity confinement" (orbits of pole-like behaviour in the terminology of Joshi). The idea is the following: given a mapping (usually rational, but not necessarily so) it may happen that, depending on the initial conditions, a singularity appears at some iteration. For a generic nonintegrable mapping, the singularity propagates *ad infinitum* under the iteration of the mapping. However, if the mapping is integrable, the singularity disappears after some iterations: it is confined. Of course, the precise meaning of "singularity" must be (and has been) refined.

Although no precise conjecture was formulated at the time, here is what the authors had in mind: "a difference equation which is integrable through spectral methods has the singularity confinement property". We believe that under this form, the conjecture holds true in the sense that no counterexample has been found to date. One would thus be tempted to consider singularity confinement as the discrete equivalent of the Painlevé property. However, soon after the initial discovery, it became clear that singularity confinement was not a sufficient integrability condition. Given the situation, and despite singularity confinement's success in the detection of the discrete analogues of the Painlevé equations [8], it became clear that singularity confinement could not play the role of the Painlevé property for discrete systems. Still, its necessary character was an indication that it should be part of the discrete Painlevé property when the latter is finally discovered.

At this point, it would be useful to examine what the singularity confinement is missing. We have formulated the first remark on the insufficient character of singularity confinement in [9]. We have shown indeed that any mapping of the form $x_{n+1} = P(x_n)/Q(x_n)$ where P, Q are polynomials and the degree of P is not larger than that of Q does satisfy the singularity confinement requirement. On the other hand, the mapping, solved for x_n in terms of x_{n+1}, leads (in general) to a number of branches which increase exponentially under the iteration. In order to face this difficulty, the singularity confinement was complemented by the requirement of preimage nonproliferation. (A further remark can be made here. Let us, for simplicity, consider a three-point mapping where both x_{n-1} and x_{n+1} enter through powers higher than unity. Its evolution leads in general to an exponentially increasing number of images and preimages. Such mappings are not integrable. Sometimes one solution can be constructed, and this is considered by some authors as an argument in order to claim integrability. The error lies in the fact that this solution is the only one that we know how to describe while the description of the whole system with its ever increasing number of branches is beyond reach.)

However, the preimage nonproliferation requirement turned out to be an insufficient fix of singularity confinement. This was shown in ample detail by examples of Viallet and Hietarinta [10] who have constructed whole families of mappings which, while confining, exhibit chaotic behaviour and are thus clearly nonintegrable. These authors went out to propose a new criterion for discrete integrability. We are going to return later to this criterion and discuss why it cannot be considered as a discrete Painlevé property.

The decisive step in the right direction was accomplished when Ablowitz and collaborators (AHH) [11] decided to interpret discrete equations as delay equations in the complex domain. As was shown by Yanagihara [12], such equations possess nontrivial solutions which are meromorphic in the complex variable. The authors of [11] conjectured that the behaviour at infinity would be the key for the integrability of discrete systems. They were guided in this by the continuous limit of difference equations to differential ones.

The main tool for the study of the behaviour at infinity of the solutions of a given mapping is the theory of Nevanlinna [13] which introduces the notion of *order* of a meromorphic function. We shall present the essentials of this theory in the following section. The conjecture is that the equations, the solutions of which are of infinite order, are not integrable. However, the practical implementation of the Nevanlinna approach leads only to a necessary condition for a finite order, which is often not restrictive enough. To complement this, we need something more. As we shall see, singularity confinement is appropriate here. (Ablowitz *et al.* use a different criterion, namely the absence of the Digamma ψ function, in the series expansion of the solution. We feel that in this way they restrict themselves to only one type of "bad" functions, while more may well exist.)

In what follows, we shall present a brief summary of the Nevanlinna theory and of the basic statements we shall use in order to investigate the integrability of a family of second-order mappings which include the discrete Painlevé equations. We shall proceed in a three-tiered strategy. First, apply the finite-order criterion to autonomous mappings in order to eliminate the ones that cannot be integrable. Second, use singularity confinement among the remaining mappings in order to further eliminate nonintegrable ones. Finally, use again singularity confinement in order to obtain the right deautonomisation of the remaining systems. In the conclusion, we shall argue why we think that this combined criterion is indeed the discrete analogue of the Painlevé property.

2. A ROUGH SKETCH OF THE NEVANLINNA THEORY

As we stated in the introduction, we expect the integrability of a mapping to be conditioned by the behaviour of its solutions when the (complex) independent variable goes to infinity. The main tool for the study of the value distribution of entire and meromorphic functions is the Nevanlinna characteristic (and various quantities related to the latter). The Nevanlinna characteristic of a function f, denoted by $T(r;f)$, measures the 'affinity' of f for the value ∞. It is usually represented as the sum of two terms: the frequency of poles and the contribution from the arcs $|z|=r$ where $|f(z)|$ is large. From the characteristic, one can define the order of a meromorphic function: $\sigma = \limsup_{r\to\infty} \log T(r;f)/\log r$. When f is rational, $T(r;f) \propto \log r$ and $\sigma = 0$. However, a zero-order function is not necessarily rational. Indeed, any T of the form $T(r;f) \propto \phi(r)\log r$, where $\phi(r)$ is a slowly enough growing function of r, will lead to $\sigma = 0$. On the other hand, if $T(r;f)/\log r$ remains finite, then f must be rational. When f is of the type $e^{P_n(z)}$ where P_n is a polynomial of degree n, one finds $T \propto r^n$ and $\sigma = n$. A fast growing function like e^{e^z} leads to $T \propto e^r$, and thus, $\sigma = \infty$.

