Topology and its Applications 159 (2012) 1489–1496

Contents lists available at SciVerse ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

A subanalytic triangulation theorem for real analytic orbifolds

Marja Kankaanrinta

Department of Mathematics, P.O. Box 400137, University of Virginia, Charlottesville, VA 22904-4137, USA

ARTICLE INFO

Article history:
Received 17 May 2011
Received in revised form 17 January 2012
Accepted 17 January 2012

MSC:
57R05
57R18

Keywords:
Subanalytic
Triangulation
Orbifold

ABSTRACT

Let X be a real analytic orbifold. Then each stratum of X is a subanalytic subset of X. We show that X has a unique subanalytic triangulation compatible with the strata of X. We also show that every C^r-orbifold, $1 \leqslant r \leqslant \infty$, has a real analytic structure. This allows us to triangulate differentiable orbifolds. The results generalize the subanalytic triangulation theorems previously known for quotient orbifolds.

© 2012 Elsevier B.V. All rights reserved.

1. Introduction

This paper concerns with subanalytic triangulations of real analytic orbifolds. To every point x in a real analytic orbifold X of dimension n one can associate a local group G_x, which is unique up to isomorphism. The sets $X_{(H)} = \{x \in X \mid G_x \cong H\}$, where H is a finite group, form a subanalytic stratification for X. The main result, Theorem 6.6, says that every real analytic orbifold X has a unique subanalytic triangulation compatible with the strata of X.

Orbit spaces of real analytic manifolds by real analytic proper almost free actions of Lie groups are called *real analytic quotient orbifolds*. Subanalytic triangulations of orbit spaces of proper real analytic G-manifolds where G is a Lie group are already known, see [8] and Theorem 7.7 in [6], as well as Theorem 3.3 in [9] for the case of a compact Lie group. These results cover the case of quotient orbifolds. The result of this paper generalizes the previous results by providing a subanalytic triangulation for all real analytic orbifolds.

We also show that every C^r-orbifold, $1 \leqslant r \leqslant \infty$, can be given a real analytic structure, by first giving a C^∞ structure to all C^r-orbifolds, $1 \leqslant r < \infty$, and then equipping every C^∞-orbifold with a real analytic structure (Theorems 7.4 and 8.2). By using the subanalytic triangulation result for real analytic orbifolds, we obtain a "subanalytic" triangulation compatible with the strata for any C^r-orbifold X, $1 \leqslant r \leqslant \infty$. "Subanalytic" triangulations are known to exist for orbit spaces, see [9].

2. Preliminaries

We begin by recalling the definition and some basic properties of an orbifold.

E-mail address: mk5aq@virginia.edu.

0166-8641/$ – see front matter © 2012 Elsevier B.V. All rights reserved.
doi:10.1016/j.topol.2012.01.010

Definition 2.1. Let X be a topological space and let $n > 0$.

1. An n-dimensional *orbifold chart* for an open subset V of X is a triple $(\tilde{V}, G, \varphi)$ such that
   (a) $\tilde{V}$ is a connected open subset of $\mathbb{R}^n$,
   (b) G is a finite group of homeomorphisms acting on $\tilde{V}$, let $\ker(G)$ denote the subgroup of G acting trivially on $\tilde{V}$.
   (c) $\varphi : \tilde{V} \to V$ is a G-invariant map inducing a homeomorphism from $\tilde{V}/G$ onto V.
2. If $V_i \subset V_j$, an *embedding* $(\lambda_{ij}, h_{ij}) : (\tilde{V}_i, G_i, \varphi_i) \to (\tilde{V}_j, G_j, \varphi_j)$ between two orbifold charts is
   (a) an injective homomorphism $h_{ij} : G_i \to G_j$, such that h_{ij} is an isomorphism from $\ker(G_i)$ to $\ker(G_j)$, and
   (b) an equivariant embedding $\lambda_{ij} : \tilde{V}_i \to \tilde{V}_j$ such that $\varphi_j \circ \lambda_{ij} = \varphi_i$. (The equivariantness means that $\lambda_{ij}(gx) = h_{ij}(g)\lambda_{ij}(x)$ for every $g \in G_i$ and every $x \in \tilde{V}_i$.)
3. An *orbifold atlas* on X is a family $\mathcal{V} = \{(\tilde{V}_i, G_i, \varphi_i)\}_{i\in I}$ of orbifold charts such that
   (a) $\{V_i\}_{i\in I}$ is a covering of X,
   (b) given two charts $(\tilde{V}_i, G_i, \varphi_i)$ and $(\tilde{V}_j, G_j, \varphi_j)$ and a point $x \in V_i \cap V_j$, there exists an open neighborhood $V_k \subset V_i \cap V_j$ of x and a chart $(\tilde{V}_k, G_k, \varphi_k)$ such that there are embeddings $(\lambda_{ki}, h_{ki}) : (\tilde{V}_k, G_k, \varphi_k) \to (\tilde{V}_i, G_i, \varphi_i)$ and $(\lambda_{kj}, h_{kj}) : (\tilde{V}_k, G_k, \varphi_k) \to (\tilde{V}_j, G_j, \varphi_j)$.
4. An atlas $\mathcal{U}$ is called a *refinement* of another atlas $\mathcal{W}$ if for every chart in $\mathcal{U}$ there exists an embedding into some chart of $\mathcal{W}$. Two orbifold atlases having a common refinement are called *equivalent*.

Definition 2.2. An n-dimensional *orbifold* is a paracompact Hausdorff space X equipped with an equivalence class of n-dimensional orbifold atlases.

