On Embedding of Graphs into Euclidean Spaces of Small Dimension

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Communicated by the Editors

Received June 4, 1987

The scalar product dimension \( d(G) \) of a graph \( G \) is defined to be the minimal number \( d \) such that vertices \( x \) of \( G \) can be represented by vector \( x \in \mathbb{R}^d \) with the property that \( xy \) is an edge of \( G \) iff \( \langle x, y \rangle \geq t \) for some real threshold \( t \). In this paper, we prove that \( d(G) < c \rho^2 \log n \) (where \( \rho \) is an absolute constant) if \( \rho(G) \leq \rho \), where \( \rho(G) \) is the edge density of \( G \)—the maximum of average degrees of subgraphs of \( G \). © 1992 Academic Press, Inc.

1. Introduction

Let \( G = (V, E) \) be a finite graph without loops and multiple edges. We shall consider embeddings \( x \rightarrow \bar{x} \in \mathbb{R}^d \) (\( x \in V \)) of \( G \) into euclidean space \( \mathbb{R}^d \) such that, for some threshold \( t \in \mathbb{R} \),

\[
xy \in E \quad \text{iff} \quad \langle x, y \rangle \geq t \quad (x, y \in V, x \neq y),
\]

where \( \langle x, y \rangle \) is the scalar product of vectors \( x, y \). The least \( d \geq 1 \) such that \( G \) admits an embedding into \( \mathbb{R}^d \) will be denoted by \( d(G) \) [7].

Recall that if \( G \) has \( n \) vertices then (trivially) \( d(G) \leq n \); the best known general upper bound is \( d(G) \leq n - \sqrt{n} \) while the best known lower bound for \( \max_G d(G) \) (over all graphs on \( n \) vertices) is \( \max_G d(G) \geq \lfloor n/2 \rfloor \) is obtained via \( d(K_{n/2, n/2}) = n/2 \) (see [6, 7]).

In the current paper, we present some sublinear upper bounds for classes of graphs with special properties.
We prove that \(d(G) \leq c \rho^2 \log n\) (\(c\) an absolute constant) if \(\rho(G) \leq \rho\), where \(\rho(G)\) is the edge density of \(G\) defined to be the maximum of average degrees of subgraphs of \(G\). This generalizes a result of [3], \(d(G) \leq cd^2 \log n\), where \(d\) is the maximum degree of \(G\). Note that \(\rho(G) \leq d\) and that \(\rho(G)\) can be essentially smaller. A simple example below shows that such an inequality is no longer true if \(\rho\) is replaced by the average degree \(\delta\) of \(G\).

We further present upper bounds for \(d(G)\) for other classes of graphs. For example, we show that any graph \(G\) not containing \(K_{2,k}\), \(k \leq n^{1/3}\), satisfies \(d(G) \leq c \sqrt{k} \sqrt{n} \log n\). Similarly, upper bounds for graphs without short cycles are derived. These results are corollaries of Theorem 1, the proof of which is based on random representations of graphs.

2. Random Unit Vectors

The following lemma uses arguments similar to those of the proof of Theorem 6.1 in [7].

**Lemma 1.** Let \(\bar{x}, \bar{y}\) be independent random unit vectors in \(\mathbb{R}^m\), where \(m = \lfloor 12(1/t^2) \ln(n/t) \rfloor\), \(0 < t \leq \frac{1}{2}\). Then

\[
\text{Prob}(\|\bar{x}\|, \|\bar{y}\| > t) < \frac{1}{n^3}.
\]

**Proof.** Let \(S\) be the unit sphere in \(\mathbb{R}^m\) and \(\mu\) the \((m-1)\)-dimensional measure on \(S\). For \(\bar{x} \in S\) put

\[
D(t) = \{\bar{y} \in S; \|\bar{x}\| \leq t\}.
\]