An explicit expression of the Nevanlinna characteristic can be given in terms of the counting and proximity functions related to the two contributions we mentioned above. We have

$$T(r;f) = N(r;f) + m(r;f), \tag{2.1}$$

where

$$N(r;f) = \int_0^r \frac{n(t;f) - n(0;f)}{t}\, dt + n(0;f)\log r \tag{2.2}$$

is the pole-counting contribution where $n(r;f)$ is the number of poles of f, including multiplicities, for $|z| \le r$. The proximity function $m(r;f)$ is given by

$$m(r;f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f\left(re^{i\theta}\right)\right| d\theta, \tag{2.3}$$

where $\log^+ g = \max(0, \log g)$. We must point out that the affinity of f for ∞, as measured by T, is the same as its affinity for 0 or any finite value a, up to terms which may be of $\mathcal{O}(\log r)$ when r is *small*, but which, when r is sufficiently large (depending on the function f and the value a) remain bounded. In what follows, we shall introduce the symbols $\asymp$, $\preceq$, and $\prec$, which denote equality, inequality, and strict inequality, respectively, *up to a function of r which remain bounded when* $r \to \infty$. The two basic relations which reproduce the statement on the affinity of f for ∞, 0, or a are

$$T\left(r;\frac{1}{f}\right) \asymp T(r;f), \tag{2.4}$$

$$T(r;f-a) \asymp T(r;f). \tag{2.5}$$

Using those two identities, we can easily prove that the characteristic function of a homographic transformation of f (with constant coefficients) is equal to $T(r;f)$ up to a bounded quantity. It is straightforward to prove that

$$T\left(r;f^n\right) \asymp |n| T(r;f), \tag{2.6}$$

and from a theorem due to Valiron [14], we have

$$T\left(r;\frac{P(f)}{Q(f)}\right) \asymp \sup(p,q) T(r;f), \tag{2.7}$$

where P and Q are polynomials in f with constant coefficients, of degrees p and q, respectively, provided the rational expression P/Q is irreducible.

Let us give also some useful classical inequalities

$$T(r;fg) \preceq T(r;f) + T(r;g), \tag{2.8}$$

$$T(r;f+g) \preceq T(r;f) + T(r;g). \tag{2.9}$$

Another inequality, which will be used in Section 3, is

$$T(r;fg+gh+hf) \preceq T(r;f) + T(r;g) + T(r;h). \tag{2.10}$$

The proof of this inequality is easy, using the fact that $T(r;f)$ is the sum of the pole counting and the proximity contributions. For the former, we remark that the density of poles of $fg+gh+hf$ cannot be higher than the sum of those of f, g, and h. As a consequence, for the part under the integral in (2.2) we have the desired inequality. The $n(0,f)\log r$ term, for $r<1$, may introduce a contribution going against the inequality we wish to prove, but whenever $r>1$, the contribution is in the right direction and we thus have indeed $N(r;fg+gh+hf) \preceq N(r;f)+N(r;g)+N(r;h)$. For the proximity function, we start by remarking that $|fg+gh+hf| < 3\sup(|fg|,|gh|,|hf|)$. Taking the $\log^+$ of both members, we can strengthen the inequality by adding to the r.h.s. the $\log^+$ of the one of the $|f|$, $|g|$, $|h|$ that does not enter in each term of the sup. We have thus: $\log^+|fg+gh+hf| < \log 3 + \log^+|f| + \log^+|g| + \log^+|h|$ which leads to $m(r;fg+gh+hf) \preceq m(r;f)+m(r;g)+m(r;h)$. Adding the two inequalities for N and m, we find (2.10).

The latter inequality can easily be generalised to

$$T\left(r;\sum_{J\subseteq I}\alpha_J\left(\prod_{j\in J} f_j\right)\right) \preceq \sum_{i\in I} T(r;f_i) \tag{2.11}$$

for constant α_J.

One last property of the Nevanlinna characteristic was obtained by Ablowitz *et al.* [11]. In our notation it reads

$$T(r; f(z \pm 1)) \preceq (1+\epsilon)T(r+1; f(z)). \tag{2.12}$$

This relation (which is valid for r large enough for any given ϵ) makes it possible to have access to the characteristic, and thus, the order of the solution of some difference equations. Let us sketch here the general procedure, which will be applied to specific cases in the following section.

