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DOMINATING CLIQUES IN GRAPHS

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A set of vertices is a *dominating set* in a graph if every vertex not in the dominating set is adjacent to one or more vertices in the dominating set. A *dominating clique* is a dominating set that induces a complete subgraph. Forbidden subgraph conditions sufficient to imply the existence of a dominating clique are given. For certain classes of graphs, a polynomial algorithm is given for finding a dominating clique. A forbidden subgraph characterization is given for a class of graphs that have a connected dominating set of size three.

Introduction

A set of vertices in a simple, undirected graph is a *dominating set* if every vertex in the graph which is not in the dominating set is adjacent to one or more vertices in the dominating set. The *domination number* of a graph G is the minimum number of vertices in a dominating set. Several types of dominating sets have been investigated, including independent dominating sets, total dominating sets, and connected dominating sets, each of which has a corresponding domination number. Dominating sets have applications in a variety of fields, including communication theory and political science. For more background on dominating sets see [3, 5, 15] and other articles in this issue.

For arbitrary graphs, the problem of finding the size of a minimum dominating set in the graph is an NP-complete problem [9]. The dominating set problem remains NP-complete even for some specific classes of graphs, including chordal graphs [2], split graphs and bipartite graphs [7, 1]. The problem remains NP-complete for some types of graphs even when the type of domination is extended. The total dominating set problem is NP-complete for bipartite graphs [17]. The connected domination problem has been shown to be NP-complete for arbitrary graphs [9] and for bipartite graphs [17].

However, there are classes of graphs for which there exist linear algorithms to locate a minimum cardinality dominating set. Cockayne, Goodman and Hedetniemi [4] presented such an algorithm for trees and this has been generalized by Natarajan and White [16] to weighted trees. Booth and Johnson [2], in investigating chordal graphs, have presented a linear algorithm for locating a minimum dominating set in directed path graphs (which include interval graphs), given an appropriate path representation. Pfaff, Laskar and Hedetniemi [18] have presented a linear algorithm for the total domination problem in series-parallel graphs.

A *complete subgraph* or a *clique* is an induced subgraph such that there is an edge between each pair of vertices in the subgraph. In this paper, characterizations of classes of graphs that contain dominating sets that induce a complete subgraph are given in terms of forbidden subgraphs. For a certain class of graphs, a polynomial time algorithm is given for finding a dominating set that induces a complete subgraph.

Dominating sets that induce a complete subgraph have a great diversity of applications. In setting up the communications links in a network one might want a strong core group that can communicate with each other member of the core group and so that everyone outside the group could communicate with someone within the core group. A group of forest fire sentries that could see various sections of a forest might also be positioned in such a way that each could see the others in order to use triangulation to locate the site of a fire. In addition, the properties of dominating sets are useful in identifying structural properties of a social network [13, 14] and in computing the threshold dimension of certain classes of graphs [6].

Clique dominated graphs

A *clique dominating set* or a *dominating clique* is a dominating set that induces a complete subgraph. A *clique dominated graph* is a graph that contains a dominating clique.

The smallest size dominating clique possible in a graph would be a single vertex. Wolk [19] presents a forbidden subgraph characterization of a class of graphs which have a dominating clique of size one. He called such a dominating clique a *central vertex* or *central point.* In the following theorem and throughout this paper the notation P_n denotes the path on n distinct vertices and C_n denotes the cycle on n vertices.

Theorem 1 (Wolk [19]). *If G is a finite connected graph with no induced P_4 or C_4, then G has a dominating vertex.*

This theorem can be extended to get forbidden subgraph conditions sufficient to imply the existence of a dominating set that induces a complete subgraph, a dominating clique. This is presented in the next theorem.

Theorem 2. *If G is a finite graph that is connected and has no induced P_5 or C_5, then G has a dominating clique.*

Proof. By induction on n, the number of vertices in G.

(i) The proposition is clearly true for $n = 1$.

(ii) Assume that any finite connected graph with n vertices, $n \geq 1$, that has no induced P_5 or C_5 has a dominating clique. Let G be a finite graph with $n+1$ vertices, $n \geq 1$, that is connected and has no induced P_5 or C_5. Let v be a vertex of G that is not a cutpoint. Such a vertex exists [11]. Let G' be the subgraph of G induced by all vertices of G except v. Since G' is a finite graph with n vertices that is connected and has no induced P_5 or C_5 it has, by the induction hypothesis, a dominating set that induces a clique. Let K' be a dominating set of G' that induces a clique. In G, if v is adjacent to any vertex in K', then K' will also be a dominating set of G that induces a clique.

