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New coordinates for de Sitter space and de Sitter radiation

Maulik K. Parikh

Department of Physics, Columbia University, New York, NY 10027, USA

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Abstract

We introduce a simple coordinate system covering half of de Sitter space. The new coordinates have several attractive properties: the time direction is a Killing vector, the metric is smooth at the horizon, and constant-time slices are just flat Euclidean space. We demonstrate the usefulness of the coordinates by calculating the rate at which particles tunnel across the horizon. When self-gravitation is taken into account, the resulting tunneling rate is only approximately thermal. The effective temperature decreases through the emission of radiation.

1. Motivation

At the heart of Einstein's theory of gravity is local diffeomorphism invariance: coordinate systems are unimportant, only diffeomorphism invariants matter. However, a poor choice of local coordinates can sometimes obscure the nature of the global aspects of spacetime, such as horizons or causal boundaries. Indeed, the true nature of the "coordinate singularity" at the Schwarzschild radius eluded Einstein himself, and was only fully illuminated with the discovery of coordinate systems that were regular at the horizon. Coordinate systems that cover larger patches of spacetime are especially useful to have in dealing with physical phenomena that are in some sense nonlocalized.

In this note, we present a simple new coordinate system for de Sitter space, covering the causal future/past of an observer. The new coordinates, which we shall call Painlevé–de Sitter coordinates, are a cross between static coordinates and planar coordinates, and inherit the strengths of each of these. For example, like static coordinates but unlike planar coordinates, the new coordinates have a direction of time that is a Killing vector, making them well-adapted to thermodynamics. On the other hand, like planar coordinates, but unlike static coordinates, Painlevé–de Sitter coordinates continue smoothly through the horizon, and constant time slices are just flat Euclidean space. Painlevé–de Sitter coordinates differ also from Eddington–Finkelstein type coordinates in that the coordinates are all either timelike or spacelike, rather than null.

The combination of a Killing time direction and regularity at the horizon is particularly powerful as it allows one to study across-horizon physics as seen by an observer. A natural application is de Sitter radiation. Heuristically, one envisions de Sitter radiance as arising in much the same way that Hawking radiation does. That is to say, a particle-pair forms just inside

E-mail address: mkp@phys.columbia.edu (M.K. Parikh).

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the horizon, one member of the pair tunnels across the horizon, and the virtual pair becomes real. To show that this is actually what happens, one would like to compute the amplitude for traversing the horizon; because of their regularity, Painlevé–de Sitter coordinates make the calculation feasible. By directly evaluating the imaginary part of the action, we obtain the emission amplitude associated with a tunneling particle. In the s-wave limit, it is in fact possible to extend the computation to include the effects of self-gravitation. As a result, the de Sitter spectrum turns out to be only approximately thermal. In particular it has a cutoff at high energies. The back-reaction is such that, unlike Schwarzschild black holes, de Sitter space lowers its temperature as it radiates.

2. Painlevé–de Sitter coordinates

Many different coordinate systems are known for de Sitter space (see, e.g., [1,2]). One commonly used metric is the static metric, analogous to the familiar Schwarzschild metric for an uncharged black hole. This metric covers a static patch, that part of de Sitter space that an observer at the origin can interact with. The time coordinate, t_s, corresponds to a timelike Killing vector, which makes it suitable for thermodynamics; thermal equilibrium requires among other things that the spatial metric be at equilibrium. However, the static metric has the limitation that it is only valid upto the horizon; it therefore covers only a very small region of the full space. Another drawback is that in a static background one cannot expect to get radiation, a phenomenon that is manifestly time-reversal asymmetric. Indeed, in early calculations of Hawking radiation from black holes, time-reversal symmetry had to be broken by hand through the introduction of a collapsing surface.

We shall now illustrate the method for obtaining Painlevé-type coordinates, valid across the horizon. Consider then a general static metric of the form

$$ds^2 = -\big(1 - g(r)\big)\, dt_s^2 + \frac{dr^2}{1 - g(r)} + r^2\, d\Omega_{D-2}^2. \tag{1}$$

Here $g(r) = 1$ corresponds to a horizon; we shall assume for simplicity that there is only one horizon. For de Sitter space, $g(r) = r^2/l^2$ and $r = l$ marks the horizon. $r = 0$ could be the worldline of an observer at the origin.