The discrete equations we shall examine here are three-point mappings of the general form

$$A(x_n, x_{n-1}, x_{n+1}) = B(x_n) \tag{2.13}$$

where, in general, A is polynomial and B is rational, with coefficients which *do not* depend on the independent variable n (something we shall come back to later). Moreover, in the cases we shall consider, A is linear separately in $x_{n\pm1}$. Following the approach of AHH we consider equation (2.13) as a delay equation in the complex domain and evaluate the Nevanlinna characteristic of both members of the equality, using (2.12) and (2.7). We find

$$u(1+\epsilon)T(r+1; x) + vT(r; x) \succeq wT(r; x) \tag{2.14}$$

(with $u = 2$ if A is linear in $x_{n\pm1}$), for appropriate values of v and w. From (2.14) we have

$$T(r+1; x) \succeq \frac{w-v}{u(1+\epsilon)} T(r; x). \tag{2.15}$$

Now, if $w > u+v$, for r large enough one can always choose ϵ small enough that $\lambda \equiv (w-v)/u(1+\epsilon)$ becomes strictly greater than unity. The precise meaning of (2.15) is that for r large enough, we have

$$T(r+1; x) \geq \lambda T(r; x) - C \tag{2.16}$$

for some C independent of r. The case C negative is trivial: $T(r+k; x) \geq \lambda^k T(r; x)$. For positive C we have

$$T(r+1; x) - \frac{C}{\lambda - 1} \geq \lambda \left(T(r; x) - \frac{C}{\lambda - 1} \right). \tag{2.17}$$

Thus, whenever $T(r; x)$ is an unbounded growing function of r (i.e., $T \succ 0$), then for some r large enough, the right-hand side of this inequality becomes strictly positive, and iterating (2.17) we see that $T(r+k; x)$ diverges at least as fast as λ^k, and thus, $\log T(r; x) > r \log \lambda$ and the order σ of x is infinite. Thus, according to the AHH hypothesis, the mapping cannot be integrable. The only way out is if $T(r; x)$ is a constant which means that x is itself a constant, since the slowest possible growth of the Nevanlinna characteristic for a nonconstant meromorphic function is $T(r; f) \asymp \log r$, for f a homographic function of z. Given that the mapping is rational, there can only be a finite number of constant solutions. We could in principle have had an infinite number of constant solutions if the identity $A(x_n, x_n, x_n) \equiv B(x_n)$ were true. However this would imply $w \leq u + v$. Thus, when $w > u + v$ in (2.14), the only possible finite-order solutions are (a finite number of) constant solutions, *all* the remaining ones having $\sigma = \infty$.

The advantage of working with autonomous mappings lies in the fact that we can control precisely the corrective terms in the inequalities for T. Had we worked with nonautonomous systems, we would have had unbounded corrective terms. For instance, if the coefficients depend rationally on z, there would be corrective terms of order $\mathcal{O}(\log r)$ and we would have been unable to exclude (finite-order) rational solutions. Though one could suspect that the generic solution is not rational, one could not easily prove this fact in the nonautonomous case. However in our approach we consider nonautonomous equations as obtained from autonomous ones through a deautonomisation procedure. This procedure will never transform a $\sigma = \infty$ solution into a finite σ one. So the generic solution will have $\sigma = \infty$ whenever $w > u + v$ in (2.14) even in the nonautonomous case. The rational solutions that we cannot exclude can only come, through the deautonomisation procedure, from the finite- (in effect, zero-)order constant solutions, of which there is a finite number.

3. THE THREE-TIERED APPROACH TO DISCRETE INTEGRABILITY

In this section, we shall show how the criterion of noninfinite order of the solution of a given difference equation can be complemented so as to become a discrete integrability predictor. By the latter we mean that the conditions we will obtain will be necessary (for integrability through spectral methods) and, although they cannot be shown to be sufficient in general, they will be constraining enough for the approach to have an undeniable heuristic value in integrability prediction.

The first step, given a mapping, is to use the Nevanlinna characteristic techniques, which we sketched in Section 2, in order to estimate the rate of growth of the solutions. Since for nonautonomous equations this rate depends on the rate of growth of the coefficients of the equation, we opt for a simple approach: at this first step we consider *only autonomous* mappings, i.e., mappings whose coefficients are constants. As we shall see in the examples that follow, this first step puts severe constraints on the discrete equations at hand. However, usually, these constraints are not restrictive enough so as to fix completely the form of the mapping, hence the necessity of the second step. (At this point our approach diverges from that of AHH.) Once the constraints of the first step are implemented, we pursue, using singularity confinement, in order to constrain further our discrete equation. Thus, all autonomous equations that do not satisfy confinement are rejected at this second step.

The third step consists of the deautonomisation of the system using once again the confinement criterion. We thus obtain a mapping which (hopefully) satisfies the Nevanlinna criterion for low-growth of the solutions and the singularity confinement as well. The major difficulty lies in the fact that the practical evaluation of the Nevanlinna characteristic gives a clear-cut answer as to mappings the solutions of which must be (generically) of infinite order, but this does not mean that all the remaining ones have their generic solution of finite order. Particular care is needed in the application of this criterion, lest one proclaim of finite order systems which have in fact infinite order solutions.

In what follows, we are going to examine our pet systems, namely discrete Painlevé equations (d-$\mathbb{P}$) [15]. Our starting point will be functional forms related to the various members of the family of "standard" d-$\mathbb{P}$. Before proceeding, we must point out that for a large number of d-$\mathbb{P}$, integrability is well established through the existence of a Lax pair. For the remaining equations, although this is certainly not a proof, there exists an independent strong indication of integrability obtained through algebraic entropy-low growth techniques [16]. For most of the discrete Painlevé equations, we have presented a geometrical description [17] (based on affine Weyl groups) which makes possible the construction of their solutions starting from those of the nonautonomous Hirota-Miwa equations, which constitutes (at least in the eyes of the present authors) a further indication of integrability.