Suppose that in G, v is not adjacent to any vertex in K'. Since G is connected, v must be adjacent to some vertex of G. Let x be any vertex of G that is adjacent to v.

Let $K = \{x\} \cup (N(x) \cap K')$. It will be shown that K is a dominating set of G that induces a clique. By construction K induces a clique. Suppose K is not a dominating set of G. Then there must be a vertex u that is not adjacent to any vertex in K. However, since K' is a dominating set of G', u must be adjacent to some vertex in K'. Let a be a vertex in K' that is adjacent to u. Let b be a vertex in K other than x. Such a vertex exists since x itself is not in K' but x must be adjacent to some vertex in K'. See Fig. 1. If u is not adjacent to v then v–x–b–a–u is an induced P_5, a contradiction to the assumption that G has no induced P_5. If u is adjacent to v then v–x–b–a–u is an induced C_5, a contradiction to the assumption that G has no induced C_5.

Therefore, G has a dominating set that induces a clique. □

It should be clear that the converse of Theorem 2 is not true. For example, the graph in Fig. 2 has a dominating clique of size one and an induced P_5.

The following theorem establishes a relationship between the forbidden subgraph conditions sufficient for a graph to contain a dominating clique and the size of a dominating clique in the graph. The notation K_{n+p} denotes the complete

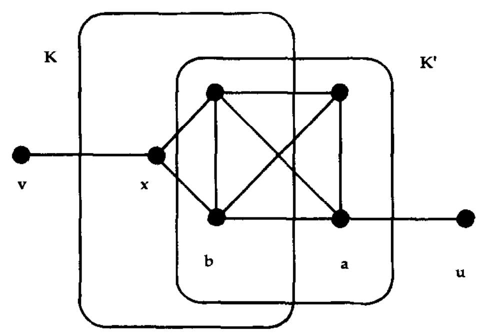

Fig. 1.

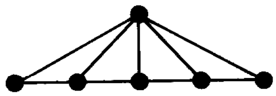

Fig. 2.

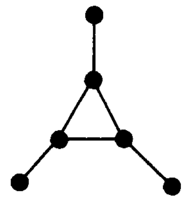

Fig. 3.

graph on n vertices with n pendants, one at each vertex of the complete graph. K_{3+p} is shown in Fig. 3.

Theorem 3. *If G is a finite graph that is connected and has no induced* P_5, C_5 *or* $K_{(k+1)+p}$, $k \geq 2$, *then G has a dominating clique of size* $\leq k$.

Proof. By Theorem 2, G has a dominating clique. Let K be a minimum dominating clique of G, and let $m =$ the size of k. If $m \leq k$, then K is a dominating clique of G of size $\leq k$. Suppose $m > k$. Since K is a minimum dominating clique, each vertex, x_i, $1 \leq i \leq m$, in K must be adjacent to at least one vertex, y_i, that is not in K and that is not adjacent to any other vertex in K. Let this set of vertices y_i be called S.

Case 1. At least $k+1$ of the vertices in S form an independent set. Then these $k+1$ vertices in S together with their neighbors in K form an induced $K_{(k+1)+p}$, a contradiction.

Case 2. S does not contain $k+1$ independent vertices but S does not induce a complete subgraph. Let y_1 and y_2 be vertices in S that are not adjacent to each other and let y_3 be a vertex in S that is not adjacent to both y_1 and y_2. By symmetry it is sufficient to consider that y_3 is not adjacent to y_1. Then y_3–x_3–x_2–y_2–y_1 is an induced P_5 or C_5, a contradiction. See Fig. 4.

Case 3. S induces a complete subgraph. Since $\{x_1, y_1\}$ is not a dominating edge of the graph, there must be a vertex, say z, that is not adjacent to either x_1 or y_1. However, since K is a dominating clique, z must be adjacent to some vertex, say x_2, in K. Then z–x_2–x_1–y_1–y_3 is an induced P_5 *or* C_5, a contradiction. See Fig. 5.

Therefore, G has a dominating clique of size $\leq k$. □

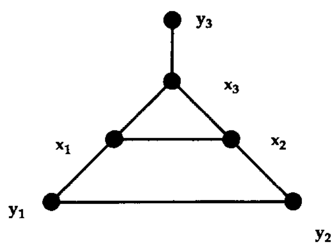

Fig. 4.

