Journal of Combinatorial Theory, Series B 102 (2012) 924–947

Contents lists available at SciVerse ScienceDirect

Journal of Combinatorial Theory, Series B

www.elsevier.com/locate/jctb

On graphs with no induced subdivision of K_4

Benjamin Lévêque [a], Frédéric Maffray [b], Nicolas Trotignon [c,1]

[a] CNRS, LIRMM, 161 rue Ada, 34392 Montpellier Cedex 05, France
[b] CNRS, Laboratoire G-SCOP, 46 avenue Félix Viallet, 38031 Grenoble Cedex, France
[c] CNRS, LIP, ENS Lyon, INRIA, Université de Lyon, France

ARTICLE INFO

Article history:
Received 13 January 2010
Available online 17 May 2012

Keywords:
Induced subgraph
Series-parallel graphs
Subdivision of K_4
Structure theorem

ABSTRACT

We prove a decomposition theorem for graphs that do not contain a subdivision of K_4 as an induced subgraph where K_4 is the complete graph on four vertices. We obtain also a structure theorem for the class $\mathcal{C}$ of graphs that contain neither a subdivision of K_4 nor a wheel as an induced subgraph, where a wheel is a cycle on at least four vertices together with a vertex that has at least three neighbors on the cycle. Our structure theorem is used to prove that every graph in $\mathcal{C}$ is 3-colorable and entails a polynomial-time recognition algorithm for membership in $\mathcal{C}$. As an intermediate result, we prove a structure theorem for the graphs whose cycles are all chordless.

1. Introduction

We use the standard notation from [1]. Unless otherwise specified, we say that a graph G *contains* H when H is isomorphic to an induced subgraph of G. Denote by K_4 the complete graph on four vertices. A subdivision of a graph G is obtained by subdividing edges of G into paths of arbitrary length (at least one). We say that H is an *ISK4 of a graph G* when H is an induced subgraph of G and H is a subdivision of K_4. A graph that does not contain any subdivision of K_4 is said to be *ISK4-free*. Our main result is Theorem 1.1, saying that every ISK4-free graph is either in some basic class or has some special cutset. In [12], it is mentioned that deciding in polynomial time whether a given graph is ISK4-free is an open question of interest. This question was our initial motivation. But our theorem

E-mail addresses: benjamin.leveque@lirmm.fr (B. Lévêque), frederic.maffray@g-scop.inpg.fr (F. Maffray), nicolas.trotignon@ens-lyon.fr (N. Trotignon).

[1] Partially supported by *Agence Nationale de la Recherche* under reference ANR 10 JCJC 0204 01 and by PHC Pavle Savić grant, jointly awarded by EGIDE, an agency of the French Ministère des Affaires étrangères et européennes, and Serbian Ministry for Science and Technological Development.

0095-8956/$ – see front matter
http://dx.doi.org/10.1016/j.jctb.2012.04.005

does not lead to a polynomial-time recognition algorithm so far. The main reason is that at some step we use cutsets (namely star cutsets and double star cutsets) that are difficult to use in algorithms. We leave as an open question the existence of a more powerful decomposition theorem.

A consequence of our work is a complete structural description of the class $\mathcal{C}$ of graphs that contain no ISK4 and no wheel. Note that this class is easily seen to be the class of graphs with no K_4 and subdivision of a wheel as an induced subgraph. We give a recognition algorithm for this class, a coloring algorithm, and we prove that every graph in this class is 3-colorable.

Before stating our main results more precisely, we introduce some definitions and notation.

A *hole* of a graph is an induced cycle on at least four vertices. A *wheel* is a graph that consists of a hole H plus a vertex $x \notin H$, called the *hub* of the wheel, that is adjacent to at least three vertices of the hole. An edge of the wheel that is incident to x is called a *spoke*. A vertex v of a graph is *complete* to a set of vertices $S \subseteq V(G) \setminus v$ if v is adjacent to every vertex in S. A vertex v is *anticomplete* to a set of vertices S if v is adjacent to no vertex in S. Two disjoint sets A, B are *complete* to each other if every vertex of A is complete to B. A graph is called *complete bipartite* (resp. *complete tripartite*) if its vertex-set can be partitioned into two (resp. three) non-empty stable sets that are pairwise complete to each other. If these two (resp. three) sets have size p, q (resp. p, q, r) then the graph is denoted by $K_{p,q}$ (resp. $K_{p,q,r}$).

Given a graph H, the *line graph* of H is the graph $L(H)$ with vertex-set $E(G)$ and edge-set $\{ef\colon e \cap f \neq \emptyset\}$. The graph H is called a *root* of $L(H)$.

We denote the path on vertices $x_1, \dots, x_n$ with edges $x_1x_2, \dots, x_{n-1}x_n$ by $x_1 - \cdots - x_n$. We also say that P is an (x_1, x_n)-path. We denote by $x_i - P - x_j$ the subpath of P with extremities x_i, x_j. A path or a cycle is *chordless* if it is an induced subgraph of the graph that we are working on.

Given two graphs G, G', we denote by $G \cup G'$ the graph whose vertex set is $V(G) \cup V(G')$ and whose edge set is $E(G) \cup E(G')$.

For any integer $k \geqslant 0$, a *k-cutset* in a graph is a subset $S \subset V(G)$ of size k such that $G \setminus S$ is disconnected. A *proper* 2*-cutset* of a graph G is a 2-cutset $\{a, b\}$ such that $ab \notin E(G)$, $V(G) \setminus \{a, b\}$ can be partitioned into two non-empty sets X and Y so that there is no edge between X and Y and each of $G[X \cup \{a, b\}]$ and $G[Y \cup \{a, b\}]$ is not an (a, b)-path.

A *star-cutset* of a graph is a set S of vertices such that $G \setminus S$ is disconnected and S contains a vertex adjacent to every other vertex of S.

A *double star cutset* of a graph is a set S of vertices such that $G \setminus S$ is disconnected and S contains two adjacent vertices u, v such that every vertex of S is adjacent at least one of u, v. Note that a star-cutset is either a double star cutset or consists of one vertex.

A multigraph is called *series-parallel* if it arises from a forest by applying the following operations repeatedly: adding a parallel edge to an existing edge; subdividing an edge. A *series-parallel graph* is a series-parallel multigraph with no parallel edges.

Our main result is the following, which is proved in Section 9.

Theorem 1.1. *Let G be an ISK*4*-free graph. Then either*:

- *G is series-parallel*;
- *G is the line graph of a graph with maximum degree at most three*;
- *G has clique-cutset, a proper* 2*-cutset, a star-cutset or a double star cutset.*

The proof of the theorem above follows a classical idea. We consider a basic graph H and prove that if a graph in our class contains H, then either the whole graph is basic, or some part of the graph attaches to H in a way that entails a decomposition. Then, for the rest of the proof, the graphs under consideration can be considered H-free. We consider another basic graph H', and so on. The basic graphs that we consider are $K_{3,3}$, then some substantial line graph, then prisms, and finally the octahedron and wheels. The idea of considering a maximal line graph in such a context was first used in [5]. The same idea is essential in proof of the Strong Perfect Graph Conjecture [3].

Given a graph G, an induced subgraph K of G, and a set C of vertices of $G \setminus K$, the *attachment* of C over K is $N(C) \cap V(K)$, which we also denote by $N_K(C)$. When a set $S = \{u_1, u_2, u_3, u_4\}$ induces a square in a graph G with u_1, u_2, u_3, u_4 in this order along the square, a *link* of S is an induced

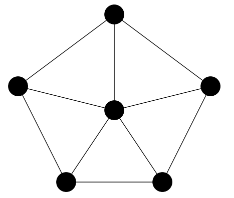

Fig. 1. Example of an ISK4-free graph with chromatic number 4.

path P of G with ends p, p' such that either $p = p'$ and $N_S(p) = S$, or $N_S(p) = \{u_1, u_2\}$ and $N_S(p') = \{u_3, u_4\}$, or $N_S(p) = \{u_1, u_4\}$ and $N_S(p') = \{u_2, u_3\}$, and no interior vertex of P has a neighbor in S. A link with ends p, p' is said to be *short* if $p = p'$, and *long* if $p \neq p'$. A *rich square* (resp. *long rich square*) is a graph K that contains a square S as an induced subgraph such that $K \setminus S$ has at least two components and every component of $K \setminus S$ is a link (resp. a long link) of S. Then S is called a central square of K. A rich square may have several central squares; for example $K_{2,2,2}$ is a rich square with three central squares.

In the particular case of wheel-free graph we have the following structure theorem. Note that a rich square is wheel-free if and only if it is long. A graph is *chordless* if all its cycles are chordless. It is easy to check that a line graph $G = L(R)$ is wheel-free if and only if R is chordless.

Theorem 1.2. *Let G be an* {*ISK*4*, wheel*}*-free graph. Then either*:

- *G is series-parallel*;
- *G is the line graph of a chordless graph with maximum degree at most three*;
- *G is a complete bipartite graph*;
- *G is a long rich square*;
- *G has clique-cutset or a proper* 2*-cutset.*

The structure of chordless graphs is elucidated in the following theorem, which will be proved in Section 10. Let us say that a graph G is *sparse* if for every edge uv of G we have either $\deg(u) \leqslant 2$ or $\deg(v) \leqslant 2$.

Theorem 1.3. *Let G be a chordless graph. Then either G is sparse or G admits a* 1*-cutset or a proper* 2*-cutset.*

Theorems 1.2 and 1.3 can be used to derive a tight bound on the chromatic number of {ISK4, wheel}-free graphs.

Theorem 1.4. *Any* {*ISK*4*, wheel*}*-free graph is* 3*-colorable.*

Theorem 1.4 will be proved in Section 11. This theorem is tight as shown by the graph on Fig. 1.

Gyárfás [8] defines a graph G to be *χ-bounded* with *χ-bounding function f* if for all induced subgraphs G' of G we have $\chi(G') \leqslant f(\omega(G'))$. A class of graphs is *$\chi$-bounded* if there exists a χ-bounding function that holds for all graphs of the class. Scott [15] conjectured that for any graph H, the class of those graphs that do not contain any subdivision of H as an induced subgraph is χ-bounded. This conjectured was disproved by Pawlik et al. [13]. It still remains to determine for which H's the statement conjectured by Scott is true. As noted by Scott [16], some of our results can be combined with a theorem of Kühn and Osthus [10] to prove his conjecture in the particular case of K_4. Note that being χ-bounded for the class of ISK4-free graphs means having the chromatic number bounded by a constant (because K_4 is a particular ISK4).

Theorem 1.5. *(See Kühn and Osthus [10].) For every graph H and every $s \in \mathbb{N}$ there exists $d = d(H, s)$ such that every graph G of average degree at least d contains either a $K_{s,s}$ as a subgraph or an induced subdivision of H.*

Theorem 1.6. *(See Scott [16].) There exists a constant c such that any ISK4-free graphs is c-colorable.*

Theorem 1.6 will be proved in Section 3. In fact, we do not know any example of an ISK4-free graph whose chromatic number is 5 or more. We propose the following conjecture.

Conjecture 1.7. *Any ISK4-free graph is 4-colorable.*

Our results yield several algorithms described in Section 12.

Theorem 1.8. *There exists an algorithm of complexity $O(n^2m)$ that decides whether a given graph is {ISK4, wheel}-free.*

There exists an algorithm of complexity $O(n^2m)$ whose input is a graph with no ISK4 and no wheel and whose output is a 3-coloring of its vertices.

2. Series-parallel graphs

Theorem 2.1. *(See Dirac [6], Duffin [7].) A graph is series-parallel if and only if it contains no subdivision of K_4 as a (possibly non-induced) subgraph.*

A *branch-vertex* in a graph G is a vertex of degree at least 3. A *branch* is a path of G of length at least one whose ends are branch-vertices and whose internal vertices are not (so they all have degree 2). Note that a branch of G whose ends are u, v has at most one chord: uv. An induced subdivision H of K_4 has four vertices of degree three, which we call the *corners* of H, and six branches, one for each pair of corners.

A *theta* is a connected graph with exactly two vertices of degree three, all the other vertices of degree two, and three branches, each of length at least two. A *prism* is a graph that is the line graph of a theta.

Lemma 2.2. *Let G be an ISK4-free graph. Then either G is a series-parallel graph, or G contains a prism, a wheel or a $K_{3,3}$.*

Proof. Suppose that G is not series-parallel. By Theorem 2.1, G contains a subdivision H of K_4 as a possibly non-induced subgraph. Let us choose a minimal such subgraph H. So H can be obtained from a subdivision H' of K_4 by adding edges (called *chords*) between the vertices of H'. Since G is ISK4-free, there is at least one such chord e in H. Let H' have corners a, b, c, d and branches P_{ab}, P_{ac}, P_{ad}, P_{bc}, P_{bd}, P_{cd} with the obvious notation. Note that, by the minimality of H, the six paths P_{ab}, P_{ac}, P_{ad}, P_{bc}, P_{bd}, P_{cd} are chordless in H.

Suppose that e is incident to one of a, b, c, d, say $e = ax$. Then x lies in none of P_{ab}, P_{ac}, P_{ad} by the minimality of H. Moreover P_{ab}, P_{ac}, P_{ad} have all length one, for otherwise, by deleting the interior vertices of one of them, we obtain a subdivision of K_4, which contradicts the minimality of H. If H has a chord e' that is not incident to a, then e' is a chord of the cycle $C = P_{bd} \cup P_{cd} \cup P_{bc}$. Since C is a cycle with one chord e' and since the branches P_{bd}, P_{cd}, P_{bc} are chordless, we may assume up to symmetry that C contains a cycle C' that goes through e', c, d and not through b. If x is in C', then $C' \cup \{a\}$ is a subdivision of K_4, which contradicts the minimality of H. So, up to the symmetry between P_{bc} and P_{bd}, we may assume that x is in $P_{bd} \setminus C'$. Then $C' \cup x{-}P_{bd}{-}d \cup \{a\}$ forms a subdivision of K_4, which contradicts the minimality of H. Hence, every chord of H is incident to a. This means that H is a wheel with hub a and the lemma holds. From now on, we assume that no chord of H is adjacent to a, b, c, d.