To obtain the new line element, define a new time coordinate, t, by $t_s = t + f(r)$. The function f is required to depend only on r and not t, so that the metric remains stationary, i.e., time-translation invariant. Stationarity of the metric automatically implements the desirable property that the time direction be a Killing vector. What other conditions should we impose on f? Our key requirement is that the metric be regular at the horizon. We can implement this as follows. We know that a radially free-falling observer who falls through the horizon does not detect anything abnormal there; we can therefore choose as our time coordinate the proper time of such an observer. As a corollary, we demand that constant-time slices be flat. We then obtain the condition

$$\frac{1}{1 - g(r)} - \big(1 - g(r)\big)\big(f'(r)\big)^2 = 1. \tag{2}$$

For de Sitter space, $g(r) = r^2/l^2$, and the solution to Eq. (2) yields

$$t_s = t \pm \frac{l}{2} \ln\big(1 - r^2/l^2\big). \tag{3}$$

As usual, the fact that the transformation is singular at $r = l$ merely indicates that the original coordinate t_s was ill-defined there. Choosing the minus sign for now, gives us the new coordinates, which we shall call Painlevé–de Sitter coordinates. The desired line element is

$$ds^2 = -\left(1 - \frac{r^2}{l^2}\right) dt^2 - 2\frac{r}{l}\, dt\, dr + dr^2 + r^2\, d\Omega^2. \tag{4}$$

The Painlevé–de Sitter metric has a number of attractive features. First, none of the components of either the metric or the inverse metric diverge at the horizon. Second, by construction constant-time slices are just flat Euclidean space. Third, the generator of t is a Killing vector. "Time" becomes spacelike across the horizon, but is nevertheless Killing; this fact can be exploited to compute global charges such as mass [3,4] in a natural way. Finally, an observer precisely at the origin does not make any distinction between these coordinates and static coordinates; indeed, the function f that distinguishes the two time coordinates vanishes there.

We call this the Painlevé–de Sitter metric because the analogous line element for four-dimensional Schwarzschild black holes is

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 - 2\sqrt{\frac{2M}{r}}\, dt\, dr + dr^2 + r^2\, d\Omega_2^2. \tag{5}$$

This superb though relatively unknown line element was discovered by Painlevé [5] many years ago; it was rediscovered in a modern context by Kraus and Wilczek [6]. Similar coordinate systems have been found for black holes in anti-de Sitter space [7,8].

These coordinates cover the observer's causal future, or half of the full space. The metric is necessarily nonstatic (that is, not time-reversal invariant) as the causal patch is itself not time-reversal invariant. To obtain the other half of the space, corresponding to the causal past of an observer at the antipode, one chooses the opposite (plus) sign for the off-diagonal component *and* reverses the sense of time, so that t increases to the past. Choosing the other sign in Eq. (3) without reversing the direction of time gives a metric that covers the causal past of the original observer. The relation between Painlevé–de Sitter coordinates and planar coordinates is made clear by the transformation

$$r = \rho e^{t/l} \quad \Rightarrow \quad ds^2 = -dt^2 + e^{2t/l}\left(d\rho^2 + \rho^2\, d\Omega_{D-2}^2\right). \tag{6}$$

We see that the new coordinates have the same radial coordinate as static coordinates and the same time coordinate as planar coordinates.

We conclude this section by writing down the radial geodesics in these coordinates. The null radial geodesics obey

$$\frac{dr}{dt} = \frac{r}{l} \pm 1, \tag{7}$$

where the plus (minus) sign corresponds to rays that go away from (towards) the observer. When the particle is beyond the horizon ($r > l$), both ingoing and outgoing trajectories correspond to increasing r, and the particle cannot (classically) cross the horizon. The general solution is

$$r(t) = l\left(e^{t/l} \mp 1\right). \tag{8}$$

Radially-directed null rays leaving the observer at $t = 0$ reach the horizon at $t = l \ln 2$, and reach future null infinity at $t = \infty$. Turning now to massive particles, the radial geodesic equation implies

$$U^r = \frac{dr}{d\tau} = \left(1 - \frac{r^2}{l^2}\right) m^2 p_t, \tag{9}$$

where τ is the proper time. Here (by stationarity) $-p_t$ is a constant of the motion, being equal to the energy measured by an observer at $r = 0$. Note that if we set $p_t (= mU_t) \equiv -m$, then using $U^2 = -1$ we find that

$$U^t = 1 \quad \Rightarrow \quad \tau = t + c, \tag{10}$$

so the Painlevé–de Sitter time coordinate is nothing more than the proper time along a radial geodesic worldline, such as that of a free-falling observer. Indeed, that is precisely how we arrived at these coordinates in the first place.