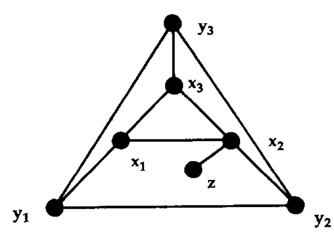

Fig. 5.

We can now establish conditions under which a graph must have a dominating clique of size two, a dominating edge.

Corollary 3.1. *If G is a finite graph with two or more vertices that is connected and has no induced P_5, C_5 or K_{3+p} then G has a dominating edge.*

Proof. By Theorem 3, with $k = 2$, G has a dominating clique of size one or two. Since any edge containing a dominating vertex is a dominating edge, G has a dominating edge. □

A *bipartite graph* is a graph whose vertex set can be partitioned into two subsets V_1 and V_2 such that every edge of G joins a vertex from V_1 with a vertex from V_2. A *split graph* is a graph such that there is a partition of the vertex set into a complete graph and an independent set. There is no restriction on edges between vertices of the complete graph and the independent set. The following corollaries relate Theorem 3, and particularly Corollary 3.1, to these well-known classes of graphs.

Corollary 3.2. *If G is a connected bipartite graph that does not contain an induced P_5 then G has a dominating edge.*

Proof. Since all of the cycles of a bipartite graph are even [11], a bipartite graph cannot have an induced C_5 or K_{3+p}. Therefore, by Corollary 3.1, G has a dominating edge. □

Corollary 3.3. *If G is a connected split graph that does not contain an induced K_{3+p} then G has a dominating edge.*

Proof. Since a split graph contains no induced $2K_2$, C_4 or C_5 [8], by Corollary 3.1, a connected split graph that does not contain an induced K_{3+p} has a dominating edge. □

Parameters

The *diameter* of a connected graph is the maximum possible distance between any two vertices of the graph. The diameter of any clique dominated graph is less than or equal to three. For graphs that have a dominating clique a parameter similar to those previously defined for various types of dominating sets can be associated with the size of a minimum dominating clique. For a graph G, $\beta(G)$ denotes the domination number of the graph, the size of a minimum dominating set, and $i(G)$ denotes the cardinality of a minimum independent dominating set of the graph. The connected domination number of a graph G is denoted by

$\beta_c(G)$ and the total domination number of a graph that has a total dominating set is denoted by $\beta_t(G)$. If a dominating clique exists in a graph G, let $\beta_k(G)$ denote the *clique domination number,* the cardinality of a minimum dominating clique of the graph G. Some elementary properties of the clique domination number can now be presented.

Property 1. *If G is a clique dominated graph then $\beta(G) \leq \beta_c(G) \leq \beta_k(G)$.*

Proof. Since every dominating clique is a connected dominating set it follows that the size of the smallest connected dominating set is less than or equal to the size of the smallest dominating clique. The size of the smallest dominating set is, in turn, less than or equal to the size of the smallest connected dominating set. □

Property 2. *If G is a clique dominated graph with p vertices and maximum degree $<p-1$, then $\beta(G) \leq \beta_t(G) \leq \beta_c(G) \leq \beta_k(G)$.*

Proof. Since any nonsingleton connected dominating set is a total dominating set, the inequality in Property 1 can be extended for graphs that contain a total dominating set. □

Property 3. *If G is a connected graph that has no induced P_4 or C_4 then $\beta(G) = i(G) = \beta_k(G) = 1$.*

Proof. By Theorem 1, G has a dominating vertex. □

Property 4. *If G is a connected graph that has no induced P_5, C_5 or K_{3+p} then $\beta(G) = \beta_k(G) \leq 2$.*

Proof. By Corollary 3.1, G has a dominating edge. □

Connected split graphs clearly have a dominating clique. The complete graph of the partition of the vertices is a dominating clique. However, this clique may not be a minimum dominating clique. The following property relates the domination number, the independent domination number and the clique domination number of a connected split graph.