Suppose that e is between interior vertices of two branches of H with a common end, P_{ab} and P_{ad} say. Put $e = uv$, where $u \in P_{ab}$, $v \in P_{ad}$. Vertices a and u are adjacent, for otherwise the deletion of the interior vertices of $a{-}P_{ab}{-}u$ produces a subdivision of K_4, which contradicts the minimality of H. Similarly, a and v are adjacent, and P_{bc}, P_{bd}, P_{cd} all have length one. So H' is a prism, whose triangles are auv, bcd. If $H = H'$, the lemma holds, so let us assume that $H' \neq H$. Then H has a chord

e' that is not an edge of H'. Up to symmetry, we assume that e' has an end u' in $uP_{ab}b$ and an end v' in $vP_{ad}d$. Note that $u' \neq b$ and $v' \neq d$. Since $e \neq e'$ we may assume $u \neq u'$. Then the deletion of the interior vertices of $aP_{ab}u'$ gives a subdivision of K_4, which contradicts the minimality of H.

Finally, suppose that e is between two branches of H with no common end, P_{ad} and P_{bc} say. Put $e = uv$, $u \in P_{ad}$, $v \in P_{bc}$. If P_{ab} has length greater than one, then by deleting its interior we obtain a subdivision of K_4, which contradicts the minimality of H. So, P_{ab}, and similarly P_{ac}, P_{bd}, P_{cd}, all have length one. The same argument shows that ua, ud, vb, vc are edges of H. Hence H is isomorphic to $K_{3,3}$. □

3. Complete bipartite graphs

Here we decompose ISK4-free graphs that contain a $K_{3,3}$.

Lemma 3.1. *Let G be an ISK4-free graph, and H be a maximal induced $K_{p,q}$ in G, such that $p, q \geqslant 3$. Let v be a vertex of $G \setminus H$. Then the attachment of v over H is either empty, or consists of one vertex or of one edge or is $V(H)$.*

Proof. Let $A = \{a_1, \dots, a_p\}$ and $B = \{b_1, \dots, b_q\}$ be the two sides of the bipartition of H. If v is adjacent to at most one vertex in A and at most one in B, then the lemma holds. Suppose now, up to symmetry, that v is adjacent to at least two vertices in A, say a_1, a_2. Then v is either adjacent to every vertex in B or to no vertex in B, for otherwise, up to symmetry, v is adjacent to b_1 and not to b_2, and $\{a_1, a_2, b_1, b_2, v\}$ is an ISK4. If v has no neighbor in B, then v sees every vertex in A, for otherwise $va_3 \notin E(G)$ say, and $\{a_1, a_2, a_3, b_1, b_2, v\}$ is an ISK4. So, v is complete to A and anticomplete to B, which contradicts the maximality of H. If v is complete to B, then v is adjacent to at least two vertices in B and symmetrically we can prove that v is complete to A. So, the attachment of v is $V(H)$. □

Lemma 3.2. *Let G be an ISK4-free graph that contains a $K_{3,3}$, and let H be a maximal induced $K_{p,q}$ of G with $p, q \geqslant 3$. Let U be the set of those vertices of $V(G) \setminus H$ that are complete to H. Let C be a component of $G \setminus (H \cup U)$. Then the attachment of C over H is either empty or consists of one vertex or of one edge.*

Proof. Suppose the contrary. So we may assume up to symmetry that there are vertices c_1, c_2 in C such that $|N(\{c_1, c_2\}) \cap D| \geqslant 2$ where D is one of A, B. Since C is connected, there is a path $P = c_1 - \cdots - c_2$ in C from c_1 to c_2. We choose c_1, c_2 such that P is minimal. Up to symmetry, we may assume that $c_1a_1, c_2a_2 \in E(G)$. By Lemma 3.1, we have $c_1 \neq c_2$. If a_3 has a neighbor in P, then by Lemma 3.1 this neighbor must be an interior vertex of P, but this contradicts the minimality of P. So, a_3 has no neighbor in P. If no vertex in B has neighbors in P, then $V(P) \cup \{a_1, a_2, a_3, b_1, b_2\}$ induces an ISK4. If exactly one vertex in B, say b_1, has neighbors in P, then $V(P) \cup \{a_1, a_2, a_3, b_2, b_3\}$ induces an ISK4. If at least two vertices in B, say b_1, b_2, have neighbors in P, then by Lemma 3.1 and by the minimality of P we may assume that $N(b_1) \cap V(P) = \{c_1\}$ and $N(b_2) \cap V(P) = \{c_2\}$. But then $V(P) \cup \{a_1, a_3, b_1, b_2\}$ induces an ISK4. In every case there is a contradiction. □

Let us say that a complete bipartite or complete tripartite graph is *thick* if it contains an induced $K_{3,3}$.

Lemma 3.3. *Let G be an ISK4-free graph that contains $K_{3,3}$. Then either G is a thick complete bipartite or complete tripartite graph, or G has a clique-cutset of size at most three.*

Proof. Let H be a maximal $K_{p,q}$ in G, with $p, q \geqslant 3$, and let U be the set of those vertices that are complete to H. Note that U is a stable set because if U contains an edge uv, then $\{u, v, a_1, b_1\}$ is an ISK4. If $V(G) = V(H) \cup U$, then G is either a complete bipartite graph (if $U = \emptyset$) or complete tripartite graph (if $U \neq \emptyset$). Now suppose that $V(G) \neq V(H) \cup U$, and let C be any component of $G \setminus (H \cup U)$. We claim that $|N(C) \cap U| \leqslant 1$. Else, consider two vertices u, v in $N(C) \cap U$ and a minimal path P in

C from a neighbor of u to a neighbor of v. By Lemma 3.2, we may assume that a_3 and b_3 have no neighbor in C (hence in P). Then $P \cup \{u, v, a_3, b_3\}$ is an ISK4, a contradiction. This proves our claim. By Lemma 3.2, $N(C) \cap (V(H) \cup U)$ is a clique-cutset of G of size at most three. □

Proof of Theorem 1.6. Let $c = d(K_4, 6) \geqslant 3$ be the constant of Theorem 1.5 with $H = K_4$ and $s = 6$. We claim that any ISK4-free graphs is c-colorable. Suppose on the contrary that there exists an ISK4-free graph G with $\chi(G) > c$, and suppose G is minimal with this property, i.e. $\chi(H) \leqslant c$ for every proper induced subgraph H of G.

We claim that the degree of every vertex is at least c. Suppose on the contrary that G contains a vertex v of degree $\deg(v) \leqslant c - 1$, then $\chi(G) \leqslant \max(\chi(G - v), \deg(v) + 1) \leqslant c$, a contradiction. So the average degree of G is at least $c = d(K_4, 6)$.

By Theorem 1.5 the graph G contains a $K_{6,6}$ as a possibly non-induced subgraph. Let A, B be the two sides of the $K_{6,6}$. The graph $G[A]$ contains no triangle, otherwise this triangle plus a vertex of B forms a K_4. Similarly $G[B]$ contains no triangle. From the well-known fact that any graph on 6 vertices contains either a triangle or a stable set on 3 vertices, both $G[A]$ and $G[B]$ contain a stable set of size 3. So G contains an induced $K_{3,3}$.

By Lemma 3.3, the graph G admits a clique cutset K. Hence $V(G) \setminus K$ is partitioned into non-empty sets X_1, X_2 such that there are no edges between X_1 and X_2. A coloring of G can be easily obtained by combining a coloring of $G[K \cup X_1]$ and $G[K \cup X_2]$, showing that $\chi(G) \leqslant \max(\chi(G[K \cup X_1]), \chi(G[K \cup X_2])) \leqslant c$. □

4. Cyclically 3-connected graphs

A *separation* of a graph H is a pair (A, B) of subsets of $V(H)$ such that $A \cup B = V(H)$ and there are no edges between $A \setminus B$ and $B \setminus A$. It is *proper* if both $A \setminus B$ and $B \setminus A$ are non-empty. The *order* of the separation is $|A \cap B|$. A *k-separation* is a separation (A, B) such that $|A \cap B| \leqslant k$. A separation (A, B) is *cyclic* if both $H[A]$ and $H[B]$ has cycles. A graph H is *cyclically* 3-*connected* if it is 2-connected, not a cycle, and there is no cyclic 2-separation. Note that a cyclic 2-separation of any graph is proper.

Here we state simple lemmas about cyclically 3-connected graphs that will be needed in the next section. Most of them are stated and proved implicitly in [4, Section 7]. But they are worth stating separately here: they are needed for the second time at least and writing down their proof now may be convenient for another time. A cyclically 3-connected graph has at least four vertices and K_4 is the only cyclically 3-connected graph on four vertices. As any 2-connected graph that is not a cycle, a cyclically 3-connected graph is edge-wise partitioned into its branches.

Lemma 4.1. *Let H be a cyclically* 3*-connected graph. For every proper* 2*-separation (A, B) of H, $A \cap B$ consists of two non-adjacent vertices, one of $H[A]$, $H[B]$ is a path, and thus is included in a branch of H, and the other contains a cycle.*

Proof. Since (A, B) is proper, $A \cap B$ is a cutset, and so it has size two since H is 2-connected. We put $A \cap B = \{a, b\}$. Since (A, B) is not cyclic, up to symmetry, $H[A]$ has no cycle. Note that $H[A]$ contains a path P from a to b, for otherwise one of a, b is a cutvertex of H, which contradicts H being 2-connected. Actually, $H[A] = P$, for otherwise $H[A]$ is a tree with at least one vertex c of degree 3, and c is a cutvertex of this tree, so c is also a cutvertex of H, a contradiction again. We have $ab \notin E(H)$ because (P, B) is proper. Since (P, B) is a separation, every internal vertex of P has degree two in H, so P is included in a branch of H as claimed. So, $ab \notin E(H)$ because (P, B) is proper. If B has no cycle, then by the same proof as for A, $H[B]$ is a path. So, H is a cycle, a contradiction. □

Lemma 4.2. *Let H be a cyclically* 3*-connected graph and a, b be two adjacent vertices of H. Then $\{a, b\}$ is not a cutset of H.*

Proof. Follows directly from Lemma 4.1. □

Lemma 4.3. *Let H be a cyclically 3-connected graph, a, b be two branch-vertices of H, and P_1, P_2, P_3 be three induced paths of H whose ends are a, b. Then either*:

- *P_1, P_2, P_3 are branches of H of length at least two and $H = P_1 \cup P_2 \cup P_3$ (so H is a theta)*;
- *there exist distinct integers $i, j \in \{1, 2, 3\}$ and a path S of H with an end in the interior of P_i, an end in the interior of P_j and whose interior is disjoint from $V(P_1 \cup P_2 \cup P_3)$; and $P_1 \cup P_2 \cup P_3 \cup S$ is a subdivision of K_4.*

Proof. Put $H' = P_1 \cup P_2 \cup P_3$. Suppose that $H = H'$. If P_1 is of length one, then $(V(P_1 \cup P_2), V(P_1 \cup P_3))$ is a cyclic 2-separation of H. So P_1, and similarly P_2, P_3 are of length at least two and the first outcome of the lemma holds. Now assume $H \neq H'$. If the second outcome of the lemma fails, then no path like S exists. In particular there is no edge between the interior of any two of the three paths, and the interiors of the three paths lie in distinct components of $H \setminus \{a, b\}$. Since H is connected and $H \neq H'$, there is a vertex in $V(H) \setminus V(H')$ with a neighbor c in one of P_1, P_2, P_3. Since H is 2-connected, $\{c\}$ is not a cutset of H and there exists a path R from c to some other vertex c' in H'. Since no path like S exists, R must have its two ends in the same branch of H', say in P_1. It follows that P_1 has an interior vertex, and we call C the component of $H \setminus \{a, b\}$ that contains the interior of P_1 union the component that contains the interior of R. Now, we put $A = \{a, b\} \cup V(H) \setminus C$, $B = C \cup \{a, b\}$ and we observe that (A, B) is a cyclic 2-separation of H, a contradiction. □

Lemma 4.4. *Let H be a cyclically 3-connected graph and let a, b be two branch-vertices of H such that there exist two distinct branches of G between them. Then H is a theta.*

Proof. Let P_1, P_2 be two distinct branches of H whose ends are a, b. Put $A = V(P_1 \cup P_2)$, $B = (V(H) \setminus A) \cup \{a, b\}$, and observe that (A, B) is a 2-separation of H. Since H is not a cycle, B contains at least three vertices, and $H[B]$ contains a shortest path P_3 from a to b since H is 2-connected. We apply Lemma 4.3 to P_1, P_2, P_3. Since P_1, P_2 are branches, the second outcome cannot happen. So H is a theta. □

Lemma 4.5. *A graph H is cyclically 3-connected if and only if it is either a theta or a subdivision of a 3-connected graph.*

Proof. A 3-connected graph has at least four vertices. So, thetas and subdivisions of 3-connected graphs are cyclically 3-connected. Conversely, if H is a cyclically 3-connected graph, then let H' be the multigraph on the branch-vertices of H obtained as follows: for every branch of H with ends a, b, we put an edge ab in H'. If H' has a multiple edge, then there are two vertices a, b of H and two branches P, Q of H with ends a, b. So, by Lemma 4.4, H is a theta. Now assume that H' has no multiple edge. Then H' is a graph and H is a subdivision of H'. Since H is 2-connected, H' is also 2-connected. We claim that H' is 3-connected. For suppose that H' has a proper 2-separation (A, B). Since H' has minimum degree at least three, it is impossible that $H'[A]$ is a path. Since H' is 2-connected, $H'[A]$ cannot be a tree and so it must contain a cycle. Symmetrically, $H'[B]$ must contain a cycle. Let A' be the union of A and of the set of vertices of degree two of H that arise from subdividing edges of $H'[A]$. Let B' be defined similarly. If $H'[A \cap B]$ is an edge and vertices of H arise from the subdivision of that edge, we put them in A'. Now we observe that (A', B') is a cyclic 2-separation of H, a contradiction. This proves our claim. It follows that H is a subdivision of a 3-connected graph. □

Lemma 4.6. *Let H be a cyclically 3-connected graph and a, b be two distinct vertices of H. If no branch contains both a, b, then $H' = (V(H), E(H) \cup \{ab\})$ is a cyclically 3-connected graph and every graph obtained from H' by subdividing ab is cyclically 3-connected.*

Proof. The graph H' is clearly 2-connected and not a cycle. So we need only prove that H' has no cyclic 2-separation. Suppose it has a cyclic 2-separation $\{A, B\}$. Up to symmetry we may assume that

a, b lie in A, because there is no edge between $A \setminus B$ and $B \setminus A$. Since (A, B) is cyclic in H', B has a cycle in H' and so in H. Hence, by Lemma 4.1, A induces a path of H and so it is included in a branch of H, contrary to our assumption.