The sets $V \in \mathcal{V}$ are called *basic open sets* in X.

An orbifold X is called *reduced*, if for every orbifold chart $(\tilde{V}, G, \varphi)$ of X, the group G acts effectively on $\tilde{V}$.

An orbifold is called a C^r-*orbifold*, $1 \leqslant r \leqslant \omega$ (where C^∞ means smooth and C^ω means real analytic), if each G_i acts via C^r-diffeomorphisms on $\tilde{V}_i$ and if each embedding $\lambda_{ij} : \tilde{V}_i \to \tilde{V}_j$ is differentiable of degree r.

An n-dimensional orbifold X is called *locally smooth*, if for each $x \in X$ there is an orbifold chart $(\tilde{U}, G, \varphi)$ with $x \in U = \varphi(\tilde{U})$ and such that the finite group G acts on $\tilde{U} \cong \mathbb{R}^n$ via an orthogonal representation.

By the differentiable slice theorem (see Proposition 2.2.2 in [11] and, for the real analytic case, Theorem 2.5 in [6]), every C^r-orbifold, $1 \leqslant r \leqslant \omega$, is locally smooth.

While it is well known that every reduced smooth orbifold is a smooth quotient orbifold, it is not known whether the corresponding real analytic result holds, i.e., whether every reduced real analytic orbifold is a real analytic quotient orbifold.

We assume that every orbifold has only countably many connected components. It follows that our orbifolds are second countable. Moreover, all orbifolds are paracompact, and for any orbifold, the dimension equals the covering dimension (i.e., the topological dimension).

In Section 6 we will make use of the following result, originally due to J. Milnor.

Theorem 2.3. *Let X be a paracompact space with covering dimension n and let $\{U_\alpha\}$ be an open cover of X. Then there is an open cover $\{O_{i\beta} \mid \beta \in B_i, i = 0, \dots, n\}$ of X refining $\{U_\alpha\}$ such that $O_{i\beta} \cap O_{i\beta'} = \emptyset$ if $\beta \neq \beta'$.*

Proof. [10], Theorem 1.8.2. □

In the proof of the previous theorem each set B_i is the set of unordered $(i+1)$-tuples from the indexing set of the U_α. Thus, if the indexing set of the U_α is countable, it follows that we can assume each B_i to be countable.

A map $f : X \to Y$ between two C^r-orbifolds is called a C^r *orbifold map* if for every $x \in X$ there are orbifold charts $(\tilde{U}, G, \varphi)$ and $(\tilde{V}, H, \psi)$, where $x \in U$ and $f(x) \in V$, a homomorphism $\theta : G \to H$ and a θ-equivariant C^r-map $\tilde{f} : \tilde{U} \to \tilde{V}$ such that the following diagram commutes:

$$\begin{array}{ccc} \tilde{U} & \xrightarrow{\tilde{f}} & \tilde{V} \\ \downarrow & & \downarrow \\ \tilde{U}/G & \longrightarrow & \tilde{V}/H \cdot \\ \downarrow & & \downarrow \\ U & \xrightarrow{f|U} & V \end{array}$$

A C^r-map $f : X \to Y$ is called a C^r-*diffeomorphism*, if it is a bijection and if the inverse map f^{-1} is a C^r-map.

3. Orbifold stratification

Let X be a C^r-orbifold, $1 \leqslant r \leqslant \omega$. Let $x \in X$ and let $(\tilde{V}, G, \varphi)$ and $(\tilde{U}, H, \psi)$ be orbifold charts such that $x \in \varphi(\tilde{V}) \cap \psi(\tilde{U})$. Let $\tilde{x} \in \tilde{V}$ and $\tilde{y} \in \tilde{U}$ be such that $x = \varphi(\tilde{x}) = \psi(\tilde{y})$. Let $G_{\tilde{x}}$ and $H_{\tilde{y}}$ be the isotropy subgroups at $\tilde{x}$ and $\tilde{y}$, respectively. Then $G_{\tilde{x}}$ and $H_{\tilde{y}}$ are isomorphic. It follows that it is possible to associate to every point $x \in X$ a finite group G_x, well-defined up to an isomorphism of groups, and called the *local group* of x.

For a finite group H, we let

$$X_{(H)} = \{x \in X \mid G_x \cong H\}.$$

The sets $X_{(H)}$ are called the *strata* of X.

We point out, that if X is a C^r quotient orbifold, i.e., the orbit space of a C^r-manifold by a proper almost free action of a Lie group G, then the local groups are defined not only up to isomorphism but up to conjugacy by an element in G.

4. Subanalytic subsets of real analytic orbifolds

For subanalytic subsets of real analytic manifolds, see for example [1] and [2]. Subanalytic subsets of real analytic orbifolds were introduced in [7]. We recall the definitions.

Definition 4.1. Let X be a real analytic orbifold. A subset A of X is called *subanalytic* if for every point x of X there is an orbifold chart $(\tilde{V}, G, \varphi)$ of X such that $x \in V = \varphi(\tilde{V})$ and $\varphi^{-1}(A \cap V)$ is a subanalytic subset of $\tilde{V}$.