Property 5. *If G is a connected split graph then $\beta(G) = \beta_k(G) \leq i(G)$.*

Proof. For any graph G, $\beta(G) \leq i(G)$ and for any clique dominated graph G, $\beta(G) \leq \beta_k(G)$. Let D be a minimum dominating set of a connected split graph G whose vertex set can be partitioned into clique K and independent set S. Let $\beta(G) = m$. To show that $\beta(G) = \beta_k(G)$, a dominating clique of size m must be found. If D is a dominating clique then clearly G has a dominating clique of size

m. If D is not a dominating clique, form a half dominating clique of the connected split graph by replacing each $x \in D$ such that $x \in S$ by $y \in N(x)$. Such a y exists since G is connected. Since each neighbor of a member of S must be a member of K, and since D is a minimum dominating set, the set formed in this way must be a dominating clique of size m. Therefore, $\beta(G) = \beta_k(G) \leqslant i(G)$. □

A polynomial algorithm

Determining whether or not a graph has a central vertex can be accomplished easily by checking the degree of each vertex or by checking each vertex to see if the *closed neighborhood of the vertex,* the union of the vertex and its neighborhood, is the vertex set of the entire graph. The proof by induction that established that a finite connected graph with no induced P_5 or C_5 has a dominating clique suggests an algorithm for finding a dominating clique in such graphs. This algorithm will be shown to run in $O(n^3)$ time where n is the number of vertices in the graph. This algorithm finds a dominating clique (which may be a minimum or maximum dominating clique, or neither a minimum nor maximum dominating clique) in a finite connected graph which does not contain an induced P_5 or C_5. If the algorithm is run on a finite connected graph which does contain an induced P_5 or C_5 then the algorithm will either terminate, saying an induced P_5 or C_5 exists, or find a dominating clique anyway. Thus, it is not necessary to first check if the graph has an induced P_5 or C_5.

In the following algorithm, K represents the set that is currently under consideration as a dominating clique of the graph, T is the set of vertices that have already been considered, and W is the set of vertices that have not yet been considered. The notation $A(K)$ denotes the set of all vertices adjacent to the present clique K, while $A(T)$ denotes the set of all vertices adjacent to the set T. When a vertex of the graph is considered, the present clique K is tested to see if it dominates this vertex. If K does dominate this vertex, then another vertex is chosen. If K does not dominate this vertex, a new clique K is formed which will dominate that vertex. In this case, the clique being replaced is called $K1$ and the neighbors of this set are represented by $A(K1)$. If the algorithm continues until all vertices of the graph have been considered and dominated by K, then K is returned as a dominating clique. If at any stage a new clique is formed which fails to dominate a vertex that had been dominated by the previous clique, then the algorithm terminates because an induced P_5 or C_5 was found.

Algorithm DC Dominating clique of a connected graph.
Input: The adjacency lists, $A(v)$, $v \in V$, of a connected graph $G = (V, E)$.

Output: A set of vertices K that induces a dominating clique in G or "The graph contains an induced P_5 or C_5."

```
begin
  K ← ∅;
  T ← ∅;
  W ← V;
  choose v ∈ W;
  K ← {v};
  W ← W − {v};
  T ← {v};
  A(K) ← A(v);
  A(T) ← A(v);
  flag ← 0;
  while W ≠ ∅ and flag = 0
    begin
      choose v ∈ A(T) ∩ W;
      if v ∈ A(K) then K ← K
        else
          begin
            choose x ∈ A(v) ∩ T;
            K ← {x} ∪ (N(x) ∩ K)
          end;
      T ← T ∪ {v};
      W ← W − {v};
      A(T) ← ⋃_{v∈T} A(T);
      A(K1) ← A(K);
      A(K) ← ⋃_{v∈K} A(v);
      if A(K1) ⊈ A(K) then flag ← 1
    end;
  if flag = 1 then
    return "The graph contains an induced P5 or C5."
    else return K
end.
```

Theorem 7. *Algorithm* DC *is correct and runs in* $O(n^3)$ *time.*

Proof. If G does not contain an induced P_5 or C_5 then, by the proof of Theorem 2, the set K formed in Algorithm DC is a dominating clique. If G does contain an induced P_5 or C_5 then it may be that each K that is formed dominates all previously dominated vertices, as well as the latest chosen vertex, so that the last

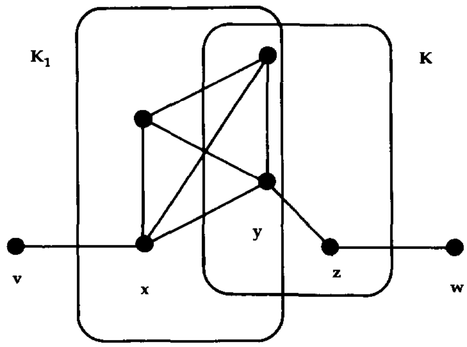

Fig. 6.