By Lemma 4.5, H' is a subdivision of a 3-connected graph since it cannot be a theta because of the edge ab. So, every graph that we obtain by subdividing ab is a subdivision of a 3-connected graph, and so is cyclically 3-connected. □

Lemma 4.7. *Let H be a cyclically 3-connected graph, let Z be a cycle of H and a, b, c, d be four distinct vertices of Z that lie in this order on Z and such that $ab \in E(Z)$ and $cd \in E(Z)$. Let P be the subpath of Z from a to d that does not contain b, c, and let Q be the subpath of Z from b to c that does not contain a, d. Suppose that the edges ab, cd are in two distinct branches of H. Then there is a path R with an end-vertex in P, an end-vertex in Q, no interior vertex in Z, and R is not from a to b or from c to d.*

Proof. Suppose there does not exist a path like R. Then $\{a, c\}$ is a cutset of H that separates b from d. By Lemma 4.1, we may assume up to symmetry that $a-P-d-c$ is included in a branch of H. Also $\{b, d\}$ is a cutset, so one of $b-a-P-d$, $b-Q-c-d$ is included in a branch of H. If it is $b-Q-c-d$, then $\{a, b\}$ is a cutset of H contradictory to Lemma 4.2. So it is $b-a-P-d$, and $b-a-P-d-c$ is included in a branch of H. Hence, ab, cd are in the same branch of H, which contradicts our assumption. □

Lemma 4.8. *Let H be a subdivision of a 3-connected graph. Let C be a cycle and e an edge of H such that C and e are edgewise disjoint. Then there exists a subdivision of K_4 that is a subgraph of H and that contains C and e.*

Proof. Since H is 2-connected, there exist two vertex-disjoint paths $R = x-\cdots-x'$ and $S = y-\cdots-y'$ between C and e, with $e = xy$ and $x', y' \in C$. Let P_1, P_2 be the two edge-disjoint paths of C with end-vertices x', y'. Let $P_3 = x-\cdots-x'-y'-\cdots-y$. Then P_1, P_2, P_3 are three edge-disjoint paths between x' and y', so at most one of them is an edge.

Vertices x', y' have degree at least three in H, so they are also vertices of the 3-connected graph of which H is a subdivision. So $H \setminus \{x', y'\}$ is connected. Let P be a shortest path connecting two paths among $P_1 \setminus \{x', y'\}$, $P_2 \setminus \{x', y'\}$, $P_3 \setminus \{x', y'\}$. Then $P_1 \cup P_2 \cup P_3 \cup P$ is a subdivision of K_4 satisfying the lemma. □

5. Line graph of substantial graphs

A *flat branch* in a graph is a branch such that no triangle contains two vertices of it. So a non-flat branch is an edge that lies in a triangle. Note that any branch of length zero is flat. Moreover, every branch of length at least two is flat.

A *triangular* subdivision of K_4 is a subdivision of K_4 that contains a triangle. A *square theta* is a theta that contains a square, in other words, a theta with two branches of length two. A *square prism* is a prism that contains a square, in other words, a prism with two flat branches of length one. Note that a square prism is the line graph of a square theta. A *square subdivision of K_4* is a subdivision of K_4 whose corners form a (possibly non-induced) square. An induced square in a graph is *even* if an even number of edges of the square lie in a triangle of the graph. It easily checked that the line graph of a subdivision H of K_4 contains an even square if and only if H is a square subdivision of K_4; in that case the vertices in any even square of $L(H)$ arise from the edges of a square on the branch-vertices of H. It is easily checked that a prism contains only even squares.

A *diamond* is a K_4 minus one edge. A *closed diamond* is any graph obtained from a K_4 by subdividing only one edge. In a closed diamond that is not a K_4, the four corners induce a diamond, there is a unique branch P of length at least two, and we say that P *closes* the diamond.

If X, Y are two paths in a graph G, a *connection between X, Y* is a path $P = p-\cdots-p'$ such that p has a neighbor in X, p' has a neighbor in Y, no interior vertex of P has a neighbor in $X \cup Y$, and if $p \neq p'$, then p has no neighbor in Y and p' has no neighbor in X.

The line graph of K_4 is isomorphic to $K_{2,2,2}$ and is usually called the *octahedron*. It has three even squares. For every even square S of an octahedron G, the two vertices of $G \setminus S$ are both links of S. Note also that when K is a square prism with a square S, then $V(K) \setminus S$ is a link of S.

Given a graph G, a graph H such that $L(H)$ is an induced subgraph of G, and a connected induced subgraph C of $V(G) \setminus L(H)$, we define several types that C may have, according to its attachment over $L(H)$:

- C is of type *branch* if the attachment of C over $L(H)$ is included in a flat branch of $L(H)$;
- C is of type *triangle* if the attachment of C over $L(H)$ is included in a triangle of $L(H)$;
- C is of type *augmenting* if C contains a connection $P = p - \cdots - p'$ between two distinct flat branches X, Y of $L(H)$ such that $N_X(p)$ is an edge of X, $N_Y(p')$ is an edge of Y, and there is no edge between $L(H) \setminus (X \cup Y)$ and P. We say that P is an *augmenting path* for C;
- C is of type *square* if $L(H)$ contains an even square S, C contains a link P of S, and there is no edge between $L(H) \setminus S$ and P. We say that P is a *linking path* for C.

Note that the types may overlap: a subgraph C may be of more than one type. Since we view a vertex of G as a connected induced subgraph of G, we may speak about the type of a vertex with respect to $L(H)$.

Lemma 5.1. *Let G be a graph that contains no triangular ISK4. Let K be a prism that is an induced subgraph of G and let C be a connected induced subgraph of $G \setminus K$. Then C is either of type branch, triangle, augmenting or square with respect to K.*

Proof. Let $X = x - \cdots - x'$, $Y = y - \cdots - y'$, $Z = z - \cdots - z'$ be the three flat branches of K denoted in such a way that xyz and $x'y'z'$ are triangles. Call X, Y, Z and the two triangles of K the *pieces* of K. Suppose that C is not of type branch or triangle and consider an induced subgraph P of C minimal with respect to the property of being a connected induced subgraph, not of type branch or triangle.

$$\begin{aligned}&P \text{ is a path, no internal vertex of } P \text{ has neighbors in } K,\\ &\text{and } N_K(P) \text{ is not included in a branch or triangle of } K.\end{aligned} \tag{1}$$

If P is not a path, then either P contains a cycle or P is a tree with a vertex of degree at least three. In either case, P has three distinct vertices a_1, a_2, a_3 such that $P \setminus a_i$ is connected for each $i = 1, 2, 3$ (if P has a cycle, take any three vertices of Z; if P is a tree, take three leaves of P). For each $i = 1, 2, 3$, by the minimality of P, the attachment of $P \setminus a_i$ over K is included in a piece X_i of K, and a_i has a neighbor y_i in $V(K) \setminus X_i$. So we have $\{y_1, y_2\} \subseteq X_3$, $\{y_1, y_3\} \subseteq X_2$, $\{y_2, y_3\} \subseteq X_3$. But this is impossible because no three pieces X_1, X_2, X_3 of K have that property. Thus P is a path. If P has length zero, then the claim holds since, by the assumption, P is not of type branch or triangle. So, we may assume that P has length at least one. Let P have ends p, p'. Suppose that the claim fails. Then by the minimality of P, we have $N_K(P \setminus p') \subset A$ and $N_K(P \setminus p) \subset B$, where A, B are distinct pieces of K; moreover, some interior vertex of P must have a neighbor in K. So the attachment of the interior of P over K is not empty and is included in $A \cap B$. Since two distinct flat branches of K are disjoint and two distinct triangles of K are disjoint, we may assume that $N_K(p) \subseteq \{x, y, z\}$, $N_K(p') \subseteq X$, and some interior vertex of P is adjacent to x. Note that p has at most two neighbors in $\{x, y, z\}$, because G has no K_4, and that p must have at least one neighbor in $\{y, z\}$, for otherwise P is of type branch. If $py, pz \in E(G)$, then, since some interior vertex of P is adjacent to x, P contains a path that closes the diamond $\{x, y, z, p\}$, a contradiction. So we may assume up to symmetry that $pz \in E(G)$ and $py \notin E(G)$. Vertex p' has a neighbor in $X \setminus x$, for otherwise P is of type triangle. Let w be the neighbor of p' closest to x' along X. Then $z - p - P - p' - w - X - x'$, $z - Z - z'$ and $z - y - Y - y'$ form a triangular ISK4, a contradiction. This proves claim (1).

Let p, p' be the two ends of P. We distinguish between two cases.

Case 1: *P is a connection between two flat branches of K and has no neighbor in the third flat branch.* We may assume that p has a neighbor in X, p' has a neighbor in Y, and none of p, p' has neighbors in Z. Let

x^L (resp. x^R) be the neighbor of p closest to x (resp. to x') along X. Let y^L (resp. y^R) be the neighbor of p' closest to y (resp. to y') along Y. If both x^Lx^R, y^Ly^R are edges, then C is of type augmenting and the lemma holds. So let us assume up to symmetry that $x^Lx^R \notin E(G)$. Suppose that $x^L \neq x^R$. We may assume $y^L \neq y'$ (else $y^R \neq y$ and the argument is similar). Then $p-x^L-X-x$, $p-x^R-X-x'-z'-Z-z$, $p-P-p'-y^L-Y-y$ form a triangular ISK4, a contradiction. So $x^L = x^R$. If y^Ly^R is an edge, then $X \cup Y \cup P$ is a triangular ISK4. So $y^Ly^R \notin E(G)$, and consequently $y^L = y^R$ (just like we obtained $x^L = x^R$). Suppose that x^L is not equal to x or x'. We may assume that $y^L \neq y'$ (else $y^R \neq y$ and the argument is similar). Then x^L-X-x, $x^L-p-P-p'-y^L-Y-y$ and $x^L-X-x'-z'-Z-z$ form a triangular ISK4, a contradiction. So x^L is one of x, x', and, similarly, y^L is one y, y'. We may assume $x^L = x$ and $y^L = y'$, for otherwise (1) is contradicted. Then $x-X-x'$, $x-p-P-p'-y'$, $x-z-Z-z'$ form a triangular ISK4, a contradiction.

Case 2: *We are not in Case* 1. Suppose first that one of p, p' has at least two neighbors in a triangle of K. Then we may assume up to symmetry that $px, py \in E(G)$, and $pz \notin E(G)$ because G contains no K_4. By (1) and up to symmetry, p' must have a neighbor in $Y \setminus y$ or in Z. Note that either $p = p'$ or $N_K(p) = \{x, y\}$, for otherwise p would contradict the minimality of P. If p' has a neighbor in Z, then let w be such a neighbor closest to z along Z. Then $p-P-p'-w-Z-z$ closes the diamond $\{p, x, y, z\}$, a contradiction. So, p' has no neighbor in Z, and so it has neighbors in $Y \setminus y$. Let w^L (resp. w^R) be the neighbor of p' closest to y (resp. to y') along Y. Note that $w^R \neq y$ by (1). If p' has no neighbor in X, then $x-X-x'$, $x-p-P-p'-w^R-Y-y'$ and $x-z-Z-z'$ form a triangular ISK4. So p' has a neighbor in X, and we denote by v^L (resp. v^R) such a neighbor closest to x (resp. to x') along X. Since we are not in Case 1, we have $p \neq p'$. If either $v^L \neq x'$ or $w^L \neq y'$, then p' contradicts the minimality of P. So assume $v^L = v^R = x'$ and $w^L = w^R = y'$. If X has length at least two, then $p-P-p'-x'-z'-Z-z$ closes the diamond $\{p, x, y, z\}$. So X has length one, and similarly Y has length one. But then P is a link of the even square $\{x, y, x', y'\}$ of K, so C is of type square.

Now we assume that both p, p' have at most one neighbor in a triangle of K. At least one of p, p' (say p) must have neighbors in more than one branch of K, for otherwise we are in Case 1. So $p = p'$ by the minimality of P, and p has neighbors in X, Y, Z, for otherwise we are again in Case 1. We may assume that $py, pz \notin E(G)$. Let x^R, y^R, z^R be the neighbors of p closest to x', y', z' along X, Y, Z respectively. Then $p-x^R-X-x'$, $p-y^R-Y-y'$, $p-z^R-Z-z'$ form a triangular ISK4, a contradiction. □

Lemma 5.2. *Let G be a graph that contains no triangular ISK4. Let H be a subdivision of K_4 such that $L(H)$ is an induced subgraph of G. Let C be a connected induced subgraph of $G \setminus L(H)$. Then C is either of type branch, triangle, augmenting or square with respect to $L(H)$.*

Proof. Let a, b, c, d be the four corners of H. See Fig. 2. The three edges incident to each vertex $x = a, b, c, d$ form a triangle in $L(H)$, which we label T_x. In $L(H)$, for every pair $x, y \in \{a, b, c, d\}$ there is one path with an end in T_x and an end in T_y, and no interior vertex in the triangles, and we denote this path by P_{xy}. Note that $P_{xy} = P_{yx}$, and the six distinct paths so obtained are vertex disjoint. Some of these paths may have length 0. In the triangle T_x, we denote by v_{xy} the vertex that is the end of the path P_{xy}. Thus the flat branches of $L(H)$ are the paths of length at least one among P_{ab}, P_{ac}, P_{ad}, P_{bc}, P_{bd}, P_{cd}. Note that $L(H)$ may have as many as four triangles other than T_a, T_b, T_c, T_d. The branch-vertices of $L(H)$ are v_{ab}, v_{ac}, v_{ad}, v_{ba}, v_{bc}, v_{bd}, v_{ca}, v_{cb}, v_{cd}, v_{da}, v_{db} and v_{dc}. The subgraph $L(H)$ has no other edges than those in the four triangles and those in the six paths. Let every flat branch and every triangle of $L(H)$ be called a *piece* of $L(H)$.

Suppose that C is not of type branch or triangle with respect to $L(H)$, and consider an induced subgraph P of C minimal with respect to the property of being a connected induced subgraph not of type branch or triangle.