Subanalytic orbifold maps are defined in the same way as C^r-maps, $1 \leqslant r \leqslant \omega$:

Definition 4.2. A map $f : X \to Y$ between two real analytic orbifolds is called *subanalytic* if for every $x \in X$ there are orbifold charts $(\tilde{U}, G, \varphi)$ and $(\tilde{V}, H, \psi)$, where $x \in U$ and $f(x) \in V$, a homomorphism $\theta : G \to H$ and a θ-equivariant subanalytic map $\tilde{f} : \tilde{U} \to \tilde{V}$ making the following diagram commute:

$$\begin{array}{ccc} \tilde{U} & \xrightarrow{\tilde{f}} & \tilde{V} \\ \downarrow & & \downarrow \\ \tilde{U}/G & \longrightarrow & \tilde{V}/H \cdot \\ \downarrow & & \downarrow \\ U & \xrightarrow{f|U} & V \end{array}$$

Lemma 4.3. *Let X be a real analytic orbifold. Then all the strata $X_{(H)}$ are subanalytic subsets of X.*

Proof. This follows from the fact that if G is a finite group acting real analytically on a real analytic manifold M, then the sets $\{x \in M \mid G_x = gHg^{-1}$, for some $g \in G\}$ are subanalytic subsets of M, see e.g. Lemma 3.2 in [9]. Then $\{x \in M \mid G_x \cong H\}$ is subanalytic as a finite union of such subanalytic sets. □

The following lemmas follow immediately from the corresponding results for subanalytic maps from a real analytic manifold to a Euclidean space, see e.g. Lemmas 4.25 and 4.28 in [6].

Lemma 4.4. *Let X be a real analytic orbifold and let $h : X \to \mathbb{R}$ and $f : X \to \mathbb{R}^n$, where $n \in \mathbb{N}$, be subanalytic maps. Then the product $hf : X \to \mathbb{R}^n$ is a subanalytic map. If $h(x) \neq 0$ for every $x \in X$, then the quotient $f/h : X \to \mathbb{R}^n$ is subanalytic.*

Let $\psi : X \to \mathbb{R}$ be a map. By the *support of* ψ, denoted by $\operatorname{supp}(\psi)$, we mean the closure of the set $\{x \in X \mid \psi(x) \neq 0\}$.

Lemma 4.5. *Let X be a real analytic orbifold and let $\psi_i : X \to \mathbb{R}$, $i \in \mathbb{N}$, be subanalytic maps such that $\{\operatorname{supp}(\psi_i)\}_{i \in \mathbb{N}}$ is locally finite. Then the map*

$$\psi : X \to \mathbb{R}, x \mapsto \sum_{i=1}^{\infty} \psi_i(x),$$

is subanalytic.

5. Subanalytic partitions of unity for orbifolds

In this section we prove the existence of subanalytic partitions of unity in the orbifold case.

Lemma 5.1. *Let $(\tilde{V}, G, \varphi)$ be a chart of a real analytic orbifold X. Let A and B be disjoint closed subsets of $V = \varphi(\tilde{V})$. Then there is a subanalytic map $f : V \to \mathbb{R}$ such that $f|A = 0$, $f|B = 1$ and $f(V) \subset [0, 1]$.*

Proof. It is well known (see e.g. Proposition 5.4 in [6]) that there is a G-invariant subanalytic map $\tilde{f} : \tilde{V} \to \mathbb{R}$ such that $\tilde{f}|\varphi^{-1}(A) = 0$, $\tilde{f}|\varphi^{-1}(B) = 1$ and $\tilde{f}(\tilde{V}) \subset [0, 1]$. This map induces a subanalytic map $f : V \to \mathbb{R}$ such that $f \circ \varphi = \tilde{f}$. Clearly, f has the desired properties. □

Definition 5.2. Let X be a real analytic orbifold. A *subanalytic partition of unity* is a collection $\{\lambda_i\}$ of subanalytic maps $X \to \mathbb{R}$ with the following properties:

1. $\lambda_i(x) \geqslant 0$ for every $x \in X$,
2. $\{\text{supp}(\lambda_i)\}$ is a locally finite cover of X, and
3. $\sum_i \lambda_i(x) = 1$ for every $x \in X$.

A partition $\{\lambda_i\}$ of unity is said to be *subordinate* to an open cover $\{U_j\}$ of X, if for every λ_i there is a U_j such that $\text{supp}(\lambda_i) \subset U_j$.

Theorem 5.3. *Let X be a real analytic orbifold and let $\mathcal{U}$ be an open cover of X. Then X has a subanalytic partition of unity subordinate to $\mathcal{U}$.*

Proof. Since X is paracompact, $\mathcal{U}$ has a locally finite refinement by basic open sets. Thus we may assume that $\mathcal{U}$ consists of basic open sets and it suffices to find a partition of unity subordinate to such $\mathcal{U}$. Since X is paracompact, $\mathcal{U} = \{U_j\}$ has locally finite refinements $\{W_j\}$ and $\{V_j\}$ by open sets W_j and V_j, respectively, such that $\bar{V}_j \subset W_j$ and $W_j \subset U_j$ for every j. According to Lemma 5.1, there exists for every j a subanalytic map $f_j : X \to \mathbb{R}$ such that $f_j(X) \subset [0, 1]$, f_j is identically one on $\bar{V}_j$ and $\text{supp}(f_j) \subset W_j$. Since $\{V_j\}$ is a cover of X, it follows that the maps

$$\lambda_i : X \to \mathbb{R}, x \mapsto \frac{f_i(x)}{\Sigma_j f_j(x)},$$

are well defined. The conditions (1), (2) and (3) of Definition 5.2 clearly hold. The maps λ_i are subanalytic by Lemmas 4.5 and 4.4. □

Notice that the maps λ_i in Theorem 5.3 are constructed in such a way that $\text{supp}(\lambda_i) \subset U_i$, for every i.