K is a dominating clique of the graph. Suppose this is not so. Then a K is formed by Algorithm DC that does not dominate some vertex, say v, that was dominated by the previous K. Let x be a vertex of the previous K that dominated v. Let w be the latest chosen vertex and let z be a vertex in the new K that dominates w. Since z was dominated by the previous K, by construction, the new K must contain some vertex, say y, that is adjacent to z and that was in the previous K. See Fig. 6.

Since x and y were in the previous K, x is adjacent to y. Since v is not dominated by the new K, v cannot be adjacent to z and v cannot be adjacent to y. If x were adjacent to z then, by the construction of the new K, x would be in the new K. Thus, x cannot be adjacent to z. If x or y were adjacent to w then the previous K would have dominated w. So x is not adjacent to w and y is not adjacent to w. Thus, if v is not adjacent to w then v–x–y–z–w is an induced P_5 and if v is adjacent to w then v–x–y–z–w is an induced C_5. The message returned by Algorithm DC is therefore correct.

Since adding vertex number k to the set of chosen vertices requires $O(k^2)$ steps, the algorithm runs in $O(n^2 + (n+1)^2 + \cdots + 1^2) = O(n^3)$ time. □

To illustrate Algorithm DC, consider the graph in Fig. 7. This is a connected graph with no induced P_5 or C_5. Initially $K = \emptyset$, $T = \emptyset$, and $W = \{a, b, c, d, e, f\}$. Choose, for example, $v = a$. Then $K = \{a\}$, $T = \{a\}$, and $W = \{b, c, d, e, f\}$. $A(K) = A(T) = \{b, c\}$. Flag $= 0$.

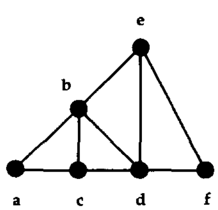

Fig. 7.

W is not the empty set until all vertices have been chosen and flag remains equal to one unless an induced P_5 or C_5 is found. Since W is not empty and flag $= 0$, choose a vertex adjacent to the one originally chosen. Suppose vertex b is chosen next. Since b is already dominated by K, K remains the same. At this stage, $K = \{a\}$, $T = \{a, b\}$, and $W = \{c, d, e, f\}$. $A(T) = \{a, b, c, d, e\}$. $A(K1) = \{b, c\}$. $A(K) = \{b, c\}$. Since $A(K1) \subseteq A(K)$, flag remains $= 0$.

Again, since W is not empty and flag $= 0$, choose a vertex that has not yet been considered and is adjacent to a vertex that has been chosen. Suppose vertex c is chosen next. Since c is also already dominated by K, K still remains the same. Now, $K = \{a\}$, $T = \{a, b, c\}$, and $W = \{d, e, f\}$. $A(T) = \{a, b, c, d, e\}$. $A(K1) = \{b, c\}$. $A(K) = \{b, c\}$. Since $A(K1) \subseteq A(K)$, flag remains $= 0$.

Neither condition to terminate the while loop has been met. Choose another vertex that has not yet been considered and is adjacent to a vertex that has been chosen. Suppose vertex d is chosen next. Vertex d has not been dominated by K, so a new K must be formed. Choose a vertex that is adjacent to d and that has already been considered. Vertices b and c both meet these conditions. Suppose vertex b is chosen. A new K is formed by uniting vertex b with the neighbors of vertex b that are in the previous K. At this stage, $K = \{a, b\}$, $T = \{a, b, c, d\}$, and $W = \{e, f\}$. $A(T) = \{a, b, c, d, e, f\}$. $A(K1) = \{b, c\}$. $A(K) = \{a, b, c, d, e\}$. Since $A(K1) \subseteq A(K)$, flag remains $= 0$.

All of the vertices of the graph have not yet been considered and an induced P_5 or C_5 has not been found. Choose a vertex that has not yet been chosen. Suppose vertex e is chosen. Since e is already dominated by the present set K, this set remains the same. Now, $K = \{a, b\}$, $T = \{a, b, c, d, e\}$, and $W = \{f\}$. $A(T) = \{a, b, c, d, e, f\}$. $A(K1) = \{a, b, c, d, e\}$. $A(K) = \{a, b, c, d, e\}$. Since $A(K1) \subseteq A(K)$, flag remains $= 0$.

W is still not empty and flag $= 0$. Since it is the only vertex that has not been chosen, vertex f must now be chosen. Vertex f is not already dominated by K, so a new K must be formed. Choose a vertex adjacent to f that has already been considered. Suppose this is vertex d. The new K is formed by uniting vertex d and the neighbors of vertex d that are in the previous K. Now, $K = \{b, d\}$, $T = \{a, b, c, d, e, f\}$, and $W = \emptyset$. $A(T) = \{a, b, c, d, e, f\}$. $A(K1) = \{a, b, c, d, e\}$. $A(K) = \{a, b, c, d, e, f\}$. Since $A(K1) \subseteq A(K)$, flag remains $= 0$.