Following the three-tiered approach we sketched above, we start with autonomous systems. Since in every case examined, the application of the Nevanlinna criterion combined to that of singularity confinement leads to precisely the QRT [18] forms, we shall not proceed to the third step, namely deautonomisation. As a matter of fact, the deautonomisation of the QRT mappings, associated with the standard family of d-P was given in full detail in [19].

We start with the equations of the d-$\mathrm{P}_{\mathrm{I/II}}$ family. They have been examined in detail by AHH, but we present here the results for the sake of completeness. They offer us also the occasion to present the differences between our arguments and the ones of AHH. The starting point is the *autonomous* equation

$$x_{n+1} + x_{n-1} = \frac{P(x_n)}{Q(x_n)}. \tag{3.1}$$

Whenever the condition $w > u + v$, obtained in Section 2, is satisfied, we know that the generic solution will have infinite order, since only a finite number of constant solutions can have finite (in fact, zero) order. For equation (3.1), $u = 2$, $v = 0$, and w is the maximum of the degrees of P and Q. Thus, w can be at most two for the order of the generic solution not to be infinite. There is a subtle difference in our reasoning compared to that of AHH. The latter authors conclude that if P and Q depend rationally on n, if $w > 2$, *and* if the solution is not rational, *then* the order is infinite, but they cannot exclude the possibility that for some choice of P, Q with $w \geq 3$ all solutions may be rational. For us, using constants P and Q we can conclude that if $w > 3$, the generic solution has $\sigma = \infty$, without any other assumption on its rationality. The only solutions with finite order are constants, and there exists a finite number of them. Then upon deautonomisation, the generic $\sigma = \infty$ solution cannot recover a finite order, while some very special finite-order rational solutions may arise from the constant solutions of the autonomous case.

Let us examine all mappings with $w \leq 2$. We start by rewriting (3.1) in the case of quadratic numerator and denominator as

$$x_{n+1} + x_{n-1} = -\frac{\eta x_n^2 + \epsilon x_n + \zeta}{\alpha x_n^2 + \beta x_n + \gamma}. \tag{3.2}$$

The singularity confinement analysis of (3.2) (in the case where α, β are not both zero) is straightforward. The resulting constraint is that the mapping must belong to the QRT family, i.e., $\eta = \beta$ or $\alpha = \eta = 0$. The final step is the deautonomisation of (3.2), which leads to the well-known forms of d-$\mathrm{P_{II}}$ and three alternate forms of d-$\mathrm{P_I}$. One notable exception to integrability for (3.2) is the polynomial mapping

$$x_{n+1} + x_{n-1} = P(x_n) \tag{3.3}$$

where P is a quadratic polynomial, i.e., $\alpha = \beta = 0$. By a slightly different approach, we can indeed show that this mapping cannot be integrable. For the sake of simplicity, let us consider the mapping $x_{n+1} + x_{n-1} = x_n^2$. The affinity of x to ∞ as measured by the Nevanlinna characteristic $T(r; x_n)$ is due to an arc of length l on the $|z| = r$ circle where x_n has a very large value Ω and/or to the presence of N poles within the circle of radius r. (In this particular case, one knows from Yanagihara [12] that there are entire solutions, but we give the argument in a general setting. The fact that the second contribution, coming from poles, can be absent, does not affect our reasoning.) From the r.h.s. of the mapping, it is clear that x_n^2 will have a value Ω^2 on the arc of length l and/or N double poles within the circle of radius r. This means that the contributions to the affinity of either x_{n+1} or x_{n-1} will necessarily come from a value Ω^2 (on a segment smaller than l) and/or (a number less than N of) of double poles. Iterating further, we find that the affinity to ∞ of x_{n+k} will be associated to a large value Ω^{2^k} (on a segment of decreasing length) and/or (a decreasing number of) poles of order 2^k. This growth (even in the presence of a slowly growing T) is equally unacceptable as a characteristic function $T \sim \lambda^k$ and following the AHH conjecture, this polynomial mapping cannot be integrable. (The same argument would preclude integrability of any nonlinear polynomial mapping of the type (3.1), but the standard proof suffices if the degree is higher than two.)