P is a path, no internal vertex of P has neighbors in $L(H)$ and
$N_{L(H)}(P)$ is not included in a flat branch or in a triangle of $L(H)$. (1)

If P is not a path, then, as in the proof of claim (1) in Lemma 5.1, P has three distinct vertices a_1, a_2, a_3 such that $P \setminus a_i$ is connected for each $i = 1, 2, 3$. For each $i = 1, 2, 3$, by the minimality of P, the

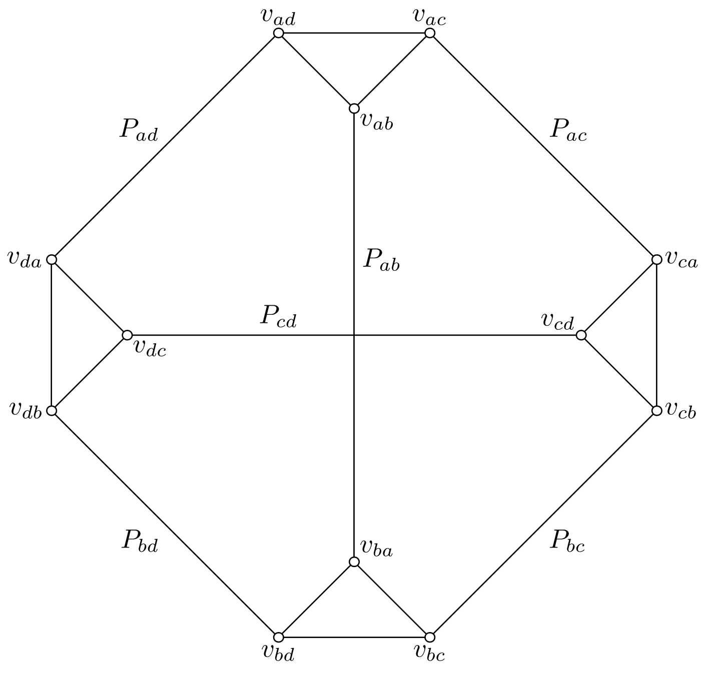

Fig. 2. The line graph of a subdivision of K_4.

attachment of $P \setminus a_i$ over K is included in a piece X_i of K, and a_i has a neighbor y_i in $V(K) \setminus X_i$. So we have $\{y_1, y_2\} \subseteq X_3$, $\{y_1, y_3\} \subseteq X_2$, $\{y_2, y_3\} \subseteq X_3$. This is possible in $L(H)$ only if each of X_1, X_2, X_3 is a triangle and $\{y_1, y_2, y_3\}$ is also a triangle. But then the attachment of P is $\{y_1, y_2, y_3\}$, so P is of type triangle, a contradiction. So P is a path. If P has length zero, then the claim holds since, by the assumption, P is not of type branch or triangle. So, we may assume that P has length at least one. Let P have ends p, p'. Suppose that the claim fails. Then by the minimality of P, $N_{L(H)}(p) \subset A$ and $N_{L(H)}(p') \subset B$, where A, B are distinct pieces of $L(H)$. Also some interior vertex of P must have a neighbor in $L(H)$. By the minimality of P, the attachment of the interior of P over $L(H)$ is included in $A \cap B$. Since two distinct flat branches of $L(H)$ are disjoint, we may assume that $A = T_d$ and either $B = P_{ad}$ or P_{ad} has length zero and $B = T_a$. In either case, $A \cap B = \{v_{da}\}$. Note that p has at most two neighbors in T_d, because G has no K_4, and that p must have at least one neighbor in $\{v_{db}, v_{dc}\}$, for otherwise the attachment of P is included in B and P is of type branch or triangle. Note that p' has neighbors in $B \setminus v_{da}$, for otherwise P is of type triangle. If $pv_{db}, pv_{dc} \in E(G)$, then since some interior vertex of P is adjacent to v_{da}, P contains a subpath that closes the diamond $T_d \cup \{p\}$, a contradiction. So, up to symmetry, we assume $pv_{db} \in E(G)$ and $pv_{dc} \notin E(G)$.

We observe that $P \cup P_{ac} \cup B$ contains an induced path Q from p to v_{ca}, and no vertex of Q has neighbors in $V(P_{cd}) \cup V(P_{bd}) \cup V(P_{bc})$. If possible, choose Q so that it does not contain v_{ab}. Now Q, P_{cd}, P_{bd}, P_{bc}, form a triangular ISK4 (whose triangle is T_c and fourth corner is v_{db}) except if Q goes through v_{ab} and P_{ab} has length zero (so $v_{ab} = v_{ba}$). In the latter situation, we must have $N_B(p') = \{v_{ab}\}$ by the choice of Q, so $B = T_a$ and P_{ad} has length zero. If P_{bd} has length at least 1, then $v_{db}-P-p'-v_{ba}$, $v_{db}-P_{bd}-v_{bd}$ and $v_{db}-v_{dc}-P_{cd}-v_{cd}-v_{cb}-P_{bc}-v_{bc}$ form a triangular ISK4. So P_{bd} has length zero. But then $\{v_{da}, v_{db}, v_{ab}\}$ is a triangle and is the attachment of P over $L(H)$, so P is of type triangle with respect to $L(H)$, a contradiction. This proves claim (1).

One of P_{ab}, P_{ac}, P_{ad}, P_{bc}, P_{bd}, P_{cd} has length at least 1. (2)

Suppose that P_{ab}, P_{ac}, P_{ad}, P_{bc}, P_{bd}, P_{cd} all have length zero. Then $L(H)$ is the octahedron ($K_{2,2,2}$). Note that $L(H)$ has no flat branch. For convenience, we rename its vertices x, x', y, y', z, z' so that $xx', yy', zz' \notin E(L(H))$ and all other pairs of distinct vertices are edges. If P has at most one neighbor in every pair $\{x, x'\}$, $\{y, y'\}$, $\{z, z'\}$, then $N_{L(H)}(P)$ is a subset of a triangle, a contradiction. So, we may assume up to symmetry that p is adjacent to x and p' to x'. Let S be the square of $L(H)$ induced by y, y', z, z'. Vertex p cannot be adjacent to the two vertices of an edge of S, for that would yield (with x) a K_4 in G. So we may assume $py, py' \notin E(G)$. If pz, pz' are both in $E(G)$, then p itself is a vertex not of type branch or triangle, so $p = p'$ by the minimality of P, and since $S' = \{x, x', z, z'\}$ is an even square of $L(H)$ and $N_{L(H)}(P) = S'$, C is of type square. Hence we may assume up to symmetry that $pz' \notin E(G)$, so p has at most one neighbor in S. Similarly, p' has at most one neighbor in S. If any edge uv of S has no neighbor of p or p', then P closes the diamond induced by $\{u, v, x, x'\}$, a contradiction. So every edge of S has a neighbor of p or p', which implies $pz \in E(G)$ and $p'z' \in E(G)$. Then P is a link of the square $\{x, z, x', z'\}$ of $L(H)$, so C is of type square. This proves claim (2).

By (2) we may assume up to symmetry that P_{ab} has length at least one. So the vertices of $P_{ad}, P_{bd}, P_{ab}, P_{ac}, P_{bc}$ induce a prism K in G, whose triangles are T_a, T_b and whose flat branches are $P_{ab}, P_{ac} \cup P_{bc}$ and $P_{ad} \cup P_{bd}$. We apply Lemma 5.1 to K and P, which leads to the following four cases.

Case 1: *P is of type branch with respect to K*. Suppose first that $N_K(P) \subseteq V(P_{ab})$. By (1), P has neighbors in P_{cd}, and we may assume that p has a neighbor in P_{ab}, p' has a neighbor in P_{cd}, and no proper subpath of P has this property. Let v^L (resp. v^R) be the neighbor of p closest to v_{ab} (resp. to v_{ba}) along P_{ab}. Up to the symmetry between P_{ab} and P_{cd} we may assume $v^L v^R \notin E(G)$, for otherwise C is of type augmenting with respect to $L(H)$ and the lemma holds. Let w^R the neighbor of p' closest to v_{cd} along P_{cd}. If $v^L = v^R$, then $v^L - P_{ab} - v_{ab} - v_{ac} - P_{ac} - v_{ca}$, $v^L - P_{ab} - v_{ba} - v_{bc} - P_{bc} - v_{cb}$, $v^L - p - P - p' - w^R - P_{cd} - v_{cd}$ form a triangular ISK4, a contradiction. If $v^L \neq v^R$, then $p - v^L - P_{ab} - v_{ab} - v_{ac} - P_{ac} - v_{ca}$, $p - v^R - P_{ab} - v_{ba} - v_{bc} - P_{bc} - v_{cb}$, $p - P - p' - w^R - P_{cd} - v_{cd}$ form a triangular ISK4, a contradiction.

Now we may assume up to symmetry that $N_K(P) \subseteq V(P_{ad}) \cup V(P_{bd})$. Suppose that P has a neighbor in each of P_{ad}, P_{bd} and P_{cd}. Let v^a, v^b, v^c be the neighbors of P closest to v_{da}, v_{db} and v_{dc} respectively along these paths. Then $V(P) \cup V(v^a - P_{ad} - v_{da}) \cup V(v^b - P_{bd} - v_{db}) \cup V(v^c - P_{cd} - v_{dc})$ induces a triangular ISK4 (whose corners are v_{da}, v_{db}, v_{dc} and one of p, p'), a contradiction. So, P has no neighbor in at least one of P_{ad}, P_{bd}, P_{cd}.

If P has no neighbor in P_{bd}, then by (1), we may assume that p has a neighbor in P_{ad}, p' has a neighbor in P_{cd}, and no proper subpath of P has such a property. Let v^R be the neighbor of p closest to v_{ad} along P_{ad}. Suppose that p' has a unique neighbor w in P_{cd}. If $v^R = v_{da}$, then $w \neq v_{dc}$ by (1) and $w - P_{cd} - v_{dc}$, $w - p' - P - p - v_{da}$, $w - P_{cd} - v_{cd} - v_{cb} - P_{bc} - v_{bc} - v_{bd} - P_{bd} - v_{db}$ form a triangular ISK4. If $v^R \neq v_{da}$, then $w - p' - P - p - v^R - P_{ad} - v_{ad}$, $w - P_{cd} - v_{cd} - v_{ca} - P_{ac} - v_{ac}$, $w - P_{cd} - v_{dc} - v_{db} - P_{bd} - v_{bd} - v_{ba} - P_{ab} - v_{ab}$ form a triangular ISK4, a contradiction. So p' has at least two neighbors on P_{cd}, and in particular P_{cd} has length at least one. So P_{cd}, P_{ad}, P_{ac}, P_{bd}, P_{cb} form a prism K'. Let us apply Lemma 5.1 to K' and P. Since P has at least two neighbors in the flat branch P_{cd} of K' and at least one neighbor in P_{ad}, P is not of type branch or triangle with respect to K'. Also P is not of type square with respect to K', because $N_{K'}(P)$ is included in $V(P_{ad}) \cup V(P_{cd})$ and cannot induce an even square of K'. So P is of type augmenting with respect to K'. So $N_{K'}(p)$ is an edge of P_{ad} (and this implies that P_{ad} is a flat branch of $L(H)$), $N_{K'}(p')$ is an edge of P_{cd}, hence P is of type augmenting with respect to $L(H)$.

If P has no neighbor in P_{ad}, the situation is similar to the preceding paragraph (by symmetry).

Now suppose that P has no neighbor in P_{cd}. By (1), we may assume that p has a neighbor in P_{ad}, p' has a neighbor in P_{bd}, and no proper subpath of P has this property. Let v^R (resp. v^L) be the neighbor of p closest to v_{ad} (resp. to v_{da}) along P_{ad}. Let w^R (resp. w^L) be the neighbor of p' closest to v_{db} (resp. to v_{bd}) along P_{bd}. If both $v^L v^R$, $w^L w^R$ are edges, then C is of type augmenting with respect to $L(H)$ and the lemma holds. So let us assume, up to the symmetry between P_{ad} and P_{bd}, that $v^L v^R$ is not an edge. If $v^L \neq v^R$, then $p - v^L - P_{ad} - v_{da}$, $p - v^R - P_{ad} - v_{ad} - v_{ac} - P_{ac} - v_{ca} - v_{cd} - P_{cd} - v_{dc}$ and $p - P - p' - w - P_{bd} - v_{db}$ form a triangular ISK4, a contradiction. So $v^L = v^R$. If $w^R w^L$ is an edge, then $P_{ab} \cup P_{ad} \cup P_{bd} \cup P$ is a triangular ISK4, a contradiction. So $w^R w^L$ is not an edge, and, as above, this implies that $w^R = w^L$. We cannot have $\{v^L, w^L\} = \{v_{da}, v_{db}\}$, for otherwise $N_{L(H)}(P) \subseteq T_d$, contradictory to (1). So we may assume that $v^L \neq v_{da}$. Then $v^L - P_{ad} - v_{da}$, $v^L - P_{ad} - v_{ad} - v_{ac} - P_{ac} - v_{ca} - v_{cd} - P_{cd} - v_{dc}$ and $v^L - p - P - p' - w - P_{bd} - v_{db}$ form a triangular ISK4, a contradiction.

Case 2: *P is of type triangle with respect to K*. We assume up to symmetry that $N_K(P) \subseteq T_a$. By (1) and up to symmetry, we may assume that p has a neighbor in T_a, p' has a neighbor in P_{cd}, and no interior vertex of P has a neighbor in $L(H)$. We may assume that we are not in Case 1, so p has at least two neighbors in T_a; and p has only two neighbors in T_a, for otherwise there is a K_4 in G. Suppose that $pv_{ac}, pv_{ad} \in E(G)$ and $pv_{ab} \notin E(G)$. If p' has only one neighbor in P_{cd}, then $P_{ac} \cup P_{ad} \cup P_{cd} \cup P$ is a triangular ISK4, a contradiction. So p' has at least two neighbors in P_{cd}, which implies that P_{cd} has length at least one, and we may assume up to symmetry that the neighbor w of p' closest to d on P_{cd} is different from c. Then $v_{ac} - P_{ac} - v_{ca} - v_{cb} - P_{cb} - v_{bc}$, $v_{ac} - v_{ab} - P_{ab} - v_{ba}$ and $v_{ac} - p - P - p' - w - P_{dc} - v_{dc} - v_{db} - P_{db} - v_{bd}$ form a triangular ISK4 (whose corners are the vertices of T_b and v_{ac}), a contradiction. So $pv_{ab} \in E(G)$ and we may assume up to symmetry $pv_{ad} \notin E(G)$. Then $v_{ab} - p - P - p' - w - P_{cd} - v_{dc}$, $v_{ab} - v_{ad} - P_{ad} - v_{da}$ and $v_{ab} - P_{ab} - v_{ba} - v_{bd} - P_{bd} - v_{db}$ form a triangular ISK4, a contradiction.