Since W is now empty, the while loop is terminated. Since flag $= 0$, the set $K = \{b, d\}$ is returned as a dominating clique of the graph. This set is a dominating clique of the graph in Fig. 7. Many choices were made arbitrarily in implementing the algorithm. A different dominating clique, such as $\{b, e\}$ or $\{c, d\}$, may have been returned by the algorithm.

Threshold dimension

The class of clique dominated graphs as well as the class of connected graphs that do not have an induced P_5 or C_5 are not perfect graphs since the complement of C_7 is in both of these classes of graphs. However, Algorithm DC is a polynomial algorithm to find a dominating clique of a finite connected graph with no induced P_5 or C_5. It may be possible to use this dominating clique to find other parameters for these graphs. Since this class of graphs contains the class of connected split graphs, any polynomial algorithms which are found for clique dominated graphs apply to the class of connected split graphs. Also, the computation of any parameters which are known to be NP-complete for connected split graphs will be NP-complete for clique dominated graphs.

For example, a *threshold graph* is a graph that has no induced P_4, C_4, or $2K_2$. The threshold dimension of a graph is the minimum number of partial subgraphs of a graph that are threshold graphs and that cover the edges of the graph. Yannakakis [20] proved that determining if the threshold dimension of an arbitrary graph is less than or equal to k, for fixed $k \geqslant 3$, is NP-complete. As we now show, determining if the threshold dimension of a connected split graph is less than or equal to k, for fixed $k \geqslant 3$, is NP-complete. Therefore, determining if the threshold dimension of a clique dominated graph is less than or equal to k, for fixed $k \geqslant 3$, is NP-complete. A *chain graph* is a bipartite graph that has no induced $2K_2$. (A graph with no induced $2K_2$ is said to be *nonseparable* [10].) The *chain dimension* of a graph is the minimum number of chain subgraphs that cover the edges of the graph.

Theorem 5. *It is* NP-*complete to determine if the threshold dimension of a split graph is less than or equal to* k, *for fixed* $k \geqslant 3$.

Proof. Let G be a split graph whose vertex set can be partitioned into clique K and independent set S. Form the bipartite graph $B(G)$ by removing the edges of the clique K from G. Similarly, any bipartite graph whose vertices can be partitioned into sets V_1 and V_2 can be transformed into a split graph by adding the edges to make either V_1 or V_2, but not both, a clique. Since any vertices that induce P_4 in the split graph G must induce $2K_2$ in the bipartite graph $B(G)$, the threshold dimension of G is equal to the chain dimension of $B(G)$. Since it is NP-complete to determine if the chain dimension of a bipartite graph is less than or equal to k, for fixed $k \geqslant 3$ [18], it is NP-complete to determine if the threshold dimension of a split graph is less than or equal to k, for fixed $k \geqslant 3$. □

Corollary 5.1. *It is* NP-*complete to determine if the threshold dimension of a connected split graph is less than or equal to* k, *for fixed* $k \geqslant 3$.

Proof. This follows from Theorem 5 and Corollary 3.1 since the threshold dimension of a graph is the sum of the threshold dimensions of the components of the graph. □

Corollary 5.2. *It is* NP-*complete to determine if the threshold dimension of a clique dominated graph is less than or equal to* k, *for fixed* $k \geqslant 3$.

Proof. This follows from Corollary 5.1 by restricting the problem to allow only instances that are connected split graphs. □

However, as the following theorem shows, it is possible to determine in polynomial time if the threshold dimension of a connected split graph is $\leqslant 2$.