6. Triangulation theorem for real analytic orbifolds

We are now ready to prove the triangulation theorem for real analytic orbifolds. The corresponding result for orbit spaces is proved in Section 7 of [6]. We construct the triangulation by adapting the ideas in [6] to the orbifold case. To get started, we need a local result, which follows from a theorem of Matumoto and Shiota:

Theorem 6.1. *Let G be a compact Lie group and let M be a real analytic G-manifold. Let $\pi : M \to M/G$ be the natural projection. Then there exists a real analytic proper G-invariant map $f : M \to \mathbb{R}^n$, where $n = 2\dim(M) + 1$, such that the induced map $\bar{f} : M/G \to f(M)$ is a homeomorphism. Moreover, if another subanalytic set structure on M/G is given by an inclusion $j : M/G \to \mathbb{R}^p$, $p \in \mathbb{N}$, such that $j \circ \pi : M \to \mathbb{R}^p$ is a proper subanalytic map, then $j(M/G)$ and $f(M)$ are subanalytically homeomorphic.*

Proof. [9], Theorem 3.1. □

Recall that if M and N are real analytic manifolds, if $f : M \to N$ is a proper real analytic map and if A is a subanalytic subset of M, then the image $f(A)$ is a subanalytic set in N (Proposition 3.8 in [2]). Thus the map f in Theorem 6.1 really induces a subanalytic structure for M/G.

By a *subanalytic homeomorphism* we mean a subanalytic bijection whose inverse map is subanalytic. Assume $f : A \to B$ is a homeomorphism between subanalytic subsets A and B of two real analytic manifolds. If f is subanalytic, then the inverse map of f is automatically subanalytic and we call A and B *subanalytically homeomorphic*. The orbifold case differs form the manifold case, since a map between two orbifolds that is both subanalytic and a homeomorphism does not need to be a subanalytic homeomorphism. The reason is that the inverse map does not need to be an orbifold map, i.e., it does not necessarily have the local lifts as in Definition 4.2.

Lemma 6.2. *Let X be a real analytic orbifold. Then there exists a subanalytic map $f_0 : X \to \mathbb{R}^q$, for some $q \in \mathbb{N}$, such that f_0 is a homeomorphism onto $f_0(X)$.*

Proof. Since orbifolds are second countable, it follows that the orbifold X can be covered by basic open sets V_i, $i \in \mathbb{N}$. By Theorem 6.1, for every orbifold chart $(\tilde{V}_i, G_i, \varphi_i)$, there is a real analytic proper G_i-invariant map $\tilde{f}_i : \tilde{V}_i \to \mathbb{R}^s$, for some $s \in \mathbb{N}$, such that the induced map $\bar{f}_i : V_i \to \tilde{f}_i(\tilde{V}_i)$ is a homeomorphism. We can choose $s = 2\dim(X) + 1$.

By Theorem 2.3, $\{V_i\}_{i=1}^{\infty}$ has an open locally finite refinement $\{O_{j\beta} \mid \beta \in B_j, j = 1, \dots, k\}$, such that $O_{j\beta} \cap O_{j\beta'} = \emptyset$ if $\beta \neq \beta'$. As pointed out after Theorem 2.3, we may assume that each $B_j \subset \mathbb{N}$. For every j we denote $\bigcup_{\beta \in B_j} O_{j\beta}$ by O_j. For every j we define a map $f_j : O_j \to \mathbb{R}^{s+1}$, by setting $f_j(y) = (\bar{f}_{i_0}(y), \beta)$ if $y \in O_{j\beta}$ and i_0 is the smallest i for which $O_{j\beta} \subset V_i$. Then f_j is real analytic. Clearly, f_j is an injection.

Since X is paracompact, we can choose open covers $\{W_{j\beta} \mid \beta \in B_j, j = 1, \dots, k\}$, $\{U_{j\beta} \mid \beta \in B_j, j = 1, \dots, k\}$ and $\{Y_{j\beta} \mid \beta \in B_j, j = 1, \dots, k\}$ of X by open sets $W_{j\beta}$, $U_{j\beta}$ and $Y_{j\beta}$, respectively, such that $\bar{U}_{j\beta} \subset W_{j\beta}$, $\bar{W}_{j\beta} \subset Y_{j\beta}$ and $\bar{Y}_{j\beta} \subset O_{j\beta}$ for every j and β. We write $W_j = \bigcup_{\beta \in B_j} W_{j\beta}$, $U_j = \bigcup_{\beta \in B_j} U_{j\beta}$ and $Y_j = \bigcup_{\beta \in B_j} Y_{j\beta}$ for every j. Let $h_j : X \to [0, 1]$ be a subanalytic map which is identically one on $\bar{U}_j$ and vanishes outside W_j. Similarly, let $h'_j : X \to [0, 1]$ be a subanalytic map which is identically one on $\bar{W}_j$ and vanishes outside Y_j. Clearly, the maps

$$f_{0j} : X \to \mathbb{R}^{s+1}, \qquad f_{0j}(y) = \begin{cases} h'_j(y) f_j(y), & \text{if } y \in O_j \\ 0, & \text{if } y \in X \setminus O_j \end{cases}$$

are subanalytic. Let

$$f_0 : X \to \mathbb{R}^p, \qquad y \mapsto \big(h_1(y), \dots, h_k(y), f_{01}(y), \dots, f_{0k}(y)\big),$$

where $p = k(s + 2)$. Then f_0 is subanalytic by Proposition 8.7 in [7].