Case 3: *P is of type augmenting with respect to K*. We may assume up to symmetry that $N_K(p)$ is an edge e in $P_{ad} \cup P_{bd}$ and $N_K(p')$ is an edge e' in either P_{ab} or in $Q = P_{ac} \cup P_{bc}$. If e' is in P_{ab}, let v^R be its vertex closest to v_{ba}. If e' is in Q let v^R be its vertex closest to v_{bc}. Let u^R be the other vertex of e'.

Suppose that $e = v_{da}v_{db}$. So $T_d \cup \{p\}$ induces a diamond. Then P has no neighbor in P_{cd}, for otherwise $P \cup P_{cd}$ would contain a path that closes the diamond $T_d \cup \{p\}$. If e' is in P_{ab}, then $v_{da} - p - P - p' - v^R - P_{ab} - v_{ba} - v_{bc} - P_{bc} - v_{cb}$, $v_{da} - v_{dc} - P_{cd} - v_{cd}$ and $v_{da} - P_{ad} - v_{ad} - v_{ac} - P_{ac} - v_{ca}$ form a triangular ISK4, a contradiction (note that this holds even when P and every P_{xy} except P_{ab} has length zero). Hence e' is in Q. If v^R is in P_{ac}, then P_{ac} has length at least one and $v^R \neq v_{ac}$, so $p - P - p' - v^R - P_{ac} - v_{ca} - v_{cd} - P_{cd} - v_{dc}$ closes the diamond $\{p, v_{da}, v_{db}, v_{dc}\}$. So v^R is not in P_{ac}; and, by symmetry, u^R is not in P_{bc}, so we must have $e' = v_{ca}v_{cb}$. If one of P_{ac}, P_{ad} has length at least one, then $p - P - p' - v_{ca} - v_{cd} - P_{cd} - v_{dc}$ closes the diamond $T_d \cup \{p\}$, a contradiction. So suppose that both P_{ad}, P_{ac} have length zero, and similarly both P_{bd}, P_{bc} have length zero. Then P is a link of the even square induced by the four vertices $v_{da} = v_{ad}$, $v_{ac} = v_{ca}$, $v_{cb} = v_{bc}$ and $v_{bd} = v_{db}$ of $L(H)$, hence, C is of type square with respect to $L(H)$.

Now we may assume that $e \neq v_{da}v_{db}$, and, similarly, that $e' \neq v_{ca}v_{cb}$. We may assume up to symmetry that e is in P_{ad}. We know that e' is in either P_{ab}, P_{ac} or P_{bc}, and that no vertex of P has a neighbor in P_{bd}. Let $e = u^L v^L$ so that the vertices v_{ad}, u^L, v^L, v_{da} lie in this order on P_{ad}. Suppose that some vertex of P_{cd} has a neighbor in P and call w such a vertex closest to v_{dc}. Note that w must be adjacent to $x \in \{p, p'\}$, so x itself is a connected induced subgraph of G, not of type branch or triangle with respect to $L(H)$. This and the minimality of P imply $x = p = p'$. Put $Q_1 = p - v^L - P_{ad} - v_{da}$, $Q_2 = p - w - P_{cd} - v_{dc}$. If e' is in P_{ab}, put $Q_3 = p - v^R - P_{ab} - v_{ba} - v_{bd} - P_{bd} - v_{db}$. If e' is in Q, put $Q_3 = p - v^R - Q - v_{bc} - v_{bd} - P_{bd} - v_{db}$. Now, if w has no neighbor in Q_3, then Q_1, Q_2, Q_3 form a triangular ISK4, a contradiction. So w has a neighbor in Q_3, which means that $w = v_{cd}$ and $v^R \in P_{ac}$. Then $p - v_{cd} - v_{cb} - P_{bc} - v_{bc} - v_{ba} - P_{ab} - v_{ab}$, $p - u^R - P_{ac} - v_{ac}$ and $p - u^L - P_{ad} - v_{ad}$ form a triangular ISK4, a contradiction. So no vertex of P has a neighbor in P_{cd}. It follows that C is of type augmenting with respect to $L(H)$.

Case 4: *P is of type square with respect to K*. So P is a link of an even square S of K and has no neighbor in $K \setminus S$. We may assume up to symmetry that S contains P_{ad} and P_{bd}, so these two paths have length zero, that is, $v_{ad} = v_{da}$ and $v_{bd} = v_{db}$. If any vertex of P has a neighbor w in P_{cd}, then $p = p'$ by the minimality of P. So p is adjacent to both v_{ad}, v_{bd}. Then $T_d \cup \{p\}$ induces either a K_4 (if $w = v_{dc}$) or a diamond that is closed by a subpath of $P_{cd} \cup \{p\}$, a contradiction. Hence, no vertex of P has a neighbor in P_{cd}. Suppose that $P_{ab} \subset S$. Note that S is an even square of K, but a non-even square of $L(H)$. Then $V(P) \cup \{v_{da}, v_{ba}\}$ contains an induced path Q from v_{da} to v_{ba} such that no interior vertex of Q has a neighbor in $(L(H) \setminus S) \cup \{v_{da}, v_{ba}\}$. Then $v_{da} - Q - v_{ba} - v_{bc} - P_{bc} - v_{cb}$, $v_{da} - v_{ac} - P_{ac} - v_{ca}$ and $v_{da} - v_{dc} - P_{cd} - v_{cd}$ form a triangular subdivision of K_4, a contradiction. So $P_{ab} \not\subset S$. So S has vertices $v_{ad} = v_{da}$, $v_{db} = v_{bd}$, $v_{bc} = v_{cb}$ and $v_{ac} = v_{ca}$, and S is an even square of $L(H)$. Thus C is of type square with respect to $L(H)$ because of S and P. $\square$

Let us say that a graph is *substantial* if it is cyclically 3-connected and not a square theta or a square subdivision of K_4. The following lemma shows that type square arises only with line graphs of non-substantial graphs.

Lemma 5.3. *Let G be a graph that contains no triangular ISK4. Let H be a substantial graph such that $L(H)$ is an induced subgraph of G. Let C be a component of $G \setminus L(H)$. Then C is either of type branch, triangle or augmenting with respect to $L(H)$.*

Proof. We suppose that C is minimal with respect to the property of being not of type branch or triangle with respect to $L(H)$. Note that every vertex in H has degree at most three since $L(H)$ contains no K_4. We may assume that there are two non-incident edges e_1, e_2 of H that are members of the attachment of C over $L(H)$ and are not in the same branch of H, for otherwise all edges of the attachment of C over $L(H)$ are either in the same branch of H, and so C is of type branch or triangle, or are pairwise incident, and so C is of type triangle. Since H is 2-connected, there exists a cycle Z of H that goes through e_1, e_2, and we put $e_1 = ab$, $e_2 = cd$ so that a, b, c, d appear in this order along Z. Note that a, b, c, d are pairwise distinct. Let P be the subpath of Z from a to d that does not contain b, c, and let Q be the subpath of Z from b to c that does not contain a, d. By Lemma 4.7 there is a path R with an end-vertex in P, an end-vertex in Q and no interior vertex in C, and R is not from a to b or from c to d.

Suppose that $V(H) = V(P) \cup V(Q) \cup V(R)$. Then R must have length at least two, and H must be a theta since it is substantial, so $L(H)$ is a prism. By the preceding paragraph, the attachment of C over $L(H)$ contains at least two vertices in distinct flat branches $L(H)$, and not in a triangle of that prism. So, by Lemma 5.1, C is of type augmenting or square with respect to the prism. Moreover, type square is impossible because H is substantial; so C is of type augmenting, and the lemma holds.

Now we may assume that H has more edges than those in P, Q, R. By Lemma 4.5, H is a subdivision of a 3-connected graph. Pick any $r \in V(P) \cap V(R), r' \in V(Q) \cap V(R)$ and put $P_1 = rPabQr'$, $P_2 = rPdcQr'$, and $P_3 = R = r\text{-}\cdots\text{-}r'$. By Lemma 4.3, for some distinct $i, j \in \{1, 2, 3\}$ there exists a path S of H with an end in the interior of P_i, an end in the interior of P_j and such that the interior of S is disjoint from P_1, P_2, P_3. Since $H' = P_1 \cup P_2 \cup P_3 \cup S$ is a subdivision of K_4, we may apply Lemma 5.2 to C and $L(H')$. Note that C cannot be of type branch or triangle with respect to $L(H')$ because of the edges ab and cd. Hence C is of type square or augmenting with respect to $L(H')$, and, by the minimality of C, it is either a link of an even square of $L(H')$ or a connection between two branches of $L(H')$. We claim that the interior vertices of C have no neighbor in $L(H')$. For suppose on the contrary that there is a vertex w of $L(H')$ with a neighbor in the interior of C. If C is of type augmenting with respect to $L(H')$, then, by the minimality of C, w must lie in the intersection of two edges of distinct flat branches of $L(H')$, a contradiction since flat branches of $L(H')$ do not intersect. If C is of type square with respect to $L(H')$, then, by the minimality of C, w must lie in the intersection of two triangles of $L(H')$ that share a common vertex not in the square. But then C contains a path that closes a diamond, a contradiction. So the claim is proved. Now, we distinguish between two cases.

Case 1: *H contains a square subdivision of K_4 as a subgraph, and C is of type square with respect to its line graph.* We may assume up to a relabeling that C is of type square with respect to $L(H')$ and that $abcd$ is a square of H, $P_1 = ab$, $P_2 = dc$, R is from a to c and S is from b to d. Every vertex of H has degree at most three since $L(H)$ contains no K_4. Since H is substantial, it is not a square subdivision of K_4, so there is a vertex in $H \setminus H'$. Since H is connected and $H \neq H'$, there exists a neighbor in $V(H) \setminus V(H')$ of a vertex $e \in V(H')$, and $e \notin \{a, b, c, d\}$ because a, b, c, d have already three neighbors. So e is in the interior of one of S, R (say S). Since H is 2-connected, $\{e\}$ is not a cutset of H and there exists a path T from e to some other vertex in H'. If every such path has its two ends in S, then we put $A = V(P) \cup V(Q) \cup V(R)$, $B = (V(H) \setminus A) \cup \{b, d\}$ and we observe that (A, B) is a cyclic 2-separation of H, a contradiction. So we may assume that the other end e' of T is in the interior of R. Now let H'' be the subgraph of H obtained from $P \cup Q \cup R \cup S \cup T$ by deleting the edges of the subpath $d\text{-}S\text{-}e$. We observe that H'' is a subdivision of K_4 (whose corners are a, b, c, e'). We now apply Lemma 5.2 to C and $L(H'')$. In fact C cannot be of type branch, triangle or augmenting with respect to $L(H'')$, because C has a neighbor in three distinct branches of $L(H'')$; and C cannot be of type square because the edges ab, bc, cd, da of H do not form an even square in $L(H'')$ since d has degree two in H''. This is a contradiction.

Case 2: *We are not in Case* 1. So C is of type augmenting with respect to $L(H')$. We may assume, up to a relabeling, that the attachment of C over $L(H')$ consists of two pairs $\{e_1, e'_1\}$, $\{e_2, e'_2\}$ of adjacent vertices, where (in H) e_1, e'_1 are two incident edges of P_1 and e_2, e'_2 are two incident edges of P_2. Suppose that there is a vertex x different from e_1, e_2, e'_1, e'_2 in the attachment of C over $L(H)$. By Lemma 4.8 applied (in H) to edge x and cycle $P_1 \cup P_2$, H contains a subdivision H'' of K_4 that contains $P_1 \cup P_2 \cup \{x\}$. By Lemma 5.2, C is either of type branch, triangle, augmenting or square with respect to $L(H'')$. In fact C is not of type square as we are not in Case 1; moreover, C cannot be of type triangle or augmenting as it has at least five neighbors in $L(H'')$. So it is of type branch. But the branch of H'' containing x is edgewise disjoint from $P_1 \cup P_2$, a contradiction. So x does not exist, and we conclude that C is of type augmenting with respect to $L(H)$. □

Lemma 5.4. *Let G be a graph that contains no triangular ISK4. Let H be a substantial graph such that $L(H)$ is an induced subgraph of G and is inclusion-wise maximum with respect to that property. Then either $G = L(H)$, or G has a clique-cutset of size at most three, or G has a proper 2-cutset.*

Proof. Suppose that $G \neq L(H)$. So there is a component C of $G \setminus L(H)$. Let us apply Lemma 5.3 to C and $L(H)$. Suppose that C is of type augmenting. So there is a path P like in the definition of the type augmenting. In H the attachment of C consists of four edge ab, be, cd, df, where b, d have degree two in H. Let us consider the graph H' obtained from H by adding between b and d a path R whose length is one plus the length of P. Then H' is substantial. Indeed, it is cyclically 3-connected by Lemma 4.6, and it is not a square theta or a square subdivision of K_4 since H is not a square theta. Moreover, $L(H')$ is an induced subgraph of G, where P corresponds to the path R. This is a contradiction to the maximality of $L(H)$. So C is of type branch or triangle. If C is of type branch, then the ends of the branch that contain the attachment of C form a cutset of G of size at most two. So either this is a proper 2-cutset or it contains a clique-cutset. If C is of type triangle, then the triangle that contains the attachment of C is a clique cutset of G. □

6. Rich squares

Lemma 6.1. *Let G be an ISK4-free graph that does not contain the line graph of a substantial graph. Let K be a rich square that is an induced subgraph of G and is maximal with respect to this property. Then either $G = K$ or G has a clique-cutset of size at most three or G has a proper 2-cutset.*

Proof. Let S be a central square of K, with vertices u_1, u_2, u_3, u_4 and edges u_1u_2, u_2u_3, u_3u_4, u_4u_1. Recall that every component of $K \setminus S$ is a link of S. A link with ends p, p' is said to be *short* if $p = p'$, and *long* if $p \neq p'$. Note that long links are flat branches of K. If two long links $B_1 = p_1 - \cdots - p'_1$ and $B_2 = p_2 - \cdots - p'_2$ are such that $N_S(p_1) = N_S(p_2)$ and $N_S(p'_1) = N_S(p'_2)$, then we say that B_1, B_2 are *parallel*, otherwise they are *orthogonal*.

Suppose that $G \neq K$. Let C be a component of $G \setminus K$. We may assume that the attachment of C over K is not empty, for otherwise any vertex of K would be a cutset of G. This leads to the following three cases.