3. Tunneling across the de Sitter horizon

The great utility of having a coordinate system that is well-behaved at the horizon is that one can study across-horizon physics. In this section, we will determine the temperature of de Sitter space by directly computing the rate at which particles tunnel across the horizon. Our computation will parallel the analogous calculation for black holes, in which Hawking radiation is expressed as a tunneling phenomenon [9].

Now, because of the infinite blueshift near the horizon, the characteristic wavelength of any wavepacket is always arbitrarily small there, so that the geometrical optics limit becomes an especially reliable approximation. This is of course a big plus: the geometrical optics limit allows us to obtain rigorous results directly in the language of particles, rather than having to use the unwieldy and physically less transparent Bogolubov method that is more traditionally used. Moreover, as we shall see, the inclusion of back-reaction effects is also perhaps easier now.

In any event, since we are in the semi-classical limit, we can apply the WKB formula. This relates the tunneling amplitude to the imaginary part of the particle action at stationary phase. (The phase is $i \int L\, dt/\hbar = i \int (p\dot{x} - H)\, dt/\hbar$; since energy is real, exponential damping comes from the imaginary part of emission rate $\int p\, dx$, which in the nonrelativistic limit becomes the usual $\int dx \sqrt{2m(V(x) - E)}$. The imaginary part of the action, I, is thus given by the imaginary part of the momentum integral.) The emission rate, Γ, is the square of the tunneling amplitude:

$$\Gamma \sim \exp(-2\,\mathrm{Im}\, I/\hbar) \approx \exp(-\beta E). \tag{11}$$

On the right-hand side, we have equated the emission probability to the Boltzmann factor for a particle of energy E. To the extent that the exponent depends linearly on the energy, the thermal approximation is justified; we can then identify the inverse temperature as the coefficient β.

We will consider here the s-wave emission of massless particles. Higher partial wave emission is in any case suppressed by $\hbar$. In the s-wave, particles are really massless shells. If we imagine a shell to consist of constituent massless particles each of which travels on a radial geodesic, then we see that the motion of the shell itself must follow the radial null geodesic for a particle. That is, it obeys Eq. (7), with the minus sign. We will use these radial geodesics to compute the imaginary part of the action, as follows.

Since the calculation involves a few tricks [9], we outline it here before putting in the details. First observe that we can formally write the action as

$$\mathrm{Im}\, I = \mathrm{Im} \int_{r_i}^{r_f} p_r\, dr = \mathrm{Im} \int_{r_i}^{r_f} \int_0^{p_r} dp_r'\, dr, \tag{12}$$

where p_r is the radial momentum. We expect r_i to correspond roughly to the site of pair-creation, which should be slightly outside the horizon. (Note that the second member of the pair contributes nothing to the tunneling rate, since it is always classically allowed and therefore has real action.) We expect r_f to be a classical turning point, at which the semiclassical trajectory (i.e., instanton) can join onto a classical-allowed motion. This must be slightly within the horizon, else the particle would not be able to propagate classically from there to the observer. However, the precise limits on the radial integral are unimportant, so long as the range of integration includes the horizon.

We now eliminate the momentum in favor of energy by using Hamilton's equation

$$\left.\frac{dH}{dp}\right|_r = \frac{\partial H}{\partial p} = \frac{dr}{dt}, \tag{13}$$

where the Hamiltonian, H, is the generator of Painlevé time. Hence within the integral over r, one can trade dp for dH. Without being very careful about signs, the integral over H now just gives the particle energy E. However, substituting Eq. (7) for a particle going radially towards $r = 0$, the radial integral has a simple pole at the horizon:

$$\mathrm{Im}\, I = \mathrm{Im} \int_{r_i}^{r_f} \int_0^{E} \frac{dH}{\frac{dr}{dt}}\, dr = \mathrm{Im}\, E \int_{r_i}^{r_f} \frac{l\, dr}{r - l}. \tag{14}$$

The pole lies along the line of integration, and therefore yields πi (rather than $2\pi i$) times the residue. Again, we postpone consideration of the sign associated with the direction of the contour. We get

$$\mathrm{Im}\, I = \pi l E. \tag{15}$$

Consulting Eq. (11), we find that this corresponds to a temperature

$$T_{dS} = \frac{\hbar}{2\pi l}, \tag{16}$$

which is precisely the temperature of de Sitter space. To summarize: Painlevé–de Sitter coordinates have allowed us to compute the radiation rate directly from the particle action, with the action incurring an imaginary part from a pole at the horizon.