The next mapping we shall examine is that related to the q-$\mathrm{P_{III}}$ family

$$x_{n+1}x_{n-1} = \frac{P(x_n)}{Q(x_n)}. \tag{3.4}$$

This indeed was one examined by Ablowitz *et al.*, but the fact that they considered coefficients linear in z does not allow to apply directly their conclusions to q-$\mathrm{P_{III}}$, where the coefficients are exponential in z. Still their main result stands: all the solutions of (3.4) are of infinite order (except a finite number of constant solutions) if the maximum of the degrees of P, Q exceeds two. The main ingredient in the proof of this result is the inequality (2.8) $T(r; x_{n+1}x_{n-1}) \preceq$

$T(r;x_{n+1})+T(r;x_{n-1})$, and thus, we have again (2.14) with $u=2$ and $v=0$. The general form of (3.4) with quadratic P, Q is

$$x_{n+1}x_{n-1}=\frac{\eta x_n^2+\zeta x_n+\mu}{\alpha x_n^2+\beta x_n+\gamma}. \tag{3.5}$$

Again, the application of singularity confinement to (3.5) results in the QRT constraint $\eta=\gamma$ or $\eta=\alpha=0$. The deautonomisation of this form, presented in [19], leads to the q-$\mathrm{P_{III}}$ equation as well as mappings which are q-discrete forms of $\mathrm{P_{II}}$ and $\mathrm{P_I}$.

Next we turn to the family of d-$\mathrm{P_{IV}}$ (which was *not* among the one examined by Ablowitz *et al.*, despite their statement to the contrary),

$$(x_{n+1}+x_n)(x_n+x_{n-1})=\frac{P(x_n)}{Q(x_n)}. \tag{3.6}$$

First, we apply naïvely (2.8) and (2.9), which gives us, with the notations of (2.14), $u=2$, $v=2$, and thus, for $w>4$ we have x_n of infinite order. Thus, the only acceptable P, Q can be quartic at maximum. However we can produce a more refined estimate using the inequality (2.10). To do this we rewrite (3.6) as

$$x_{n+1}x_{n-1}+x_nx_{n+1}+x_nx_{n-1}=\frac{P(x_n)-x_n^2Q(x_n)}{Q(x_n)}. \tag{3.7}$$

(Note that since P/Q is irreducible, the r.h.s. of (3.7) is equally irreducible.) Using (2.10) we find $u=2$, $v=1$, and so we have $w\le 3$. Thus, for integrability candidates we can have for the degree of Q at maximum $q\le 3$ and for the degree of $P-x^2Q$ equally at maximum three. From what we saw above, the degree of P is $p\le 4$ and if $q=3$, then the degree of the numerator would be five, which is forbidden. Thus, we can have at most $q=2$ and $P=x^2Q+R$ where R is a polynomial at most cubic in x. The well-known discrete $\mathrm{P_{IV}}$ falls precisely in this class. As a matter of fact, the precise application of singularity confinement to the mapping

$$(x_{n+1}+x_n)(x_n+x_{n-1})=\frac{\alpha x_n^4+\eta x_n^3+\kappa x_n^2+\theta x_n+\mu}{\alpha x_n^2+\beta x_n+\gamma} \tag{3.8}$$

results in the QRT form: $\eta=\theta=0$. Further, the deautonomisation of the mapping yields d-$\mathrm{P_{IV}}$. On the other hand, if $q=1$, we have $p\le 3$. The mapping then has the form

$$(x_{n+1}+x_n)(x_n+x_{n-1})=\frac{\eta x_n^3+\kappa x_n^2+\theta x_n+\mu}{\beta x_n+\gamma}. \tag{3.9}$$

Singularity confinement leads to two distinct subcases. One corresponds $\alpha=0$ and $\eta=\theta=0$ in (3.8) while the other leads to the constraints $\eta=\beta$ and $\beta\mu=\kappa\theta$ and corresponds to the case where the r.h.s. of (3.8) is not irreducible. In the case $q=0$ we have three possibilities. Two come from $\eta=\beta=0$ in (3.9), which entails either $\kappa=0$ or $\theta=0$, the latter case being equivalent to $\alpha=\beta=0$ in (3.8). A third case is *formally* obtained by $\eta=\beta=0$ and $\kappa=\gamma$ in (3.9) and corresponds to the case where the r.h.s. of (3.9) is not irreducible. So to summarize, the r.h.s., if polynomial, must be either ax_n^2+b, $x_n^2+ax_n+b$, or ax_n+b. Note that in all three cases, $p\le 2$ rather than $p=3$. This limit on p when $q=0$ can in fact be proven by Nevanlinna-type methods, using some specific argument along lines similar to the one used for equation (3.3). The deautonomisations of these mappings were presented in detail in [19].

The next mapping we are going to examine is the family of q-$\mathrm{P_V}$,

$$(x_{n+1}x_n-1)(x_nx_{n-1}-1)=\frac{P(x_n)}{Q(x_n)}. \tag{3.10}$$

The straightforward application of (2.8) and (2.9) gives $u = 2$, $v = 2$. Thus, just as in the case of d-P_{IV}, for $w > 4$ we have a generic x_n of infinite order, and the only acceptable P, Q are quartic. However, we can rewrite (3.10) as

$$x_{n+1}x_{n-1} - \frac{x_{n+1}}{x_n} - \frac{x_{n-1}}{x_n} = \frac{P(x_n) - Q(x_n)}{x_n^2 Q(x_n)}. \tag{3.11}$$