Then

\[
\text{Prob}(\|\bar{x}\|, \|\bar{y}\| > t) = \frac{\mu(D(t))}{\mu(S)} = \frac{\int_0^{\pi/2} \cos^{m-2} \phi \, d\phi}{\int_0^{\pi/2} \cos^{m-2} \phi \, d\phi} \leq \frac{\int_0^{\pi/2} \cos^{m-2} \phi \, d\phi}{\int_0^{\pi/2} \cos^{m-2} \phi \, d\phi} < \frac{\pi(1-t^2)(m-2)/2}{t(1-t^2/4)(m-2)/2} \leq \frac{\pi}{t} \exp \left[ -\frac{(m-2) t^2}{4} \right] \leq \exp(3/16) \frac{\pi t^3}{n^3} < \frac{1}{n^3}.
\]

**Lemma 2.** Let \(n, m, K\) be positive integers and \(t \in \mathbb{R}\) such that \(K \leq n, 0 < t \leq 1/3 \sqrt{K}\), and \(m = \lfloor 12(1/t^2) \ln(n/t) \rfloor\). Then for each positive integer \(j \leq K\),

\[
\text{Prob}(\|\bar{x}_1 + \bar{x}_2 + \cdots + \bar{x}_j\|^2 \geq 2j) < \frac{j}{n^3},
\]

where \(\bar{x}_1, \ldots, \bar{x}_j\) are independent random unit vectors in \(\mathbb{R}^m\).
Proof. Put $y_j = \|\vec{x}_1 + \cdots + \vec{x}_j\|^2$. We proceed by induction on $j=1, \ldots, K$. The case $j=1$ is clear. Let $1 \leq j < K$. As $y_{j+1} = y_j + 2 \vec{x}_{j+1} \cdot (\vec{x}_1 + \cdots + \vec{x}_j)$, using the induction assumption and Lemma 1 we have

$$
\begin{align*}
\text{Prob}(y_{j+1} \geq 2(j+1)) & \leq \text{Prob}(y_j \geq 2j) \\
& + \text{Prob}(2\vec{x}_{j+1} \cdot (\vec{x}_1 + \cdots + \vec{x}_j) \geq 1 \text{ and } y_j < 2j) \\
& < \frac{j}{n^3} + \text{Prob} \left( \frac{\vec{x}_{j+1} \cdot (\vec{x}_1 + \cdots + \vec{x}_j)}{\|\vec{x}_1 + \cdots + \vec{x}_j\|} \geq \frac{1}{2\sqrt{2j}} \right) \\
& < \frac{j}{n^3} + \frac{1}{n^3}
\end{align*}
$$

because $1/(2\sqrt{2j}) > 1/(3\sqrt{K}) > t$.

Lemma 3. Let $n, m, K, t$ be as in Lemma 2. Let $\vec{x} = \vec{x}_1 + \cdots + \vec{x}_j$ and $\vec{y} = \vec{y}_1 + \cdots + \vec{y}_k$, where $\vec{x}_1, \ldots, \vec{x}_j$, $\vec{y}_1, \ldots, \vec{y}_k$ are independent random unit vectors in $\mathbb{R}^m$, $j, k < K$. Then

$$
\text{Prob}(\|\vec{x}\vec{y}\| \geq 2\sqrt{jk}t) < \frac{3K}{n^3}.
$$

Proof. Define $p = \text{Prob}(\|\vec{x}\vec{y}\| \geq 2\sqrt{jk}t)$. Let $A$ be the event that $\|\vec{x}\| \geq 2j$ or $\|\vec{y}\| \geq \sqrt{2k}$. Then $p \leq p_1 + p_2$, where $p_1 = \text{Prob}(A)$ and $p_2 = \text{Prob}(\|\vec{x}\vec{y}\| \geq 2\sqrt{jk}t \text{ and } \overline{A})$. According to Lemma 2, $p_1 < j/n^3 + k/n^3 \leq 2K/n^3$. Using Lemma 1 we get

$$
p_2 = \text{Prob} \left( \frac{|\vec{x}\vec{y}|}{\|\vec{x}\| \|\vec{y}\|} \geq \frac{2\sqrt{jk}}{\|\vec{x}\| \|\vec{y}\|} t \text{ and } \overline{A} \right) \leq \text{Prob} \left( \frac{|\vec{x}\vec{y}|}{\|\vec{x}\| \|\vec{y}\|} \geq t \right) < \frac{1}{n^3}.
$$

Thus $p < 2K/n^3 + 1/n^3 < 3K/n^3$.