Let us now do the calculation more carefully, keeping track of the signs, and including the effects of back-reaction. Perhaps it should be stressed that the reason we are interested in back-reaction is not merely to compute higher order in E effects, but because self-gravitation is central to the entire process of across-horizon tunneling. Without self-gravitation the back-of-the-envelope calculation above is puzzling: if this is tunneling, where is the barrier? Put another way, if particles created just inside the horizon have only to tunnel just across—an infinitesimal separation—what characterizes the scale of the tunneling? Recall that in the Schwinger process of electron–positron pair production in an electric field, there is a nonzero separation scale, $r \sim mc^2/qE$, between the classically allowed configurations. In the following, we will see that, as with Hawking radiation [9,10], self-gravitation resolves these issues. Back-reaction results in a shift of the horizon radius; the finite separation between the initial and final radius is the classically-forbidden region, the barrier.

How does one incorporate back-reaction? In a general situation, this is a notoriously difficult problem, calling for a theory of quantum gravity. Indeed, generically one has to worry about how to consistently quantize the gravitational waves produced by a matter source. However, for the special case of spherical gravity it is possible to integrate out gravity, at least semi-classically. This is because for spherical gravity, Birkhoff's theorem (more precisely, its generalization to a nonzero cosmological constant) states that the only effect on the geometry that the presence of a spherical shell has, is to provide a junction condition for matching the total mass inside and outside the shell. (In three dimensions, where there are no gravitational waves, it may be possible to compute the emission rates for the higher partial waves as well.)

Since the geometry is different on the two sides of the shell, one can now ask which geometry determines the motion of the shell. (Thus, self-gravitation automatically breaks the principle of equivalence.) The geometry inside the shell is empty de Sitter space, while the geometry outside is that of Schwarzschild–de Sitter space with energy E. (Technically, empty de Sitter space has a mass too [4]; what follows is unaffected by this shift.) It is the outside E-dependent metric that determines the motion of the self-gravitating shell. Consider, for simplicity, dS_3; generalization to higher dimensions is straightforward. The effective geometry whose radial geodesic determines the motion of the shell has the line element

$$ds^2_{\text{effective}} = -\left(1 - 8GE - \frac{r^2}{l^2}\right) dt_s^2 + \frac{dr^2}{1 - 8GE - r^2/l^2} + r^2\, d\phi^2. \tag{17}$$

Here we have inserted Newton's constant, and $E > 0$ is physically the energy of the shell as measured by an observer at $r = 0$. The corresponding Painlevé metric is now

$$ds^2_{\text{effective}} = -\left(1 - 8GE - \frac{r^2}{l^2}\right) dt^2 - 2\sqrt{\frac{r^2}{l^2} + 8GE}\, dt\, dr + dr^2 + r^2\, d\phi^2. \tag{18}$$

The imaginary part of the action is then

$$\operatorname{Im} I = \operatorname{Im} \int_0^H \int_{r_i}^{r_f} \frac{dr\, dH'}{\sqrt{r^2/l^2 + 8GE'} - 1}, \tag{19}$$

where we have inserted the radial geodesic derived from the effective metric. Here $r_i = l$ is the original radius of the horizon just before pair-creation, while r_f is the *new* radius of the horizon, and is equal to $l\sqrt{1 - 8GE}$. What matters is that $r_f < r_i$. The Feynman prescription for evaluating the sign of the contour is to displace the energy from E' to $E' - i\varepsilon$. Substituting $u = r^2$,

$$\operatorname{Im} I = \operatorname{Im} \int_0^H \int_{u_i}^{u_f} \frac{du}{2\sqrt{u}} \frac{l\sqrt{u + 8Gl^2(E' - i\varepsilon)} + l^2}{u - (l^2 - 8Gl^2(E' - i\varepsilon))}\, dH', \tag{20}$$

we see that the pole lies in the upper-half u-plane. Doing the u integral first we find

$$\operatorname{Im} I = -\pi l \int_0^H \frac{dH'}{\sqrt{1 - 8GE'}}. \tag{21}$$

Now the total energy of de Sitter space *decreases* when positive-energy matter is added to it [4], because of the negative gravitational binding energy. Therefore the Hamiltonian H satisfies $dH = -dE$, giving

$$\operatorname{Im} I = -\frac{\pi l}{4G}\left(\sqrt{1 - 8GE} - 1\right), \tag{22}$$

and the tunneling rate is therefore

$$\Gamma \sim \exp\left(+\frac{\pi l}{2G\hbar}\left(\sqrt{1 - 8GE} - 1\right)\right). \tag{23}$$