Since the maps f_i are embeddings, it follows that the restriction $f_{0j}|W_{j\beta}$ is an embedding for every β. Let $(x_d)_{d=1}^{\infty}$ be a sequence in X such that $f_0(x_d) \to f_0(x)$, for some $x \in X$. Then $x \in U_{j\beta}$ for some j and β and $h_j(x) = 1$. Since $h_j(x_d) \to h_j(x)$, it follows that $h_j(x_d) > 0$ for sufficiently large d. Therefore $x_d \in W_{j\beta}$ for sufficiently large d. Since $f_{0j}|W_{j\beta}$ is an embedding and $f_{0j}(x_d) \to f_{0j}(x)$, it follows that $x_d \to x$. Consequently, f_0 is injective and the inverse map $f_0^{-1} : f_0(X) \to X$ is continuous. □

Theorem 6.3. *Let X be a real analytic orbifold. Then there exists a proper subanalytic map $f : X \to \mathbb{R}^n$ such that the induced map $X \to f(X)$ is a homeomorphism. Thus $f(X)$ is a closed subanalytic subset of $\mathbb{R}^n$. If $g : X \to \mathbb{R}^p$ is any proper subanalytic map that also is a topological embedding, then $g \circ f^{-1} : f(X) \to g(X)$ is a subanalytic homeomorphism.*

Proof. Let $\{V_i\}_{i=1}^{\infty}$ be a cover of X by basic open sets. We may assume that the closure of each V_i is compact. By Theorem 5.3, there is a subanalytic partition of unity $\{\lambda_i\}$ subordinate to $\{V_i\}_{i=1}^{\infty}$. Let

$$\lambda : X \to \mathbb{R}, \qquad x \mapsto \sum_{i=1}^{\infty} 2^{-i} \lambda_i(x).$$

By Lemma 4.5, λ is subanalytic. Clearly, $0 < \lambda(x) < 1$, for every $x \in X$. By Lemma 6.2, there is a subanalytic map $f_0 : X \to \mathbb{R}^q$, for some $q \in \mathbb{N}$, such that f_0 is a homeomorphism onto the image $f_0(X)$. Let

$$f_1 : X \to \mathbb{R}^{q+1}, \qquad x \mapsto \big(f_0(x), 1\big),$$

and let

$$f : X \to \mathbb{R}^{q+1}, \qquad x \mapsto \frac{f_1(x)}{\lambda(x)}.$$

Then f is subanalytic. Since f_0 is an embedding, it follows that also f is an embedding.

We show that $f(X)$ is closed in $\mathbb{R}^{q+1}$. If the contrary is true, there is a $y \in \overline{f(X)} \setminus f(X)$. Let $(x_j)_{j=1}^{\infty}$ be a sequence in X such that $f(x_j) \to y$. Then $(\frac{1}{\lambda(x_j)})_{j=1}^{\infty}$ converges to the last coordinate y_{q+1} of y in $\mathbb{R}^{q+1}$. Since $0 < \lambda(x) < 1$ for every $x \in X$, it follows that $y_{q+1} > 0$. Then $f_1(x_j) \mapsto \frac{y}{y_{q+1}}$. Assume first $\frac{y}{y_{q+1}} \notin f_1(X)$ and let $\{U_m\}_{m=1}^{\infty}$ be a neighborhood basis of $\frac{y}{y_{q+1}}$. Since the maps λ_i have compact supports, the sets $U'_m = U_m \setminus \bigcup_{i=1}^{m} f_1(\operatorname{supp}(\lambda_i))$ also form a neighborhood basis of $\frac{y}{y_{q+1}}$. For every m there exists an $x_{j_m} \in \{x_j\}_{j=1}^{\infty}$ such that $f_1(x_{j_m}) \in U'_m$. We may choose $j_{m+1} > j_m$ for every m. Then $x_{j_m} \notin \bigcup_{i=1}^{m} \operatorname{supp}(\lambda_i)$. Thus $\lambda(x_{j_m}) \to 0$, which is impossible. It follows that $\frac{y}{y_{q+1}} \in f_1(X)$. Thus $\frac{y}{y_{q+1}} = f_1(x)$ for some $x \in X$. Consequently, $f_0(x_j) \mapsto f_0(x)$. Since f_0 is an embedding, it follows that $x_j \mapsto x$. Thus $\lambda(x_j) \mapsto \lambda(x)$ and $\lambda(x) = \frac{1}{y_{q+1}}$.

Therefore, $y = \frac{f_1(x)}{\lambda(x)} = f(x)$, and it follows that $f(X)$ is closed. Thus f is a proper subanalytic map and it follows from Theorem 9.3 in [7] that $f(X)$ is a closed subanalytic subset of $\mathbb{R}^{q+1}$.

It remains to show that the subanalytic structure on X is unique. Therefore, let $g : X \to \mathbb{R}^p$ be a proper subanalytic map that also is a topological embedding, where $p \in \mathbb{N}$. Then $g \circ f^{-1} : f(X) \to g(X)$ is a homeomorphism. Clearly, the map $(f, g) : X \to \mathbb{R}^{q+1} \times \mathbb{R}^p$ is subanalytic and proper. Theorem 9.3 in [7] indicates that the graph $\mathrm{Gr}(g \circ f^{-1}) = (f, g)(X)$ is a subanalytic subset of $\mathbb{R}^{q+1} \times \mathbb{R}^p$. Thus $g \circ f^{-1}$ is a subanalytic map. □

Let K be a simplicial complex. We denote by $|K|$ the space of K. The corresponding open simplex of any simplex $\sigma \in K$ is denoted by $\mathrm{int}\,\sigma$. The space of any countable locally finite simplicial complex K admits a linear embedding as a closed subset of some Euclidean space $\mathbb{R}^n$, see Theorem 3.2.9 in [13]. If $e : |K| \to \mathbb{R}^n$ is a closed linear embedding, then the image $e(|K|)$ is a subanalytic subset of $\mathbb{R}^n$. If $e_* : |K| \to \mathbb{R}^m$ is another closed linear embedding, then $e_* \circ e^{-1} : e(|K|) \to e_*(|K|)$ is a subanalytic homeomorphism. Hence the closed linear embeddings induce a unique subanalytic structure on $|K|$.