Case 1: *$N_K(C)$ contains vertices of a long link of S.* Let $B_1 = p_1 - \cdots - p'_1$ be such a link. We may assume up to symmetry that $N_S(p_1) = \{u_1, u_2\}$ and $N_S(p'_1) = \{u_3, u_4\}$. If C has no neighbor in $K \setminus B_1$, then $\{p_1, p'_1\}$ is a proper 2-cutset of G and the lemma holds. So C has a neighbor in $K \setminus B_1$.

Suppose that C has no neighbor in $K \setminus (S \cup B_1)$. Hence C has a neighbor in S. By Lemma 5.1 applied to the prism $S \cup B_1$ and C, we deduce that C is of type augmenting, triangle or square. If C is of type triangle, then there is a triangle cutset in G, and the lemma holds. If C is of type augmenting, let P be a shortest path of C that sees B_1 and S. Let B be a component of $K \setminus (S \cup B_1)$. Then $G[B_1 \cup B \cup P \cup \{u_1, u_3\}]$ is an ISK4, a contradiction. If C is of type square and not augmenting, then it must be that B_1 has length one and, up to symmetry, C contains a path P with one end adjacent to u_1, p_1 and the other end to u_4, p'_1. Let B be any component of $K \setminus (S \cup B_1)$. Then $G[B_1 \cup B \cup P \cup \{u_1, u_3\}]$ is an ISK4, a contradiction.

Therefore $N_K(C)$ contains vertices of a component B_2 of $K \setminus (S \cup B_1)$. Suppose that B_2 is either short or orthogonal to B_1. Then $K' = G[S \cup B_1 \cup B_2]$ is the line graph of a subdivision of K_4, and we can apply Lemma 5.2 to K' and C. Clearly, C is not of type branch or triangle with respect to K', and it is also not of type square because $B_1 \cup B_2$ contains no even square of K'. So C is of type augmenting, with a path P as in the definition of type augmenting. This implies that B_2 is a flat branch of K, and so it is a long link of S. Then $G[S \cup B_1 \cup B_2 \cup P]$ is the line graph of a substantial graph, a contradiction.

So B_2 is a long link parallel to B_1. Let $B_2 = p_2 - \cdots - p'_2$ with $N_S(p_2) = N_S(p_1)$ and $N_S(p'_2) = N_S(p'_1)$. Let $P = p_3 - \cdots - p'_3$ be a shortest path of C such that p_3 has neighbors in B_1 and p'_3 has neighbors in B_2. If no vertex of P has a neighbor in $\{u_1, u_2\}$, then $B_1 \cup B_2 \cup P$ contains a path that closes the diamond $\{p_1, p_2, u_1, u_2\}$, a contradiction. So some vertex of P has a neighbor in $\{u_1, u_2\}$ and similarly some vertex of P has a neighbor in $\{u_3, u_4\}$. By Lemma 5.1 applied to the prism $K' = G[S \cup B_1]$ and P, we deduce that P is of type augmenting with respect to K'. Let P' be a shortest subpath of P that contains neighbors of B_1 and S. One end of P' must be p_3, and $N_{B_1}(p_3) = \{q_1, q'_1\}$, where $q_1 q'_1$ is an edge of B_1 and p_1, q_1, q'_1, p'_1 appear in this order along B_1. We denote the other end of P' by p''_3, and we can assume up to symmetry that $N_K(p''_3) = \{u_2, u_3\}$. If $p'_3 \neq p''_3$, then $B_1 \cup B_2 \cup P' \cup \{u_1, u_3\}$ is a triangular ISK4, a contradiction. So $p'_3 = p''_3$. By Lemma 5.1 applied to $K'' = G[S \cup B_2]$ and p'_3, we deduce that p'_3 is of type augmenting with respect to K'', so $N_{B_2}(p'_3) = \{q_2, q'_2\}$, where q_2, q'_2 is an edge of B_2 and p_2, q_2, q'_2, p'_2 appear in this order along B_2. Then the paths $p'_3 - u_2$, $p'_3 - q_2 - B_2 - p_2$ and $p'_3 - P - p_3 - q'_1 - B_1 - p'_1 - u_4 - u_1$ form a triangular ISK4, a contradiction.

Case 2: *$N_K(C)$ does not contain any vertex of a long link of S, and contains vertices of a short link.* So there exists a vertex b_1 adjacent to all of S and to C. Suppose that C is also adjacent to a component of $K \setminus (S \cup b_1)$, that is, to a vertex $b_2 \neq b_1$ adjacent to all of S. Then $K' = G[S \cup \{b_1, b_2\}]$ is the line graph of K_4, so we can apply Lemma 5.2 to K' and C. We deduce that C is of type square with respect to K', with a linking path P. Since $K \cup P$ cannot be a rich square (which would contradict the maximality of K), we may assume up to symmetry that $N_{K'}(P) = \{u_1, u_3, b_1, b_2\}$. Since K is a maximal rich square, and $S \cup P \cup \{b_1, b_2\}$ is a rich square, $K \setminus (S \cup \{b_1, b_2\})$ must have a component B_3 (a link of S). Then $B_3 \cup P \cup \{u_2, u_4, b_1, b_2\}$ is a (non-triangular) ISK4, a contradiction. So no vertex of C has a neighbor in $K \setminus (S \cup \{b_1\})$. Let B_2 be any component of $K \setminus (S \cup \{b_1\})$. Note that $K' = S \cup B_2 \cup \{b_1\}$ is the line graph of a subdivision of K_4. By Lemma 5.2 applied to K' and C, we deduce that C is of type triangle with respect to K'. Since no vertex of C has a neighbor in a component of $K \setminus S$ (except b_1), we see that G has a triangle cutset.

Case 3: *$N_K(C)$ is included in S.* Let K' be a subgraph of K that contains S and is either the line graph of an ISK4 or a prism (take S plus a long link if possible or two short links otherwise). We can apply Lemma 5.1 or 5.2 to K' and C. If C is of type augmenting or square with respect to K' with path P, then $K \cup P$ is a rich square, a contradiction to the maximality of K. If C is of type branch or triangle, then G has a proper 2-cutset or a clique cutset. □

7. Prisms

Lemma 7.1. *Let G be an ISK4-free graph that does not contain the line graph of a substantial graph or a rich square as an induced subgraph. Let K be a prism that is an induced subgraph of G. Then either $G = K$ or G has a clique-cutset of size at most three or G has a proper 2-cutset.*

Proof. Suppose that $G \neq K$, and let C be any component of $G \setminus K$. Apply Lemma 5.1 to K and C. If C is of type branch, then the ends of the branch of K that contains the attachment of C over K is a cutset of size at most two, and either it is proper or it contains a clique cutset. If C is of type triangle, then G has a triangle cutset. If C is of type augmenting, with augmenting path P, then $P \cup K$ is either the line graph of a non-square subdivision of K_4, or a rich square, in both cases a contradiction. If C is of type square, with a linking path P, then $K \cup P$ is a rich square, a contradiction. □

Lemma 7.2. *Let G be an ISK4-free graph that contains a prism. Then either G is the line graph of a graph with maximum degree three, or G is a rich square, or G has a clique-cutset of size at most three or G has a proper 2-cutset.*

Proof. Since G contains a prism, it contains as an induced subgraph the line graph $L(H)$ of a cyclically 3-connected graph. By Lemma 4.5, H is either a theta or a subdivision of a 3-connected graph. In the latter case, if H is substantial, then the result holds by Lemma 5.4. Else, we may assume that G does not contain the line graph of a substantial graph and $L(H)$ is a rich square made of a square with two links, and then the result holds by Lemma 6.1. Hence, in the first case, we may assume that G contains no rich square and no line graph of a substantial graph. Then the result holds by Lemma 7.1. □

8. Wheels and double star cutset

A *paw* is a graph with four vertices a, b, c, d and four edges ab, ac, ad, bc.

Lemma 8.1. *Let G be a graph that does not contain a triangular ISK4 or a prism. If G contains a paw, then G has a star-cutset.*

Proof. Suppose that G does not have a star-cutset. Let X be a paw in G, with vertices a, b, c, d and edges ab, ac, ad, bc. Since G does not admit a star-cutset, the set $\{a\} \cup N(a) \setminus \{b, d\}$ is not a cutset of G, so there exists a chordless path P_1 with endvertices b, d such that the interior vertices of P_1 are distinct from a and not adjacent to a. Likewise, the set $\{a\} \cup N(a) \setminus \{c, d\}$ is not a cutset of G, so there exists a chordless path P_2 with endvertices c, d such that the interior vertices of P_2 are distinct from a and not adjacent to a. The definition of P_1, P_2 implies that there exists a path Q with endvertices b, c such that $V(Q) \subseteq V(P_1) \cup V(P_2)$, Q is not equal to the edge bc, and bc is the only chord of Q. So $V(Q)$ induces a cycle. If d is in Q, then $V(Q) \cup \{a\}$ induces a triangular subdivision of K_4, a contradiction. If d is not in Q, then the definition of P_1, P_2 implies that there exists a path R whose endvertices are d and a vertex q of Q and $V(R) \subseteq V(P_1) \cup V(P_2)$. We choose a minimal such path R. Let d' be the neighbor of q in R. The minimality of R implies that R is chordless, $(V(R) \setminus \{q\}) \cap V(Q) = \emptyset$, and d' is the only vertex of R with a neighbor in Q. If d' has only one neighbor in Q, then $V(Q) \cup V(R) \cup \{a\}$ induces a triangular subdivision of K_4 (whose corners are a, b, c, q), a contradiction. If d' has exactly two neighbors in Q and these are adjacent, then $V(Q) \cup V(R) \cup \{a\}$ induces a prism, a contradiction. If d' has at least two non-adjacent neighbors in Q, then $V(Q) \cup V(R) \cup \{a\}$ contains an induced triangular subdivision of K_4 (whose corners are a, b, c, d'), a contradiction. □

Lemma 8.2. *Let G be an ISK4-free graph that does not contain a prism or an octahedron. If G contains a wheel (H, u) with $|V(H)| = 4$, then G has a star-cutset.*

Proof. Suppose that G does not have a star-cutset. Let the vertices of H be $u_1, \dots, u_4$ in this order. If u is adjacent to only three of them, then $V(H) \cup \{u\}$ induces a subdivision of K_4. So we may assume that u is adjacent to all vertices of H. Since G does not admit a star-cutset, the set $\{u\} \cup N(u) \setminus \{u_1, u_3\}$ is not a cutset of G, so there exists a chordless path P with endvertices u_1, u_3 such that the interior vertices of P are distinct from u and not adjacent to u. Let $P = u_1 - v - \cdots - u_3$. Vertex v must be adjacent to u_2, for otherwise $\{u, u_1, u_2, v\}$ induces a paw, which contradicts Lemma 8.1. Likewise, v is adjacent to u_4. If v is not adjacent to u_3, then $\{u_1, u_2, u_3, u_4, v\}$ induces a subdivision of K_4, a contradiction. If v is adjacent to u_3, then $\{u, u_1, u_2, u_3, u_4, v\}$ induces an octahedron, a contradiction. □

Lemma 8.3. *Let G be an ISK4-free graph that does not contain a prism or an octahedron. If G contains a wheel, then G has a star-cutset or a double star cutset.*

Proof. Suppose that the lemma does not hold. Let (H,u) be a wheel in G such that $|V(H)|$ is minimum. Let $u_1,\dots,u_h$ be the neighbors of u in H in this order. If $h=3$, then $V(H)\cup\{u\}$ induces a subdivision of K_4, so we may assume that $h\geqslant 4$. By Lemma 8.2, we may assume that $|V(H)|\geqslant 5$. Let us call *fan* any pair (P,x) where P is a chordless path, x is a vertex not in P, and x has exactly four neighbors in P, including the two endvertices of P. Since $|V(H)|\geqslant 5$, we may assume up to symmetry that u_1 and u_4 are not adjacent. Letting Q be the subpath of H whose endvertices are u_1, u_4 and which contains u_2, u_3, we see that (Q,u) is a fan. Since G contains a fan, we may choose a fan (P,x) with a shortest P. Let x_1, x_2, x_3, x_4 be the four neighbors of x in P in this order, where x_1, x_4 are the endvertices of P. If x_1 is adjacent to x_2, then $\{x,x_1,x_2,x_4\}$ induces a paw, which contradicts Lemma 8.1. So x_1 is not adjacent to x_2, and similarly x_3 is not adjacent to x_4. Also x_2 is not adjacent to x_3, for otherwise $\{x,x_1,x_2,x_3\}$ induces a paw. For $i=1,2,3$, let P_i be the subpath of P whose endvertices are x_i and x_{i+1}. Let x_2', x_2'' be the two neighbors of x_2 in P, such that x_1, x_2', x_2, x_2'', x_3, x_4 lie in this order in P.

Since G does not admit a double star cutset, the set $\{x,x_2\}\cup N(x)\cup N(x_2)\setminus\{x_2',x_2''\}$ is not a cutset, and so there exists a path $Q=v_1-\cdots-v_k$ such that v_1 has a neighbor in the interior of P_1, v_k has a neighbor in the interior of P_2, and the vertices of Q are not adjacent to either x or x_2. We may choose a shortest such path Q, so Q is chordless and its interior vertices have no neighbor in $V(P_1)\cup V(P_2)$. If v_1 has at least four neighbors in P_1, then there is a subpath P_1' of P_1 such that (P_1',v_1) is a fan, which contradicts the minimality of (P,x). If v_1 has exactly three neighbors in P_1, then $V(P_1)\cup\{x,v_1\}$ induces a subdivision of K_4. So v_1 has at most two neighbors in P_1. Let $\{y_1,z_1\}$ be the set of neighbors of v_1 in P_1, such that x_1, y_1, z_1, x_2 lie in this order in P_1 (possibly $y_1=z_1$). Likewise, v_k has at most two neighbors in P_2. Let $\{y_2,z_2\}$ be the set of neighbors of v_k in P_2, such that x_2, y_2, z_2, x_3 lie in this order in P_2 (possibly $y_2=z_2$).