3. Random Embeddings

If $G = (V, E)$ is a graph and $\vec{E}$ an orientation of $E$, denote $N_+(i) = \{ j \mid (i, j) \in \vec{E} \}$, $d_+(i) = |N_+(i)|$ ($i \in V$).
THEOREM 1. Let $G = (V, E)$ be a graph, $V = \{1, \ldots, n\}$. Suppose that $E$ admits an orientation $\vec{E}$ such that for every $i, j \in V$, $i \neq j$,
\[ d_+(i) \leq K, \quad |N_+(i) \cap N_+(j)| \leq 1. \]
Put $A = \sqrt{K} + 1$. Then $d(G) \leq cA^2 \log(nA)$.

Proof. Let $t \in (0, 1/3 \sqrt{K})$, $\beta > 0$, $m = \lfloor 12(1/t^2) \ln(n/t) \rfloor$. Choose independent random unit vectors $\vec{x}_1, \ldots, \vec{x}_n$ in $\mathbb{R}^m$. To simplify the notation, write $\sum N$ for $\sum_{k \in N} \vec{x}_k$ and put
\[ \vec{x}_i = \beta \vec{x}_i + \sum N_+(i) \quad (i \in V). \]
We prove that parameters $\beta, t$ can be chosen so that the vectors $\vec{x}_i$ form a representation of $G$ in $\mathbb{R}^m$ with a positive probability. For a fixed couple $i, j \in V$, $i \neq j$,
\[ \vec{x}_i \vec{x}_j = \beta^2 \vec{x}_i \vec{x}_j + \beta \vec{x}_i \sum N_+(j) + \beta \vec{x}_j \sum N_+(i) + \sum N_+(i) \sum N_+(j). \]
Let us estimate each of the four summands.

1. By Lemma 1, with probability $p_1 > 1 - 1/n^3$,
\[ -\beta^2 t < \beta^2 \vec{x}_i \vec{x}_j < \beta^2 t. \]

2. (a) Suppose $ij \in E$, say $i \in N_+(j)$. Then $\beta \vec{x}_i \sum N_+(j) = \beta + (\beta \vec{x}_i) \sum (N_+(j) - \{i\})$. According to Lemma 3, with probability $p_2 > 1 - 3K/n^3$,
\[ \beta \vec{x}_i \sum N_+(j) > \beta - 2 \sqrt{K}t\beta. \]

(b) Suppose $ij \notin E$. Then, with probability $p_2 > 1 - 3K/n^3$,
\[ \beta \vec{x}_i \sum N_+(j) < 2 \sqrt{K}t\beta. \]

3. This is analogous to 2, but now with $j \notin N_+(i)$; hence, with probability $p_3 > 1 - 3K/n^3$,
\[ -2 \sqrt{K}t\beta < \beta \vec{x}_j \sum N_+(i) < 2 \sqrt{K}t\beta. \]

4. We have
\[ \sum N_+(i) \sum N_+(j) = \sum (N_+(i) \cap N_+(j)) \sum (N_+(i) - N_+(j)) \]
\[ + \sum (N_+(i) \cap N_+(j)) \sum (N_+(j) - N_+(i)) \]
\[ + \sum (N_+(i) - N_+(j)) \sum (N_+(j) - N_+(i)) \]
\[ + \left\| \sum N_+(i) \cap N_+(j) \right\|^2. \]
By Lemmas 3 and 2, with probability

\[ p_4 > 1 - 3K/n^3 - 3K/n^3 - 3K/n^3 - n^3 \geq 1 - 10K/n^3 \]

we have

\[ -6Kt < \sum N_+(i) \sum N_+(j) < 6Kt + 2I. \]