Definition 6.4. Let X be a real analytic orbifold. A *subanalytic triangulation* of X is a pair of a simplicial complex K and a homeomorphism $\tau : |K| \to X$ such that the inverse map $\tau^{-1} : X \to |K|$ is subanalytic. We say that the triangulation is *compatible* with the family $\{X_i\}$ of subsets of X, if each X_i is a union of some $\tau(\mathrm{int}\,\sigma)$, where $\sigma \in K$.

Theorem 6.5. *Let $\{X_i\}$ be a locally finite family of subanalytic subsets in $\mathbb{R}^n$ which are contained in a subanalytic closed set X in $\mathbb{R}^n$. Then there exists a locally finite simplicial complex K and a subanalytic homeomorphism $\tau : |K| \to X$ such that each X_i is a union of some $\tau(\mathrm{int}\,\sigma)$, where $\sigma \in K$.*

Proof. [3], Theorem on p. 180. □

Theorem 6.6. *Let X be a real analytic orbifold. Then X has a subanalytic triangulation compatible with the strata of X.*

Proof. Let $f : X \to \mathbb{R}^n$ be as in Theorem 6.3. The strata $X_{(H)}$ are subanalytic in X, by Lemma 4.3. Clearly, they form a locally finite family in X. Since f is proper and subanalytic, it follows that the sets $f(X_{(H)})$ are subanalytic in $\mathbb{R}^n$ and that they form a locally finite family. By Theorem 6.5, there exists a simplicial complex K and a subanalytic homeomorphism $\tau : |K| \to f(X)$ such that each $f(X_{(H)})$ is a union of some $\tau(\mathrm{int}\,\sigma)$, where $\sigma \in K$. Thus $f^{-1} \circ \tau : |K| \to X$ is a homeomorphism and each $X_{(H)}$ is a union of some $f^{-1} \circ \tau(\mathrm{int}\,\sigma)$, where $\sigma \in K$. The inverse map $\tau^{-1} \circ f$ of $f^{-1} \circ \tau$ is subanalytic since τ^{-1} and f are subanalytic and τ^{-1} is proper, see Corollary 9.4 in [7]. □

The subanalytic triangulation of X is unique in the sense that if $\tau_1 : |K_1| \to X$ and $\tau_2 : |K_2| \to X$ are two subanalytic triangulations of X, then $\tau_1^{-1} \circ \tau_2 : |K_2| \to |K_1|$ is a subanalytic homeomorphism by Theorem 6.3. Moreover, by Corollary 4.3 in [12], the subanalytic triangulation is unique up to a PL homeomorphism.

7. Compatible differential structures

By a C^r-*differential structure* on an orbifold X we mean a maximal C^r-atlas α on X. A C^s-differential structure β on X, $s > r$, is called *compatible* with α, if $\beta \subset \alpha$. In this case, every chart on β is a chart on α. Equivalently, it means that the identity map of X is a C^r-orbifold diffeomorphism $X(\alpha) \to X(\beta)$.

Let M be a C^k-manifold, $1 \leqslant k \leqslant \omega$. If a Lie group G acts on M via a C^k-action, we call M a C^k-G-manifold. The following results are needed to prove Theorem 7.4:

Theorem 7.1. *Let G be a finite group and let M and N be C^k-G-manifolds, $2 \leqslant k \leqslant \omega$. Then any C^r-differentiable G-equivariant map $M \to N$, $1 \leqslant r < k$, can be approximated arbitrarily well in the strong C^r-topology by a C^k-differentiable G-equivariant map.*

Proof. A special case of Theorem 1.2 in [9]. □

Theorem 7.2. *Let G be a compact Lie group and let M be a C^r-G-manifold, $1 \leqslant r \leqslant \infty$. Then, there is a C^k-G-manifold $\tilde{M}$ which is C^r G-equivariantly diffeomorphic to M, $r < k \leqslant \omega$.*

Proof. Theorem 1.3 in [9]. □

Let $1 \leqslant r \leqslant \infty$, and let M and N be C^r-G-manifolds, where G is a finite group. We denote the set of C^r-differentiable G-equivariant maps $M \to N$ equipped with the strong, i.e., the Whitney topology, by $\mathrm{C}^r_{G,\mathrm{S}}(M, N)$. For $r = \infty$, we denote the set of C^∞-differentiable G-equivariant maps $M \to N$ equipped with the Cerf topology (a topology finer than the Whitney topology) by $\mathrm{C}^\infty_{G,\mathrm{C}}(M, N)$. Then $\mathrm{C}^\omega(M, N)$ is dense in $\mathrm{C}^\infty_{\mathrm{C}}(M, N)$ and, consequently, $\mathrm{C}^\omega_G(M, N)$ is dense in $\mathrm{C}^\infty_{G,\mathrm{C}}(M, N)$, for finite G.