Suppose that $y_1\neq z_1$. Note that z_1 and z_2 are not adjacent, for that would be possible only if $z_1=x_2$ (and $z_2=x_2''$), which would contradict the definition of Q. Then $V(P_1)\cup V(z_2-P_2-x_3)\cup V(Q)\cup\{x\}$ induces a subdivision of K_4. So $y_1=z_1$. Likewise, $y_2=z_2$. But, then $V(P_1)\cup V(P_2)\cup V(Q)\cup\{x\}$ induces a subdivision of K_4. □

9. Decomposition theorems

Proof of Theorem 1.2. Let G be a graph that contains no ISK4 and no wheel. By Lemma 2.2, we may assume that G contains a $K_{3,3}$ or a prism. Note that G cannot be a thick complete tripartite graph, because such a graph contains a wheel $K_{1,2,2}$. So if G contains $K_{3,3}$, then we are done by Lemma 3.3. If G contains a prism, then we are done by Lemma 7.2. □

Proof of Theorem 1.1. By Theorem 1.2, we can assume that G is either a complete bipartite graph, a rich square or contains a wheel. Note that complete bipartite graphs and rich squares either are series-parallel or admit a star cutset or a double star cutset. So we may assume that G contains a wheel. If G contains a prism then we are done by Lemma 7.2. So, we assume that G contains no prism and in particular no line graph of a substantial graph. If G contains an octahedron, then we are done by Lemma 6.1, since an octahedron is a rich square. So we may assume that G contains no prism and no octahedron. Hence, we are done by Lemma 8.3. □

10. Chordless graphs

Most of the proof of Theorem 1.3 is implicitly given in [20] (proof of Theorem 2.2 and Claims 12 and 13 in the proof of Theorem 2.4). But the result is not stated explicitly in [20] and many details differ. For the sake of completeness and clarity we repeat the whole argument here.

Proof of Theorem 1.3. Let us assume that G has no 1-cutset and no proper 2-cutset. Note that G contains no K_4, since a K_4 is a cycle with two chords. Moreover:

We may assume that G is triangle-free. (1)

For suppose that G contains a triangle T. Then T is a maximal clique of G since G contains no K_4. We may assume that $G \neq T$ because a triangle is sparse, and that G is connected, for otherwise every vertex is a 1-cutset. So some vertex a of T has a neighbor x in $G \setminus T$. Since a is not a 1-cutset of G, there exists a shortest path P between x and a member b of $T \setminus a$. But, then $P \cup T$ is a cycle with at least one chord (namely ab), a contradiction. This proves claim (1).

We may assume that G has no clique cutset. (2)

Suppose that K is a clique cutset in G. Since G has no cutset of size one and there is no clique of size at least three by (1), K has exactly two elements a and b. Let X and Y be two components of $G \setminus \{a, b\}$. Since none of a and b is a 1-cutset of G, $X \cup \{a, b\}$ contains a path P_X with endvertices a and b; and a similar path P_Y exists in $Y \cup \{a, b\}$. But, then $P_X \cup P_Y$ forms a cycle with at least one chord (namely ab), a contradiction. This proves claim (2).

We can now prove that G is sparse. Suppose on the contrary that G has two adjacent vertices a, b both of degree at least three. Let c, e be two neighbors of a different from b, and let d, f be two neighbors of b different from a. Note that $\{c, e\}$ and $\{d, f\}$ are disjoint by (1). By (2), $\{a, b\}$ is not a cutset, so there is in $G \setminus \{a, b\}$ a path between $\{c, e\}$ and $\{d, f\}$ and consequently a path P that contains exactly one of c, e and one of d, f. Let the endvertices of P be e and f say. Thus $P \cup \{a, b\}$ forms a cycle C. Since $G \setminus \{a, b\}$ is connected, there exists a path $Q = c - \cdots - u$, where $u \in P \cup \{b, d\}$ and no interior vertex of Q is in $C \cup \{d\}$. If u is in $\{b, d\}$, then $Q \cup C$ forms a cycle with at least one chord, namely ab. So $u \in P$. Also since $G \setminus \{a, b\}$ is connected, there exists a path $R = d - \cdots - v$ where $v \in P \cup Q$ and no interior vertex of R is in $C \cup Q$.

If v is in $Q \setminus u$, then $bdRvQcaePfb$ is a cycle with at least one chord, namely ab, a contradiction. So v is in P. If e, v, u, f lie in this order on P and $v \neq u$, then $bdRvPeacQuPfb$ is a cycle with at least one chord, namely ab, a contradiction. So e, u, v, f lie in this order on P (possibly $u = v$). This restores the symmetry between c and e and between d and f. We suppose from here on that the paths P, Q, R are chosen subject to the minimality of the length of uPv.

Let $P_e = ePu \setminus u$, $Q_c = cQu \setminus u$, and $P_b = bPu \setminus u$. We show that $\{a, u\}$ is a 2-cutset of G. Suppose not; so there is a path $D = x - \cdots - y$ in $G \setminus \{a, u\}$ such that x lies in $P_e \cup Q_c$, y lies in $P_b \cup R$, and no interior vertex of D lies in $P \cup \{a\} \cup Q \cup R$. We may assume up to symmetry that x is in Q_c. If y is in the subpath $u - P - v$, then, considering path $Q' = c - Q - x - D - y$, we see that the three paths P, Q', R contradict the choice of P, Q, R because y and v are closer to each other than u and v along P. So y is not in uPv, and so, up to symmetry, y is in $R \setminus \{v\}$. But, then $xQaePfbRyDx$ is a cycle with at least one chord (namely ab), a contradiction. This proves that we can partition $G \setminus \{a, u\}$ into a set X that contains $P_e \cup Q_c$ and a set Y that contains $P_b \cup R$ such that there is no edge between X and Y, so $\{a, u\}$ is a 2-cutset. So, by (2), a and u are not adjacent. This implies that $\{a, u\}$ is proper. □

11. Forbidding wheels

Recall that a *branch* in a graph G is a path of G of length at least one whose ends are branch vertices and whose internal vertices are not (so they all have degree 2). A *subbranch* is a subpath of a branch. *Reducing* a subbranch of length at least two means replacing it by an edge.

Lemma 11.1. *Let G be a graph that contains no ISK4, no wheel and no $K_{3,3}$. Let B be a subbranch of length at least two in G, and let G' be the graph obtained from G by reducing B. Then G' contains no ISK4, no wheel and no $K_{3,3}$.*

Proof. Let e be the edge of G' that results from the reduction of B.

Suppose that G' contains an ISK4 H. Then H must contain e, for otherwise H is an ISK4 in G. Then replacing e by B in H yields an ISK4 in G, a contradiction.

Now suppose that G' contains a wheel $W = (H, x)$. Let $x_1, \ldots, x_h$ be the neighbors of x in H, with $h \geqslant 4$. Then W must contain e, for otherwise W is a wheel in G. Suppose that e is an edge in H. Then replacing e by B in H yields a wheel in G (with hub x and the same number of spokes), a contradiction. Now suppose that $e = xx_h$. So, in G, vertices x and x_h are the endvertices of B and

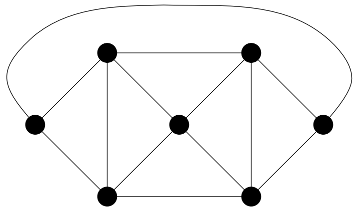

Fig. 3. Example of a rich square with chromatic number 4.

they are not adjacent. If $h \geqslant 5$, then (H, x) induces a wheel in G (with the same hub and with $h-1$ spokes). If $h = 4$, then $V(H) \cup \{x\}$ induces an ISK4 in G, a contradiction.

Finally, suppose that G' contains a $K_{3,3}$ H. Then H must contain e, for otherwise H is a $K_{3,3}$ in G. Let $e = xy$. Then x and y are the endvertices of B in G and they are not adjacent, so $V(H)$ induces an ISK4 in G, a contradiction. □

Note that the converse of Lemma 11.1 is not true. Let G be the graph with vertices $x_1, \dots, x_7$ such that $x_1, \dots, x_5$ induce a hole in this order, x_6 is adjacent to x_1, x_3, x_5, and x_7 is adjacent to x_2, x_4. Then x_2-x_7-x_4 is a branch whose reduction yields the prism on six vertices, a graph that contains no ISK4, no wheel and no $K_{3,3}$. But G contains an ISK4.

The following result is well known. See [17] for a simple greedy coloring algorithm.

Lemma 11.2. *(See Dirac [6].) Let G be a series-parallel graph. Then G is* 3*-colorable.*

Lemma 11.3. *Let G be a rich square that contains no wheel. Then G is* 3*-colorable.*

Proof. By the definition of a rich square, there is a square $S = \{u_1, u_2, u_3, u_4\}$ in G such that every component of $G \setminus S$ is a link of S. We make a 3-coloring of the vertices of G as follows. Assign color 1 to u_1, color 2 to u_2 and u_4, and color 3 to u_3. Let P be any component of $G \setminus S$. So P is a path $p_1 - \cdots - p_t$. Note that $t \geqslant 2$, for otherwise $S \cup \{p_1\}$ would induce a wheel (with four spokes). We may assume that $N_S(p_1) = \{u_1, u_2\}$ or $\{u_1, u_4\}$ and $N_S(p_t) = \{u_3, u_4\}$ or $\{u_2, u_3\}$. In either case, assign color 3 to p_1, color 1 to p_t, and, if $t \geqslant 3$, assign colors 2 and 3 alternately to $p_2, \dots, p_{t-1}$. Repeating this for every link produces a 3-coloring of the vertices of G. □

Note that Lemma 11.3 is tight, in the sense that a rich square may fail to be 3-colorable, as shown by the graph on Fig. 3. The following result also is tight since the graph represented on Fig. 3 is a line graph. The line graph of the Petersen graph is another example of a line graph of a cubic graph whose chromatic number is 4.

Lemma 11.4. *Let G be a graph that contains no ISK*4*, no wheel and such that G is a line graph. Then G is* 3*-colorable.*

Proof. Let G be the line graph of H. So we need only to prove that H is 3-edge-colorable. Since G contains no ISK4, in particular it contains no K_4, so H has maximum degree at most three. If C is a cycle of length at least four in H and e is a chord of C, then the edges of C plus edge e are vertices of G that induce a wheel in G (with hub e and four spokes), a contradiction. So every cycle of H is chordless. By Theorem 1.3, one of the following holds:

(a) The vertices of H of degree at least 3 are pairwise non-adjacent;
(b) H has a cutvertex;
(c) H has a proper 2-cutset.

We prove that our graph H is 3-edge-colorable in each case.

(a) Let $f = xy$ be any edge of H. Since H satisfies (a), we may assume that x has degree at most two and y has degree at most three in H. Thus, in G, vertex f has degree at most three. It follows from the theorem of Brooks [2] that G is 3-colorable (and so H is 3-edge-colorable).

(b) Let x be a cutvertex of H. Let $A_1, \dots, A_k$ be the components of $H \setminus x$, and let H_i be the subgraph of H induced by $V(A_i) \cup \{x\}$ for each $i = 1, \dots, k$. Since H is connected, x has a neighbor in each A_i, and we have $k \leqslant 3$ since H has maximum degree at most 3. By the induction hypothesis, each H_i admits a 3-edge-coloring. Up to renaming some color classes, we can combine these colorings so that the colors used at x are different; thus we obtain a 3-edge-coloring for H.

(c) Let $A_1, \dots, A_k$ be the components of $H \setminus \{a, b\}$. We may assume that we are not in case (b), so H is 2-connected and each of a and b has a neighbor in A_i for each $i = 1, \dots, k$. Since H has maximum degree at most 3, we may assume up to symmetry that a has only one neighbor a_1 in A_1. Suppose that b has two neighbors in A_1. Then $k = 2$ and b has only one neighbor b_2 in A_2, and then $\{a, b_2\}$ is also a proper 2-cutset of H. Thus in any case we may assume that both a, b have only one neighbor in A_1. Let b_1 be the neighbor of b in A_1. Let H_1 be the graph obtained from A_1 by adding a vertex x_1 adjacent to a_1 and b_1. Let H_2 be the graph obtained from $H \setminus A_1$ by adding a vertex x_2 adjacent to a and b. Suppose that H_1 contains a cycle C that has a chord. Then C must contain x_1. Since H is 2-connected there exists a chordless path P with endvertices a and b in $H \setminus A_1$. Then $(C \setminus x) \cup P$ is a cycle with a chord in H, a contradiction. So every cycle in H_1 is chordless. By a similar argument, every cycle in H_2 is chordless. Note that H_1 and H_2 have strictly fewer vertices than H because the cutset $\{a, b\}$ is proper. By the induction hypothesis, H_1 and H_2 have a 3-edge-coloring. In the coloring of H_1, edges x_1a_1 and x_1b_1 have different colors, and in the coloring of H_2 edges x_2a and x_2b have different colors too, so we can combine these colorings to make a 3-edge-coloring for H. □

Proof of Theorem 1.4. We prove the theorem by induction on the number of vertices of G. Suppose that G has a clique cutset K. So $V(G) \setminus K$ can be partitioned into two sets X, Y such that there is no edge between them. Since G contains no ISK4, we have $|K| \leqslant 3$. By the induction hypothesis, the two subgraphs of G induced by $X \cup K$ and $Y \cup K$ are 3-colorable. We can combine 3-colorings of these subgraphs so that they coincide on K, and consequently we obtain a 3-coloring of G. Now we may assume that G has no clique cutset. If G contains a $K_{3,3}$, then, by Lemma 3.3, G is a complete bipartite (recall that a thick complete tripartite graph contains a wheel), so it is 3-colorable. Now we may assume that G contains no $K_{3,3}$.

Suppose that G has a 2-cutset $\{a, b\}$. So $V(G) \setminus K$ can be partitioned into two sets X, Y such that there is no edge between them. Since G has no clique cutset, it is 2-connected, so there exists a chordless path P_Y with endvertices a and b and with interior vertices in Y. Let G'_X be the subgraph of G induced by $X \cup V(P_Y)$. Note that P_Y is a subbranch in G'_X. Let G''_X be obtained from G'_X be reducing P_Y (thus a and b are adjacent in G''_X). Define a graph G''_Y similarly. Since G'_X is an induced subgraph of G, it contains no ISK4, no wheel and no $K_{3,3}$. So, by Lemma 11.1, G''_X contains no ISK4, no wheel, and no $K_{3,3}$. The same holds for G''_Y. By the induction hypothesis, G''_X and G''_Y admit a 3-coloring. We can combine these two 3-colorings so that they coincide on $\{a, b\}$, and consequently we obtain a 3-coloring of G.

Now we may assume that G contains no 2-cutset. By Theorem 1.2, G is either a series-parallel graph, a rich square, a line graph, or a complete bipartite graph. Then the desired result follows from Lemmas 11.2, 11.3, 11.4, and the fact that bipartite graphs are 3-colorable. □

12. Algorithms for {ISK4, wheel}-free and chordless graphs

In this section, we give two algorithms for the class of {ISK4, wheel}-free graphs. The first one is a recognition algorithm for that class and the second is a coloring algorithm. Both are based on the results proved in the preceding sections.