Using (2.10) we find $u = 2$, $v = 1$, and thus, we must have $w \leq 3$. However, this does not mean that the degree of $P - Q$ and $x^2 Q$ must be less or equal to three because although P/Q is irreducible, $P - Q$ may have one or more x factors. We are thus led to the examination of each particular case. If $q \leq 1$, then no x factorisation is necessary and in this case, we have $p \leq 3$. If $q \leq 2$ and one x factorises, we have $P = Q + xR$ where R is cubic at maximum and $p \leq 4$. (Note that even if $q \leq 1$, $p = 4$ is allowed because R may still be cubic.) Finally if $q \leq 3$, a factor x^2 is necessary and $P = Q + x^2 S$, with S at most quadratic so that $p \leq 4$. For $q \leq 2$, this case is a subcase of the previous one with $R = xS$. But for $q = 3$, this case can be shown to have its generic solution of infinite order. Indeed one can show that, though the usual method gives (2.14) with $u = 2$, $v = 1$, a more refined calculation leads in this particular case to a situation where $u = 2$, but the "effective" v is zero, which combined with $w = 3$ leads to a growth of $T(r)$ faster than λ^r with $\lambda = 3/(2 + 2\epsilon)$ for ϵ arbitrarily small when r is large enough. First, let us remark that x does not divide Q (otherwise x would also divide $P = Q + x^2 S$, but P/Q has been assumed irreducible). Thus Q has three roots, none of which is zero. Since the degree of S is less than that of Q, the affinity of the r.h.s. of (3.11), S/Q, for infinity is entirely due to the affinity of x for each of the three roots of Q, i.e., $3T(r; x)$ (up to a bounded correction). In the l.h.s. x_{n+1}, x_{n-1} do contribute to the affinity for infinity as usually $2(1 + \epsilon)T(r + 1; x)$, i.e., $u = 2$. However, since none of the roots of Q is zero, the $1/x_n$ terms do *not* contribute to the affinity for infinity when the r.h.s. is near infinity. Thus, it is as if v were zero and only the contribution from x_{n+1}, x_{n-1} balances the contribution $3T(r; x)$ of the r.h.s.

The case of quadratic Q is the one corresponding to q-P_V. The general form of the mapping is

$$(x_{n+1}x_n - 1)(x_n x_{n-1} - 1) = \frac{\eta x_n^4 + \theta x_n^3 + \mu x_n^2 + \kappa x_n + \gamma}{\alpha x_n^2 + \beta x_n + \gamma}. \tag{3.12}$$

The application of singularity confinement leads to the constraints $\eta = \gamma$, $\theta = \kappa$, which reduce the mapping to its QRT form. The deautonomisation of the latter was presented in detail in [19]. In the case of linear Q we have *a priori*

$$(x_{n+1}x_n - 1)(x_n x_{n-1} - 1) = \frac{\eta x_n^4 + \theta x_n^3 + \mu x_n^2 + \kappa x_n + \lambda}{\beta x_n + \gamma}. \tag{3.13}$$

The constraints resulting from singularity confinement lead to three cases. Either $\eta = \lambda = \gamma$ and $\theta = \kappa$, which comes from taking $\alpha = 0$ in (3.12), or $\eta = 0$, $\lambda = \gamma$, and $\theta(\theta - \kappa) - \gamma(\gamma - \mu) = 0$, which comes from a situation where (3.12) is not irreducible and a nontrivial factor drops out, and finally $\eta = \theta = 0$, $\lambda = \mu$. The latter comes from a trivial simplification by x in (3.12) where $(\eta =)\gamma = 0$, so $\lambda = \mu$ is a consequence of $\theta = \kappa$ in (3.12) and we *do not* require $\lambda = \gamma$ in that case.

Finally, we will just list the possible forms of the r.h.s., when $q = 0$, once singularity confinement is implemented. Each form comes from (3.12) either by some special values ($\alpha = \beta = 0$, for instance), or by trivial (by x) or nontrivial simplifications. The possible forms are $x^4 + ax^3 + bx^2 + ax + 1$, $ax^3 + bx^2 + cx + 1$ with $a^2 - ac + b - 1 = 0$, $ax^2 + bx + a$, $ax^2 + bx + 1$, $ax + b$ where any number of the coefficients a, b, c may vanish.

For the remaining discrete Painlevé equations, a direct application of the Nevanlinna method is not convenient. Thus, we shall follow a slightly different approach. Instead of a single discrete equation, we consider the system

$$x_{n+1} * x_n = \frac{P(y_n)}{Q(y_n)}, \tag{3.14a}$$

$$y_n \star y_{n-1} = \frac{R(x_n)}{S(x_n)}, \tag{3.14b}$$

where $*$ and $\star$ stand for either of the operators $+$ or $\times$ (and the use of two different symbols stresses the fact that they can be chosen independently). In order to estimate the order of the solutions of (3.14), we calculate the characteristic of both members of (3.14a),(3.14b). We have

$$T(r; x_{n+1}) + T(r; x_n) \succeq wT(r; y_n), \tag{3.15a}$$

$$T(r; y_n) + T(r; y_{n-1}) \succeq \omega T(r; x_n), \tag{3.15b}$$

where w, ω are the maximum of the degrees of P, Q and R, S, respectively. Next, we compute (3.15a) once downshifted, i.e., at the point $(n-1)$, and eliminating $T(r; y)$, we find

$$T(r; x_{n+1}) + 2T(r; x_n) + T(r; x_{n-1}) \succeq w\omega T(r; x_n). \tag{3.16}$$

Using (2.12), we have

$$2(1+\epsilon)T(r+1; x_n) \succeq (w\omega - 2)T(r; x_n), \tag{3.17}$$

which means that (apart from a finite number of constant solutions) the generic x is of infinite order if $w\omega > 4$. Given this constraint, integrability candidates may only have $w\omega \leq 4$. We must thus examine the cases: $w = \omega = 2$ and $w = 1$, $\omega \leq 4$ (or the equivalent one $\omega = 1$, $w \leq 4$).