Lemma 7.3. *Let U and V be open sets in $\mathbb{R}^n$, for some $n \in \mathbb{N}$. Assume a finite group G acts C^r-differentiably both on U and on V, $1 \leqslant r < \infty$. Let W be an open G-invariant subset of U. Let $f : U \to V$ be a C^r-differentiable G-equivariant map with $V' = f(W)$ open. Then the restriction $f|W$ has a neighborhood $\mathcal{N}$ in $C^r_{G,S}(W, V')$ such that if $g_0 \in \mathcal{N}$, then the map*

$$T(g_0) = g : U \to V,$$

where

$$g(x) = g_0(x), \quad \text{if } x \in W \quad \text{and} \quad g(x) = f(x) \quad \text{if } x \in U \setminus W,$$

is a C^r-differentiable G-equivariant map, and $T : \mathcal{N} \to C^r_{G,S}(U, V)$, $g_0 \mapsto T(g_0)$, is continuous.

Proof. It is clear that $T(g_0)$ is equivariant when g_0 is equivariant. The rest of the claims follow immediately from Lemma 2.2.8 in [4]. □

According to Theorem 2.2.9 in [4], every C^r-manifold, $1 \leqslant r < \infty$, has a compatible C^s-differential structure, where $r < s \leqslant \infty$. We follow Hirsch's proof to prove the corresponding result for orbifolds:

Theorem 7.4. *Let α be a C^r-differential structure on the orbifold X, $r \geqslant 1$. For every s, $r < s \leqslant \infty$, there exists a compatible C^s-differential structure $\beta \subset \alpha$ on X.*

Proof. Let $\mathcal{B}$ denote the family of all pairs (B, β), where B is an open subset of X and β is a C^s-structure on B, compatible with the C^r-structure B inherited from X. By Lemma 7.2, the basic open sets of X have a compatible C^s-structure. Thus $\mathcal{B}$ is not empty.

Define

$$(B_1, \beta_1) \leqslant (B_2, \beta_2)$$

if and only if:

1. $B_1 \subset B_2$.
2. The C^s-structure β_1 on B_1 is the one induced from the C^s-structure β_2 on B_2.

Then $\leqslant$ defines an order in $\mathcal{B}$. Let $\mathcal{C}$ be a chain in $\mathcal{B}$. We denote by $\mathcal{C}_1$ the family of all B occurring as the first coordinate of a pair in $\mathcal{C}$, and by $\mathcal{C}_2$ the family of all β occurring as the second coordinate of a pair in $\mathcal{C}$. Let

$$B^* = \bigcup_{B \in \mathcal{C}_1} B, \quad \text{and} \quad \beta' = \bigcup_{\beta \in \mathcal{C}_2} \beta.$$

Then B^* is an open subset of X and β' is a C^s-atlas on B^*, compatible with the C^r-structure on B^*. Let β^* be the maximal C^s-atlas on B^* generated by β'. Then (B^*, β^*) is an upper bound for $\mathcal{C}$ in $\mathcal{B}$. It now follows from Zorn's lemma, that $\mathcal{B}$ has a maximal element (B, β). We show that $B = X$.

Assume that $B \neq X$. Then there is a chart $(\tilde{U}, G, \varphi)$ of X such that $U \cap (X \setminus B) \neq \emptyset$, where $U = \varphi(\tilde{U})$. We have that $U \cap B \neq \emptyset$, since $U \cap B = \emptyset$ would contradict the maximality of B. By Theorem 7.2, we can assume there is an open set $\hat{U}$ of $\mathbb{R}^n$, where $n = \dim(X)$, on which G acts C^s-differentiably, and a G-equivariant C^r-diffeomorphism $f : \hat{U} \to \tilde{U}$. Put $W' = U \cap B$, $\tilde{W}' = \varphi^{-1}(W')$ and $W = f^{-1}(\tilde{W}')$.

There now are two differential structures on W': the C^r-structure α and the compatible C^s-structure $\beta \subset \alpha$. We shall find a G-equivariant C^r-diffeomorphism $\theta : \hat{U} \to \tilde{U}$ such that the restriction $\theta|W : W \to \tilde{W}'$ is a C^s-diffeomorphism. Then θ induces a C^r-diffeomorphism $\theta' : \hat{U}/G \to U$ such that $\theta'|W/G : W/G \to W'$ is a C^s-diffeomorphism. In that case the chart $(\hat{U}, G, \theta' \circ \pi)$, where $\pi : \hat{U} \to \hat{U}/G$ is the natural projection, has C^s overlap with β. The C^s-atlas $\beta \cup (\hat{U}, G, \theta' \circ \pi)$ on $B \cup U$ is contained in α, which contradicts the maximality of (B, β).

To construct θ, we use Lemma 7.3 to obtain a neighborhood $\mathcal{N} \subset C^r_{G,S}(W, \tilde{W}')$ of $f|W : W \to \tilde{W}'$ with the following property: Whenever $g_0 \in \mathcal{N}$, the map $T(g_0) = g : \hat{U} \to \tilde{U}$ defined by

$$g(x) = g_0(x), \quad \text{if } x \in W \quad \text{and} \quad g(x) = f(x) \quad \text{if } x \in \hat{U} \setminus W,$$

is C^r-differentiable and G-equivariant, and the resulting map

$$T : \mathcal{N} \to C^r_{G,S}(\hat{U}, \tilde{U})$$

is continuous. The set $\mathrm{Diff}^r_G(\hat{U}, \tilde{U})$ of G-equivariant C^r-diffeomorphisms is open in $C^r_{G,S}(\hat{U}, \tilde{U})$. Since $T(f|W)$ is the diffeomorphism f, there is a neighborhood $\mathcal{N}_0 \subset \mathcal{N}$ of $f|W$ such that $T(\mathcal{N}_0) \subset \mathrm{Diff}^r_G(\hat{U}, \tilde{U})$. Now, by Theorem 7.1, there is a G-equivariant C^s-diffeomorphism $\theta_0 \in \mathcal{N}_0$. The required map θ is then $T(\theta_0)$. □

We point out that the result of Theorem 7.4 is true, with the same proof, for $1 \leqslant r < \infty$ and $r < s \leqslant \omega$.