Summarizing the above inequalities we see that with probability

\[ p_{ij} > 1 - 1/n^3 - 3K/n^3 - 3K/n^3 - 10/Kn^3 = 1 - 17K/n^3 \]

we have

if \( ij \in E \) then \( \tilde{x}_i \tilde{x}_j > -\beta^2 t + 2 K \sqrt{t} + 6Kt \),
if \( ij \notin E \) then \( \tilde{x}_i \tilde{x}_j < \beta^2 t + 2 K \sqrt{t} + 6Kt + 2I. \)

These hold simultaneously for all pairs \( i, j \) (\( i \neq j \)) with probability

\[ p > 1 - \left( \begin{array}{c} n \\ \end{array} \right)^2 = 1 - \frac{9K}{n}. \]

We may suppose \( K < n/9 \) for otherwise the statement of Theorem 1 is trivial. Hence the vectors \( \tilde{x}_i \) form a representation of \( G \) with positive probability provided that

\[ -\beta^2 t + \beta - 4 \sqrt{Kt} - 6Kt \geq 0. \]

i.e.,

\[ (2t) \beta^2 + (8 \sqrt{Kt} - 1) \beta + (12Kt + 2I) \leq 0. \]

Our track is to find \( t \in (0, 1/3 \sqrt{K}) \) (as large as possible) such that the above quadratic inequality has a solution \( \beta > 0 \). To this end, let \( t \) be the positive root of its discriminant \( (8 \sqrt{Kt} - 1)^2 - 8t(12Kt + 2I) = -32Kt^2 - 16(\sqrt{K} + I)t + 1; \) then \( (1/t)^2 - 16(\sqrt{K} + I)(1/t) - 32K = 0, \)

\[ \frac{1}{t} = 8(\sqrt{K} + I) + \sqrt{64(\sqrt{K} + I)^2 + 32K} < (8 + \sqrt{96})(\sqrt{K} + I). \]

Also, \( t \leq 1/3 \sqrt{K} \), as required, and the solution \( \beta = (1 - 8 \sqrt{Kt})/4t \) is positive.

We have proved the existence of a representation of \( G \) in \( R^m \), where \( m = \lceil 12(1/t^2) \ln(n/t) \rceil \leq (12(8 + \sqrt{96})^2 \ln 2) A^2 \log(nA(8 + \sqrt{96})). \)
4. Applications

We apply Theorem 1 to obtain upper bounds for \( d(G) \) for classes of graphs which admit orientations with \( K, l \) small.

Recall that for a graph \( G \), the edge density is defined by

\[
\rho(G) = \max \left\{ \frac{2|E'|}{|V'|} \mid (V', E') \text{ a subgraph of } G, V' \neq \emptyset \right\}.
\]

In the other words, \( \rho(G) \) is the maximum of average degrees of subgraphs of \( G \).

The following folkloristic lemma enables us to make \( K \leq \rho \).

**Lemma 4.** Every graph \( G = (V, E) \) admits an orientation \( \tilde{E} \) of edges with

\[
d_+(i) \leq \rho(G) \quad (i \in V).
\]

**Proof.** We proceed by induction on \( n = |V| \). The case \( n = 1 \) is clear. Let \( n > 1 \). It follows by the definition of \( \rho(G) \) that \( G \) contains a vertex \( i_0 \) with degree \( d(i_0) \leq \rho(G) \). Then orient edges of \( G - \{i_0\} \) using the induction assumption and orient edges \( i_0, j \in E \) so that \((i_0, j) \in \tilde{E}\). The proof of Lemma 4 is concluded.

**Theorem 2.** Let \( G = (V, E) \) be a graph with \( |V| = n > 1 \) and \( \rho(G) \leq \rho \). Then

\[
d(G) \leq c_0 \rho^2 \log n.
\]

**Proof.** Apply Lemma 4 and Theorem 1 with \( K = l = \rho \).

**Remark.** Note that the analogous statement with \( \rho \) replaced by average degree \( \delta(G) \) is no longer true: For example, if \( G \) is a graph on \( n \) vertices consisting of \( K_{m, m} \), \( m = \lfloor \sqrt{n} \rfloor \) and \( n - 2m \) isolated vertices then \( \delta(G) \leq 2 \) while \( d(G) \geq d(K_{m, m}) = m \) (cf. [7]).