Lemma 1. *Let G be a connected split graph whose vertices can be partitioned into clique K and independent set S. Let $B(G)$ be the bipartite graph formed from G by removing the edges of K. Then the threshold dimension of G is $\leqslant 2$ if and only if the chain dimension of $B(G)$ is $\leqslant 2$.*

Proof. If the threshold dimension of G is $\leqslant 2$, then $G = G_1 \cup G_2$ where G_1 and G_2 are threshold graphs. G_1 and G_2 are split graphs whose vertex sets can be partitioned into a clique that is a subset of K and an independent set that is a subset of S. Thus, G_1 can be partitioned into $K_1 \subseteq K$ and $S_1 \subseteq S$. Similarly, G_2 can be partitioned into $K_2 \subseteq K$ and $S_2 \subseteq S$. Form the bipartite graph $B(G_1)$ by removing the edges of K_1 from G_1. Form the bipartite graph $B(G_2)$ by removing the edges of K_2 from G_2. Since G_1 and G_2 are threshold graphs they contain no induced P_4 or $2K_2$. This implies that bipartite graphs $B(G_1)$ and $B(G_2)$ contain no $2K_2$. Therefore, $B(G_1)$ and $B(G_2)$ are chain graphs. Since $B(G_1) \cup B(G_2)$ covers the edges of $B(G)$, the chain dimension of $B(G) \leqslant 2$.

If the chain dimension of $B(G) \leqslant 2$, then $B(G) = G_1 \cup G_2$ where G_1 and G_2 are chain graphs. Form the graphs $G_1 \cup K$ and $G_2 \cup K$. The fact that the chain graphs G_1 and G_2 have no induced $2K_2$ implies that $G_1 \cup K$ and $G_2 \cup K$ have no induced P_4. Since, by construction, $G_1 \cup K$ and $G_2 \cup K$ are split graphs, they have no induced C_4 or $2K_2$. Therefore, they are threshold graphs. Since their union covers the edges of G, the threshold dimension of $G \leqslant 2$. □

Theorem 6. *There is a polynomial algorithm to determine if the threshold dimension of a connected split graph is $\leqslant 2$.*

Proof. Let G be a connected split graph whose vertex set can be partitioned into clique K and independent set S. Form the bipartite graph $B(G)$ by removing the edges of K from G. By results of Yannakakis [20] and Ibaraki and Peled [12] there is a polynomial algorithm to determine if the chain dimension of a bipartite graph is $\leqslant 2$. By Lemma 1, this same algorithm will determine if the threshold dimension of G is $\leqslant 2$ by determining if the chain dimension of $B(G)$ is $\leqslant 2$. □

A polynomial algorithm exists for determining the threshold dimension of split graphs with no K_{3+p}, and, more generally, for graphs with a dominating edge such that each induced subgraph has a dominating edge [6].

Graphs that have a dominating K_3 or P_3

Dominating vertices and dominating edges are the only connected dominating sets of size one and size two. However, connected dominating sets of size three can be either cliques or paths. In this section, a forbidden subgraph characterization for a graph to have a connected dominating set of size three is given. Let A_1, A_2, A_3, and A_4 be the graphs shown in Fig. 8. Let $A = \{P_6, C_6, K_{4+p}, A_1, A_2, A_3, A_4\}$.

Theorem 7. *If G is a finite, connected graph with three or more vertices that has none of the graphs in A as an induced subgraph, then G has a connected dominating set of size three.*

Proof. By induction on n, the number of vertices in G.

(i) The proposition is clearly true for $n = 3$.

(ii) Assume that any finite connected graph with n vertices, $n \geqslant 3$, that has none of the graphs in A as an induced subgraph has a connected dominating set of size three. Let G be a finite connected graph with $n + 1$ vertices, $n \geqslant 3$, that has none of the graphs in A as an induced subgraph. Let v be a vertex of G that is not a cutpoint. Such a vertex exists [11]. Let G' be the subgraph of G induced by all vertices of G except v. Since G' is a finite connected graph with n vertices that has none of the graphs in A as an induced subgraph it has, by the induction hypothesis, a connected dominating set of size three. Let $D = \{a, b, c\}$ be a connected dominating set of size three of G'. In G, if v is adjacent to any vertex in D, then D will also be a connected dominating set of size three of G.

Suppose that, in G, v is not adjacent to any vertex in D. Since G is connected, v must be adjacent to some vertex of G. Let x be any vertex of G that is adjacent

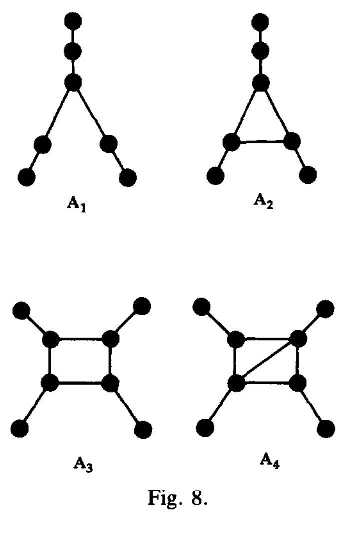

Fig. 8.