Notice that for C^r-differentiable quotient orbifolds the existence of a compatible C^s-differential structure follows immediately from the corresponding equivariant results.

8. Triangulation theorem for differentiable orbifolds

In this section we show that every smooth orbifold has a compatible real analytic structure. Lemma 8.1 is similar to Lemma 7.3 with one difference: In Lemma 7.3, the topology in the space of C^r-differentiable maps, $1 \leqslant r < \infty$, is the *Whitney* C^r*-topology*. A corresponding result for gluing C^∞-maps does not hold in the Whitney C^∞-topology. That's why the space of C^∞-maps in Lemma 8.1 must be equipped with a finer topology, the *Cerf topology*.

Lemma 8.1. *Let G be a compact Lie group and let M and N be smooth G-manifolds. Let $f : M \to N$ be a C^∞-differentiable G-equivariant map and let U be a G-invariant open subset of M. Then there exists an open neighborhood $\mathcal{N}$ of $f|U$ in $C^\infty_{G,C}(U, N)$ such that the following holds: If $h \in \mathcal{N}$ and $T(h) : M \to N$ is defined by*

$$T(h)(x) = h(x), \quad \text{if } x \in U \quad \text{and} \quad T(h)(x) = f(x), \quad \text{if } x \in M \setminus U,$$

then $T(h)$ is a C^∞-differentiable G-equivariant map. Furthermore, $T : \mathcal{N} \to C^\infty_{G,C}(M, N)$, $h \mapsto T(h)$, is continuous.

Proof. Lemma 8.1 in [5]. □

Theorem 8.2. *Let α be a C^∞-differential structure on the orbifold X. Then there exists a compatible real analytic differential structure $\beta \subset \alpha$ on X.*

Proof. The proof is similar to the proof of Theorem 7.4. The reference to Lemma 7.3 should be replaced by a reference to Lemma 8.1. □

Theorems 7.4 and 8.2 imply:

Corollary 8.3. *Let α be a C^1-differential structure on the orbifold X. Then there exists a compatible real analytic structure $\beta \subset \alpha$ on X.*

In other words:

Theorem 8.4. *Let $1 \leqslant r \leqslant \infty$. Then every C^r-orbifold is C^r-diffeomorphic to a C^ω-orbifold.*

Let X be a C^r-orbifold, $1 \leqslant r \leqslant \infty$. If there is a real analytic orbifold Y and a C^r orbifold diffeomorphism $f : Y \to X$, then a subanalytic triangulation of Y induces a triangulation of X. We call such a triangulation of X "subanalytic".

Theorem 8.5. *Let X be a C^r-orbifold, $1 \leqslant r \leqslant \infty$. Then X has a "subanalytic" triangulation.*

Proof. The result follows immediately from Theorems 8.4 and 6.6. □

References

[1] E. Bierstone, P.D. Milman, Semianalytic and subanalytic sets, Inst. Hautes Études Sci. Publ. Math. 67 (1988) 5–42.
[2] H. Hironaka, Subanalytic sets, in: Number Theory, Algebraic Geometry and Commutative Algebra, in Honor of Y. Akizuki, Kinokuniya, Tokyo, 1973, pp. 454–493.
[3] H. Hironaka, Triangulation of algebraic sets, in: Proc. Sympos. Pure Math., vol. 29, Amer. Math. Soc., 1975, pp. 165–185.
[4] M.W. Hirsch, Differential Topology, Springer-Verlag, New York–Berlin, 1976.
[5] S. Illman, The very strong C^∞ topology on $C^\infty(M, N)$ and K-equivariant maps, Osaka J. Math. 40 (2003) 409–428.
[6] M. Kankaanrinta, Proper real analytic actions of Lie groups on manifolds, Ann. Acad. Sci. Fenn. Ser. A I Math. Diss. 83 (1991) 1–41.
[7] M. Kankaanrinta, On subanalytic subsets of real analytic orbifolds, http://arxiv.org/abs/1104.4653, submitted for publication.
[8] T. Matumoto, M. Shiota, Proper subanalytic transformation groups and unique triangulation of the orbit spaces, in: Lecture Notes in Math., vol. 1217, Springer-Verlag, Berlin–Heidelberg, 1986, pp. 290–302.
[9] T. Matumoto, M. Shiota, Unique triangulation of the orbit space of a differentiable transformation group and its applications, in: Adv. Stud. Pure Math., vol. 9, 1986, pp. 41–55.
[10] R.S. Palais, The classification of G-spaces, Mem. Amer. Math. Soc. 36 (1960).
[11] R.S. Palais, On the existence of slices for actions of non-compact Lie groups, Ann. of Math. (2) 73 (1961) 295–323.
[12] M. Shiota, M. Yokoi, Triangulations of subanalytic sets and locally subanalytic manifolds, Trans. Amer. Math. Soc. 286 (1984) 727–750.
[13] E.H. Spanier, Algebraic Topology, Springer-Verlag, New York–Heidelberg–Berlin, 1966.