**Theorem 3.** Let \( G = (V, E) \) be a graph, \( |V| = n > 1 \). Suppose \( G \) does not contain \( K_{2, k} \) as a subgraph \((k \geq 2)\).

(a) If \( k \leq n^{1/3} \) then

\[
d(G) \leq c_1 \sqrt{k} \sqrt{n} \log n.
\]

(b) If \( K > n^{1/3} \) then

\[
d(G) \leq c_2 k^2 \log n.
\]
Proof. It is well known (cf. [4] or [5]) that if a graph on \( n \) vertices contains no \( K_{2,k} \) then it has at most \( \frac{1}{2} \sqrt{kn^2} \) edges. Thus in our case \( \rho(G) \leq \sqrt{kn} \) follows. By Lemma 4 we may apply Theorem 1 to \( G \) with \( K = \sqrt{kn} \) and \( I = k \) to obtain
\[
d(G) \leq c(\sqrt[4]{kn} + k)^2 \log(n(\sqrt[4]{kn} + k)).
\]

If \( k \leq n^{1/3} \) then \( \sqrt[4]{kn} + k \leq 4\sqrt[4]{k}(\sqrt[4]{n} + \sqrt[4]{k^3}) \leq 2\sqrt[4]{k} \sqrt[4]{n} \leq 2 \sqrt[4]{n}, \) hence
\[
d(G) \leq 4c \sqrt{kn} \log(n2^{3/4}).
\]

If \( k > n^{1/3} \) then \( \sqrt[4]{kn} + k \leq 2k, \) hence
\[
d(G) \leq 4ck^2 \log(n2k) \leq 4ck^2 \log(2n^2).
\]

Theorem 4. Let \( G = (V, E) \) be a graph, \( |V| = n. \)

(a) If \( G \) does not contain \( C_4 \) as a subgraph then
\[
d(G) \leq c_3 \sqrt{n} \log n.
\]

(b) If \( G \) does not contain \( C_4 \) and \( C_k \) (\( k \) even, \( k \geq 4 \)) as a subgraph then
\[
d(G) \leq c_4 kn^{2/k} \log n.
\]

(c) If \( G \) has girth \( > k, \) i.e., if \( G \) does not contain \( C_3 \) to \( C_k \) (\( k \leq 4 \)) as a subgraph then
\[
d(G) \leq c_5 n^{2/k} \log n \quad \text{for } k \text{ even},
\]
\[
d(G) \leq c_5 n^{2/(k-1)} \log n \quad \text{for } k \text{ odd}.
\]

Proof. (a) This follows by Theorem 3(a) with \( k = 2. \)

(b) According to [4] (cf. also [5]), a graph without cycles \( C_k \) (\( k \) even, \( k \geq 4 \)) has less than \( 45 kn^{1 + 2/k} \) edges, hence \( \rho(G) \leq 90 kn^{2/k}. \) If \( G, \) in addition, does not contain \( C_4 = (K_{2,2}) \) as a subgraph then we may use Theorem 1 with \( K = 90 kn^{2/k} \) and \( I = 1. \) Thus
\[
d(G) \leq c(\sqrt{90 kn^{1/k}} + 1)^2 \log(n(\sqrt{90 kn^{1/k}} + 1)) \leq c_4 kn^{2/k} \log n.
\]

(c) It is sufficient to consider the case when \( k \) is even. Using a simple averaging argument, find a subgraph \( G' = (V', E') \) of \( G \) with minimum
degree $\delta \geq \rho/2$. As $G'$ has also girth $> k$, according to [1], $|V'| \geq (\delta - 1)^{k/2}$. As $|V'| \leq n$, it follows that

$$\rho \leq 2\delta \leq 2(n^{2/k} + 1).$$

By Lemma 4 and Theorem 1 with $K = \rho$, $I = 1$, $A \leq \sqrt{2(n^{2/k} + 1) + 1}$,

$$d(G) \leq c(\sqrt{2(n^{2/k} + 1) + 1})^2 \log n A \leq c_5 n^{2/k} \log n.$$

**References**