12.1. Recognizing {ISK4, wheel}-free graphs

The recognition algorithm is based on Theorem 1.2: if a graph G is {ISK4, wheel}-free, then either G has a clique-cutset or a proper 2-cutset, or G is of one of the following four types: G is series-parallel, G is the line graph of a chordless graph with maximum degree at most three, G is a complete bipartite graph, or G is a rich square. We analyze each of these cases separately. Let us assume that G has n vertices and m edges.

Suppose that G has a clique cutset K. So $V(G) \setminus K$ can be partitioned into two sets X, Y such that there is no edge between them. Let G_X and G_Y be the subgraphs of G induced by $X \cup K$ and $Y \cup K$. We consider that G is decomposed into G_X and G_Y. These subgraphs can in turn be decomposed along clique cutsets. This is applied as long as possible, which yields a *clique cutset decomposition tree* $T_{cc}(G)$ of G. Building such a tree can be done in time $O(n+m)$, see [19,22]. If any clique cutset found during this step has size at least four, we stop with the obvious answer "G is not ISK4-free". Therefore let us assume that all the clique cutsets found by the algorithm have size at most three. Note that a graph that is either a subdivision of K_4 or a wheel has no clique cutset. It follows that G is {ISK4, wheel}-free if and only if all leaves of T_{cc} are {ISK4, wheel}-free. So our algorithm proceeds with examining the leaves of the tree.

Now suppose that G has no clique cutset and has a proper 2-cutset $\{a, b\}$. So $V(G) \setminus \{a, b\}$ can be partitioned into two sets X, Y such that there is no edge between them and each of $G[X \cup \{a, b\}]$ and $G[Y \cup \{a, b\}]$ is not an (a, b)-path. Let G_X be the subgraph of G induced by $X \cup \{a, b\}$ plus an artificial vertex adjacent to a and b, and define G_Y similarly. Thus G is decomposed into graphs G_X and G_Y. Note that G_X and G_Y have fewer vertices than G (because $\{a, b\}$ is proper), and that they have no clique cutset (because such a set would also be a clique cutset of G). These subgraphs can in turn be decomposed along proper 2-cutsets, and this is applied as long as possible, which yields a proper 2-cutset decomposition tree T_{2c} of G. Note that a graph that is either a subdivision of K_4 or a wheel has no proper 2-cutset. It follows that G is {ISK4, wheel}-free if and only if all leaves of T_{2c} are {ISK4, wheel}-free. So our algorithm proceeds with examining the leaves of the tree.

Let T be the decomposition tree that is obtained by combining $T_{cc}(G)$ and the T_{2c}'s of all leaves of T_{cc}. We show that T has $O(n)$ nodes. To do this, we define for every graph H the function $f(H) = |V(H)| - 4$. Suppose that G is decomposed by a cutset K into subgraphs G_X, G_Y as above, where K is either a clique cutset of size at most three or a proper 2-cutset. If K is a clique cutset, then we have $f(G_X) = |X| + |K| - 4$, $f(G_Y) = |Y| + |K| - 4$, and $f(G) = |X| + |Y| + |K| - 4$. It follows (because $|K| \leqslant 3$) that $f(G_X) + f(G_Y) \leqslant f(G)$. If K is a proper 2-cutset, then we have $f(G_X) = |X| + 3 - 4$, $f(G_Y) = |Y| + 3 - 4$, and $f(G) = |X| + |Y| + 2 - 4$. It follows again that $f(G_X) + f(G_Y) \leqslant f(G)$. Let T^* be the subtree of T induced by the nodes that are graphs with at least five vertices. Applying the above inequality recursively, and letting $G_1, \dots, G_\ell$ be the leaves of T^*, we obtain that $f(G_1) + \dots + f(G_\ell) \leqslant f(G)$. Since all G_i's satisfy $f(G_i) > 0$, we obtain $\ell \leqslant n$. Consequently, T^* has at most $2n - 1$ nodes. In addition, each node of T with at least five vertices may have one or two children with at most four vertices. Moreover, the size of the decomposition tree of graphs with at most four vertices is bounded by a constant. So T has $O(n)$ leaves. Recall that the leaves have fewer vertices than G.

Now we show that T can be constructed in time $O(n^2m)$. Because proper 2-cutset can be found in time $O(nm)$ as follows: for any vertices v, find the cut vertices and the blocks of $G \setminus v$ by using DFS. For any such block, check whether the corresponding cutvertex u is such that $\{u, v\}$ is a proper 2-cutset. Thus, building the tree can be done by running $O(n)$ times this subroutine (or the routine that finds a clique cutset) and therefore takes time $O(n^2m)$.

Now suppose that G has no clique cutset and no proper 2-cutset. Theorem 1.2 implies that if G contains no induced subdivision of K_4 and no wheel, then G must be either (i) series-parallel, or (ii) a complete bipartite graph, or (iii) a long rich square or (iv) the line graph of a chordless graph H with maximum degree at most three. The converse is also true, namely, if G satisfies one of (i)–(iv), then it contains no ISK4 and no wheel (this is easy to check and we omit the details). So our algorithm needs only test if G is of one of the four types.

Testing (i) can be done in time $O(n+m)$, see [21].

Testing (ii) can be done by checking with breadth-first search whether G is bipartite, and, then checking whether any two vertices on different sides of the bipartition are adjacent. This takes time $O(m+n)$.

To test (iii), note that if G is a rich square and contains no wheel, then G has exactly four vertices of degree at least four (the four vertices of the central square) and all other vertices have degree three or two. So we need only identify the four vertices of largest degree, check whether they induce a square S, and, then check whether each component of $G \setminus S$ is a path and attaches to S as in the definition of a rich square. This can be done in time $O(n+m)$.

In order to test (iv), we apply one of the algorithms in [11,14], which run in time $O(n+m)$. If G is a line graph, then any such algorithm returns a graph H such that G is the line graph of H; moreover, it is known that H is unique up to isomorphism, except when G is a clique on three vertices (where H is either K_3 or $K_{1,3}$). Then we need only check if H has maximum degree at most three, which is easy, and contains no cycle with a chord, which can be done in time $O(n^2m)$ by a method described in the next section.

Let us now evaluate the total complexity of the algorithm. Building the tree takes time $O(n^2m)$. Since for each leaf H on n' vertices and m' edges, the test performed on H takes time $O(n'^2m')$, and since the sum of the sizes of the leaves of the tree is $O(n+m)$, processing all the leaves of the tree takes time $O(n^2m)$. Hence, the recognition algorithm runs in time $O(n^2m)$.

We would have liked to make our algorithm rely on classical decomposition along 2-cutsets, but the classical algorithms, such as Hopcroft and Tarjan's decomposition into triconnected components [9]. But this algorithm does not use our "proper" 2-cutset, so we do not know how we could use it.

12.2. Recognizing and coloring chordless graphs

On the basis of Theorem 1.3, we can give a polynomial-time recognition algorithm for chordless graphs. We describe this algorithm informally. Let the input of the algorithm be a graph G with n vertices and m edges. We first decompose G along its cutsets of size one (if any). This can be done in time $O(n+m)$ using depth-first search, see [18]; depth-first search produces the maximal 2-connected subgraphs ("blocks") of G, and their number is at most n. Clearly, G contains a cycle with a chord if and only if some block of G contains a cycle with a chord. So our algorithm proceeds with examining the blocks of G.

Now suppose that G is 2-connected and has a proper 2-cutset $\{a,b\}$. So $V(G) \setminus \{a,b\}$ can be partitioned into two sets X, Y such that there is no edge between them and each of $G[X \cup \{a,b\}]$ and $G[Y \cup \{a,b\}]$ is not an (a,b)-path. Let G_X be the subgraph of G induced by $X \cup \{a,b\}$ plus an artificial vertex adjacent to a and b, and define G_Y similarly. We consider that G is decomposed into graphs G_X and G_Y. These subgraphs can in turn be decomposed along proper 2-cutsets.

This is applied as long as possible, which yields a *proper* 2*-cutset decomposition tree* T_{2c} of G, whose leaves are graphs that have no proper 2-cutset. By Theorem 1.3, if such a leaf contains no cycle with a chord then it is sparse, and it is easy to see that the converse also holds. So it suffices to check that every leaf L is sparse, which is easily done by examining the degree of the two endvertices of every edge of L.

Exactly like in the previous section, a tree using 2-cutsets as we do above has size $O(n)$. Checking the leaves of the tree takes linear time, so in total our algorithm runs in time $O(n^2m)$.

Lemma 12.1. *Recognizing a chordless graph can be performed in time $O(n^2m)$.*

Note that chordless graphs are included in the class of graphs that do not contain a cycle with a unique chord and that do not contain K_4. These graphs are shown to be 3-colorable by a polynomial time algorithm in [20], but the proof is complex. Here below, we show that this problem is very easy in the particular case of chordless graphs.

Lemma 12.2. *A* 2*-connected chordless graph has a vertex of degree at most* 2*. So, any chordless graph is* 3*-colorable and a* 3*-coloring can be found in linear time.*

Proof. If G is chordless and 2-connected then it has an ear decomposition (see [1]). The last ear added to build G cannot be an edge because such an edge would be a chord of some cycle. So, the last ear added to build G is a path of length at least 2 and its interior vertices are of degree 2. □

12.3. Coloring {ISK4, wheel}-free graphs

We present here a coloring algorithm which colors every {ISK4, wheel}-free graph with three colors. Its validity is based on Theorem 1.4 and it follows the same lines. Let G be any {ISK4, wheel}-free graph with n vertices and m edges.

We first decompose G along its clique-cutsets, as in the preceding subsection. As in the proof of Theorem 1.4, a 3-coloring of the vertices of G can be obtained simply by combining 3-colorings of each child of G in the decomposition. So let us now suppose that G has no clique cutset.

If G contains a $K_{3,3}$, then, by Lemma 3.3, G must be a complete bipartite graph. We can test that property in time $O(n+m)$, and, if G is complete bipartite, we return an obvious 2-coloring. Now let us assume that G contains no $K_{3,3}$.

If G has a proper 2-cutset, then, as in the proof of Theorem 1.4, we decompose G into two graphs G''_X and G''_Y and we can obtain a 3-coloring of the vertices of G by combining 3-colorings of G''_X and G''_Y. Moreover, we know that these two graphs contain no ISK4, no wheel and no $K_{3,3}$. These graphs can be decomposed further (possibly also by clique cutsets). As above, one can prove that the total size of the decomposition tree is $O(n)$ (we omit the details).

Finally, consider a leaf L of the decomposition tree. By Theorem 1.2, L is either a series-parallel graph, a rich square, a line graph, or a complete bipartite graph. Then Lemmas 11.2, 11.3 and 11.4 show how to construct a 3-coloring of L in polynomial time. As for the recognition, this can be implemented to run in time $O(n^2m)$.

Acknowledgment

We are grateful to Alex Scott for showing us the proof of Theorem 1.6.

References

[1] J.A. Bondy, U.S.R. Murty, Graph Theory, Grad. Texts in Math., vol. 244, Springer, 2008.
[2] R.L. Brooks, On colouring the nodes of a network, Proc. Cambridge Philos. Soc. 37 (1941) 194–197.
[3] M. Chudnovsky, N. Robertson, P. Seymour, R. Thomas, The strong perfect graph theorem, Ann. of Math. 164 (1) (2006) 51–229.
[4] M. Chudnovsky, P. Seymour, Clawfree graphs. IV. Decomposition theorem, J. Combin. Theory Ser. B 98 (5) (2008) 839–938.
[5] M. Conforti, G. Cornuéjols, Graphs without odd holes, parachutes or proper wheels: a generalization of Meyniel graphs and of line graphs of bipartite graphs, J. Combin. Theory Ser. B 87 (2003) 331–347.
[6] G.A. Dirac, A property of 4-chromatic graphs and some remarks on critical graphs, J. Lond. Math. Soc. 27 (1952) 85–92.
[7] R.J. Duffin, Topology of series-parallel networks, J. Math. Anal. Appl. 10 (1965) 303–318.
[8] A. Gyárfás, Problems from the world surrounding perfect graphs, Zastow. Mat. Appl. Math. 19 (1987) 413–441.
[9] J.E. Hopcroft, R.E. Tarjan, Dividing a graph into triconnected components, SIAM J. Comput. 2 (3) (1973) 135–158.
[10] D. Kühn, D. Osthus, Induced subdivisions in $K_{s,s}$-free graphs of large average degree, Combinatorica 24 (2) (2004) 287–304.
[11] P.G.H. Lehot, An optimal algorithm to detect a line graph and output its root graph, J. Assoc. Comput. Machin. 21 (4) (1974) 569–575.
[12] B. Lévêque, D. Lin, F. Maffray, N. Trotignon, Detecting induced subgraphs, Discrete Appl. Math. 157 (2009) 3540–3551.
[13] A. Pawlik, J. Kozik, T. Krawczyk, M. Lasoń, P. Micek, W.T. Trotter, B. Walczak, Triangle-free intersection graphs of line segments with large chromatic number, manuscript, 2012.
[14] N.D. Roussopoulos, A max $\{m, n\}$ algorithm for determining the graph H from its line graph G, Inform. Process. Lett. 2 (4) (1973) 108–112.
[15] A.D. Scott, Induced trees in graphs of large chromatic number, J. Graph Theory 24 (1997) 297–311.
[16] A.D. Scott, Personal communication, 2009.
[17] P.D. Seymour, Colouring series-parallel graphs, Combinatorica 10 (4) (1990) 379–392.
[18] R.E. Tarjan, Depth first search and linear graph algorithms, SIAM J. Comput. 1 (2) (1972) 146–160.
[19] R.E. Tarjan, Decomposition by clique separators, Discrete Math. 55 (2) (1985) 221–232.
[20] N. Trotignon, K. Vušković, A structure theorem for graphs with no cycle with a unique chord and its consequences, J. Graph Theory 63 (2010) 31–67.
[21] J. Valdes, R.E. Tarjan, E.L. Lawler, The recognition of series parallel digraphs, SIAM J. Comput. 11 (2) (1982) 298–313.
[22] S.H. Whitesides, An algorithm for finding clique cut-sets, Inform. Process. Lett. 12 (1) (1981) 31–32.