When the particle's energy is small, $8GE \ll 1$, the square root can be approximated. To linear order in GE, we recover our previous back-of-the-envelope result, Eq. (16). As a check, we note that the sign has also come out correctly. To this order then, the thermal approximation is a good one. But at higher energies the spectrum cannot be approximated as thermal. Indeed, the spectrum has an ultraviolet cutoff at $8GE = 1$, beyond which there is no radiation whatsoever. The precise expression, Eq. (23), is related to the change in de Sitter entropy. The entropy of three-dimensional de Sitter space is

$$S = \frac{2\pi r_H}{4G\hbar}, \tag{24}$$

where r_H is the horizon radius. Once the shell has been emitted, its energy causes the horizon to shrink to the new radius $r_H = l\sqrt{1 - 8GE}$. Thus, Eq. (23), can be written as the exponent of the difference in the entropies, ΔS, before and after the particle has been emitted. This also explains the UV cutoff: the horizon cannot shrink past zero. Incidentally, Eq. (23) takes the same form as the corresponding expression for Hawking radiation from a Schwarzschild black hole [9,11], once back-reaction effects have been taken into account.

Indeed, one expects Eq. (23) on general grounds. For whatever the ultimate form of the holographic description of de Sitter space, quantum field theory tells us (via Fermi's Golden Rule) that the rate must be expressible as

$$\Gamma(i \to f) = |\mathcal{M}_{fi}|^2 \times (\text{phase space factor}), \qquad (25)$$

where the first term on the right is the square of the amplitude for the process. The phase space factor is obtained by summing over final states and averaging over initial states. But the number of final states is just the final exponent of the final entropy, while the number of initial states is the exponent of the initial entropy. Hence

$$\Gamma \sim \frac{e^{S_{\text{final}}}}{e^{S_{\text{initial}}}} = \exp(\Delta S) \qquad (26)$$

in agreement with our result.

We end this section by noting that, although we have derived de Sitter radiance directly as particles tunneling across the de Sitter horizon, an alternate viewpoint is also possible. In this view, there is no de Sitter horizon and the spacetime is cutoff by a membrane living at the horizon. An observer then interprets the radiation not as tunneling particles, but rather as the spontaneous emissions of the membrane. In [11,12] it was shown that the classical equations of motion of the membrane can be derived from an action, and that Euclideanizing this action yields the correct entropy. It would be interesting to see whether the radiation can also be understood in this language.

4. On the thermal stability of de Sitter space

The signs in Eq. (23) have the consequence that three-dimensional de Sitter space is thermally stable. For after a particle has been emitted, the new horizon radius is smaller than what it had previously been. The probability for emission of a second particle is now

$$\Gamma_2 \sim \exp \Delta S = \exp\left(\frac{\pi l}{2G\hbar}\left(\sqrt{1-8G(E_1+E_2)} - \sqrt{1-8GE_1}\right)\right), \qquad (27)$$

where E_1 and E_2 are the energies of the two particles. For small E_2, this is

$$\Gamma_2 \approx \exp\left(\frac{-2\pi l}{\sqrt{1-8GE_1}}\frac{E_2}{\hbar}\right). \qquad (28)$$

The effective temperature which governs the emission probability of the second particle is therefore

$$T_2 = \frac{\hbar\sqrt{1-8GE_1}}{2\pi l}, \qquad (29)$$

which is *lower* than before. This is to be contrasted with the situation for Schwarzschild black holes, for which there is a runaway explosion as the black hole becomes smaller. In that respect, de Sitter space more closely resembles charged black holes. Note also that the change in the temperature of de Sitter space takes place because of the matter inside it, and not by any change in the cosmological constant, which of course remains constant throughout.

The decrease in the effective temperature holds out the possibility of thermally stabilizing de Sitter space [13]. We started by considering empty de Sitter space. As this radiates and lowers its temperature, the horizon volume fills up with radiation. Eventually, the radiation passes through the origin and leaves the horizon; this causes the horizon radius to increase again, raising the temperature. By detailed balance, a stable thermodynamic equilibrium can be reached.

In higher dimensions ($D > 3$), the story is somewhat different. The presence of the de Sitter radiation can now lead to the formation of a black hole, which in turn will radiate Hawking radiation outwards. Equilibrium is presumably reached only when the de Sitter and black hole horizons are coincident.

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