The case $w = \omega = 2$ leads (after the singularity confinement constraints have been implemented) to well-known integrable equations. In all cases, we present only the generic equation in which we assume that both w and ω are exactly two. The cases where one (or both) r.h.s. are homographic are treated later. Still, various subcases do exist, coming from special values of the parameters. The purely multiplicative case is

$$x_{n+1}x_n = \frac{\kappa y_n^2 + \lambda y_n + \mu}{\alpha y_n^2 + \beta y_n + \gamma}, \qquad y_n y_{n-1} = \frac{\gamma x_n^2 + \zeta x_n + \mu}{\alpha x_n^2 + \delta x_n + \kappa}. \tag{3.18}$$

When deautonomised using singularity confinement, this becomes the 'asymmetric' q-$\mathrm{P_{III}}$, discrete $\mathrm{P_{VI}}$, obtained by Jimbo and Sakai [20], or some degenerate form thereof (for $\alpha = 0$, for instance).

The purely additive one is the 'asymmetric' d-$\mathrm{P_{II}}$, discrete $\mathrm{P_{III}}$, we have studied in [21],

$$x_{n+1} + x_n = \frac{\delta y_n^2 + \epsilon y_n + \zeta}{\alpha y_n^2 + \beta y_n + \gamma}, \qquad y_n + y_{n-1} = \frac{\beta x_n^2 + \epsilon x_n + \lambda}{\alpha x_n^2 + \delta x_n + \kappa}. \tag{3.19}$$

Finally, the mixed case is a discrete $\mathrm{P_V}$ [22],

$$x_{n+1} + x_n = \frac{\delta y_n^2 + \epsilon y_n + \zeta}{\alpha y_n^2 + \beta y_n + \gamma}, \qquad y_n y_{n-1} = \frac{\gamma x_n^2 + \zeta x_n + \mu}{\alpha x_n^2 + \delta x_n + \kappa}. \tag{3.20}$$

Let us now turn to the case $\omega \leq 4$, $w = 1$. The latter means that the r.h.s. of (3.14a) is just homographic in y. Solving for y, we obtain $y_n = H(x_{n+1} * x_n)$ where H is homographic in its argument. Thus, we obtain the following mapping:

$$H(x_{n+1} * x_n) \star H(x_n * x_{n-1}) = \frac{R(x_n)}{S(x_n)}, \tag{3.21}$$

where the degrees of R and S are not larger than four. Four cases must be distinguished depending on the choices of $*$ and $\star$. When $*$ stands for $+$, the resulting equation is the autonomous form of a discrete $\mathrm{P_V}$ (once the singularity confinement constraints have been implemented in the most generic case, where R/S is irreducible with both R and S quartic). When $*$ stands for $\times$, the final equation is the autonomous form of a q-$\mathrm{P_{VI}}$. If $\star$ is taken as $\times$, the two forms are the standard ones, obtained in [23],

$$\frac{(x_{n+1}+x_n-2\zeta)}{(x_{n+1}+x_n)}\frac{(x_{n-1}+x_n-2\zeta)}{(x_{n-1}+x_n)}=\frac{\alpha(x_n-\zeta)^4+\beta(x_n-\zeta)^2+\gamma}{\alpha x_n^4+\delta x_n^2+\epsilon}, \tag{3.22}$$

$$\frac{\left(x_{n+1}x_n-\zeta^2\right)}{\left(x_{n+1}x_n-1\right)}\frac{\left(x_{n-1}x_n-\zeta^2\right)}{\left(x_{n-1}x_n-1\right)}=\frac{\alpha x_n^4+\beta x_n^3+\gamma x_n^2+\beta\zeta^2x_n+\alpha\zeta^4}{\alpha x_n^4+\delta x_n^3+\epsilon x_n^2+\delta x_n+\alpha}. \tag{3.23}$$

If $\star$ is the operator $+$, the resulting forms are just limiting cases of (3.22),(3.23). At this stage, ζ is a constant, like all the other parameters. The deautonomisation was presented in [23]. We will not comment on the hosts of special forms R and S can assume through special values of the coefficients and various factorisations.

4. TOWARDS THE DISCRETE ANALOGUE OF THE PAINLEVÉ PROPERTY

After having shown the applicability of our three-tiered approach, based on the combination of the Nevanlinna notion of growth and the singularity confinement, we can now tackle the question we set out to answer at the beginning of the paper. Namely, what would be the discrete analogue of the Painlevé property, characterising integrable systems.