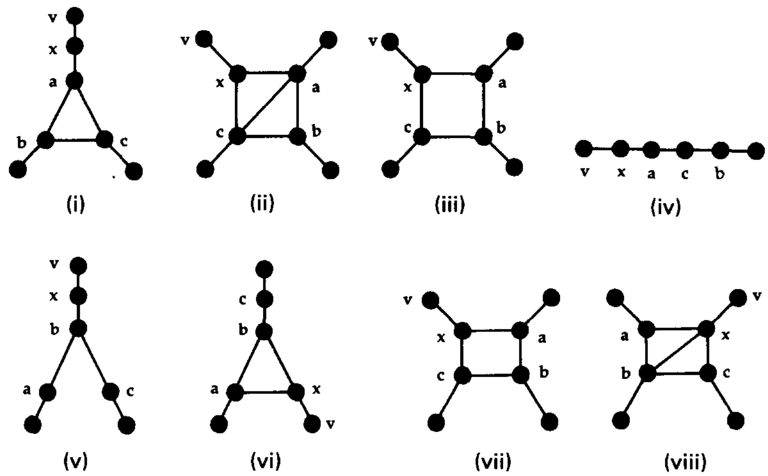

Fig. 9.

to v. Since D is a dominating set of G', x must be adjacent to some vertex in D. Up to symmetry, the set $\{v, x, a, b, c\}$ induces one of eight possible subgraphs. See Fig. 9. Now, either x and two of the vertices of D form a dominating P_3 or K_3 of G, or G has one of the subgraphs (not necessarily induced) shown in Fig. 9. Since the subgraphs formed by $\{v, x, a, b, c\}$ are induced subgraphs, if G has no edges between the pendants of the subgraphs in Fig. 9 then there is a contradiction to the assumption that G has none of the graphs in A as an induced subgraph.

Suppose G has at least one of the edges between the pendants shown in Fig. 9. Suppose further than G contains the subgraph shown in Fig. 9(i). If G has exactly one edge between the pendants then G has an induced P_6. If G has exactly two nonsymmetric edges between the pendants then G has an induced C_6. If G has two symmetric edges between the pendants or all three edges between the pendants then either $\{v, x, a\}$ is a dominating P_3 of G or there is a vertex not adjacent to any vertex in $\{v, x, a\}$ but adjacent to a and/or c. This would imply that G has an induced P_6 or C_6. See Fig. 10.

In a similar way, consideration of the edges between the pendants of each of

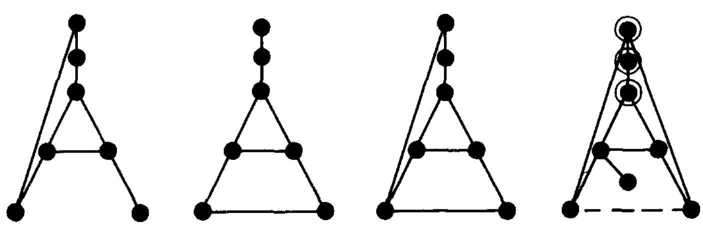

Fig. 10.

the subgraphs in Fig. 9 can be shown to lead to the conclusion that either G has a dominating P_3 or K_3 or there is a contradiction to the assumption that G has none of the graphs in A as in induced subgraph.

Therefore, G has a connected dominating set of size three. □

As noted earlier for Theorem 2, the converse of Theorem 7 is not true. However, it is possible to extend the forbidden subgraph relationships described in this paper to if and only if statements and thus to give forbidden subgraph characterizations of these classes of graphs. An example of this type of extension is given in the following corollary.

Corollary 7.1. *Let G be a finite connected graph with three or more vertices. Every connected induced subgraph of G with three or more vertices has a connected dominating set of size three if and only if G has none of the graphs in A as an induced subgraph.*

Proof. This follows easily from Theorem 7 and the fact that none of the graphs in A has a connected dominating set of size three. □

Open problems

This paper has given forbidden subgraph characterizations of graphs with a dominating clique or a connected dominating set of size three. Several problems related to this area remain open. These problems include: Is there a forbidden subgraph characterization of graphs that have a connected dominating set of size four? Is there a polynomial algorithm to locate a *minimum* dominating clique in a clique dominated graph or in a connected graph with no induced P_5 or C_5? What other parameters are computable in polynomial time for clique dominated graphs or for connected split graphs? Are there forbidden subgraph characterizations of classes of graphs that contain other specific types of dominating sets?

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