Let us first start with a most successful discrete integrability detector which has been proposed by Viallet *et al.* [24]. This method links integrability to the low-growth of the degrees of the iterates of some given initial condition. The problem with this approach is that it applies equally well to mappings which are linearisable [25] while having unconfined singularities. But as we have shown in [26], integrability through linearisation is not necessarily related to the Painlevé property. In a sense, this integrability criterion is much too efficient since it is also a linearisability detector. Thus, we feel that it cannot be the discrete analogue of the Painlevé property, but it would be very interesting to derive its continuous analogue.

The discrete Painlevé property of Conte *et al.* [27] is in fact an application to the discrete domain of the perturbative Painlevé method (and thus, the name is rather misleading). Its applicability is limited by the fact that it relies crucially on the continuous limit of the given discrete system and, as is well known, not all discrete systems do possess nontrivial continuous limits.

The method of Ablowitz *et al.* introduces the most interesting notion of growth of solutions (of some discrete system). The Nevanlinna method for the evaluation of the order of the solution is certainly an essential one. However, we feel that the fact that AHH search for solutions, the asymptotic expansions of which are free of Digamma ψ functions, introduces some limitations (beyond the computational complications). Other singularities, Digamma derivatives, Gamma functions of fractional arguments, etc., may well spoil integrability (although the ψ are the primary suspects).

What we believe (and this is certainly a subjective statement) is that the property of integrable discrete systems which comes closest to being the discrete analogue for the Painlevé property is the one described in this paper. Namely, integrable discrete systems must be of finite order, in the Nevanlinna sense, and also possess confined singularities. (Moreover, the fact that a large class of linearisable systems have unconfined singularities stresses the parallel to the Painlevé method.) Our three-tiered approach provides an efficient algorithm for the investigation of this property. Thus, it may constitute a most useful heuristic detector of integrability, as the examples presented here have amply demonstrated.

REFERENCES

1. B. Grammaticos and A. Ramani, *Integrability (and How to Detect It)*, Course at the Pondicherry 96 School, Springer LNP 495, p. 30, (1997).
2. P. Painlevé, *Acta Math.* **25**, 1, (1902).
3. M.J. Ablowitz and H. Segur, *Phys. Rev. Lett.* **38**, 1103, (1977).
4. N.J. Zabusky and M.D. Kruskal, *Phys. Rev. Lett.* **15**, 240, (1965).
5. M.J. Ablowitz, A. Ramani and H. Segur, *Lett. Nuov. Cim.* **23**, 333, (1978).
6. B. Grammaticos, A. Ramani and V. Papageorgiou, *Phys. Rev. Lett.* **67**, 1825, (1991).
7. N. Joshi, *J. Math. An. and Appl.* **184**, 573, (1994).
8. A. Ramani, B. Grammaticos and J. Hietarinta, *Phys. Rev. Lett.* **67**, 1829, (1991).
9. B. Grammaticos, A. Ramani and K.M. Tamizhmani, *Jour. Phys.* **A27**, 559, (1994).
10. J. Hietarinta and C. Viallet, *Phys. Rev. Lett.* **81**, 325, (1998).
11. M.J. Ablowitz, R. Halburd and B. Herbst, *Nonlinearity* **13**, 889, (2000).
12. N. Yanagihara, *Arch. Ration. Mech. Anal.* **91**, 169, (1985).
13. E. Hille, *Ordinary Differential Equations in the Complex Domain*, J. Wiley and Sons, New York, (1976).
14. G. Valiron, *Bull. Soc. Math. France* **59**, 17, (1931).
15. B. Grammaticos, F. Nijhoff and A. Ramani, Discrete Painlevé equations, In *The Painlevé Property: One Century Later, CRM Series in Mathematical Physics*, (Edited by R. Conte), pp. 413–516, Springer-Verlag, New York, (1999).
16. Y. Ohta, K.M. Tamizhmani, B. Grammaticos and A. Ramani, *Phys. Lett.* **A262**, 152, (1999).
17. A. Ramani, Y. Ohta, J. Satsuma and B. Grammaticos, *Comm. Math. Phys.* **192**, 67, (1998).
18. G.R.W. Quispel, J.A.G. Roberts and C.J. Thompson, *Physica* **D34**, 183, (1989).
19. A. Ramani and B. Grammaticos, *Physica* **A228**, 160, (1996).
20. M. Jimbo and H. Sakai, *Lett. Math. Phys.* **38**, 145, (1996).
21. B. Grammaticos, F.W. Nijhoff, V.G. Papageorgiou, A. Ramani and J. Satsuma, *Phys. Lett.* **A185**, 446, (1994).
22. B. Grammaticos, Y. Ohta, A. Ramani and H. Sakai, *J. Phys.* **A31**, 3545, (1998).
23. B. Grammaticos and A. Ramani, *Phys. Lett.* **A257**, 288, (1999).
24. M.P. Bellon, J.-M. Maillard and C.-M. Viallet, *Phys. Rev. Lett.* **67**, 1373, (1991).
25. A. Ramani, B. Grammaticos, S. Lafortune and Y. Ohta, *J. Phys.* **A33**, L287, (2000).
26. A. Ramani, B. Grammaticos and S. Tremblay, *J. Phys.* **A33**, 3045, (2000).
27. R. Conte and M. Musette, *Phys. Lett.* **A223**, 439, (1996).