JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **226,** 166–179 (1998)
ARTICLE NO. AY986043

On Some New Inequalities Similar to Hilbert's Inequality

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Submitted by William F. Ames

Received January 15, 1998

In the present article we establish some new inequalities similar to Hilbert's inequality involving series of nonnegative terms. The integral analogues of the main results are also given. © 1998 Academic Press

Key Words: Hilbert's inequality; series of nonnegative terms; integral analogue; Schwarz inequality; Jensen's inequality.

1. INTRODUCTION

The well-known inequality due to Hilbert can be stated as follows (see [2, p. 226]).

THEOREM A. *If $p > 1$, $p' = p/(p-1)$ and $\sum a_m^p \le A$, $\sum b_n^{p'} \le B$, the summations running from 1 to ∞, then*

$$\sum \sum \frac{a_m b_n}{m+n} < \frac{\pi}{\mathrm{Sin}(\pi/p)} A^{1/p} B^{1/p'}, \tag{1}$$

unless the sequence $\{a_m\}$ or $\{b_n\}$ is null.

The integral analogue of the Hilbert's inequality can be stated as follows (see [2, p. 226]).

0022-247X/98 $25.00

THEOREM B. *If* $p > 1$, $p' = p/(p-1)$ *and*

$$\int_0^\infty f^p(x)\,dx \le F, \qquad \int_0^\infty g^{p'}(y)\,dy \le G,$$

then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\,dx\,dy < \frac{\pi}{\operatorname{Sin}(\pi/p)} F^{1/p} G^{1/p'}, \tag{2}$$

unless $f \equiv 0$ *or* $g \equiv 0$.

The inequalities in Theorems A and B were studied extensively and numerous variants, generalizations, and extensions appeared in the literature, see [1–5, 8] and the references cited therein. The main purpose of the present article is to establish some new inequalities similar to the Hilbert's inequality given in Theorem A, involving a series of nonnegative terms. The integral analogues of our main results similar to that of those given in Theorem B are also given. The analysis used in the proofs is elementary and our results provide new estimates on these types of inequalities.

2. MAIN RESULTS

Our main result is given in the following theorem.

THEOREM 1. *Let* $p \ge 1$, $q \ge 1$ *and* $\{a_m\}$ *and* $\{b_n\}$ *be two nonnegative sequences of real numbers defined for* $m = 1, 2, \ldots, k$ *and* $n = 1, 2, \ldots, r$, *where* k, r *are the natural numbers and define* $A_m = \sum_{s=1}^m a_s$ *and* $B_n = \sum_{t=1}^n b_t$. *Then*

$$\sum_{m=1}^k \sum_{n=1}^r \frac{A_m^p B_n^q}{m+n} \le C(p,q,k,r)\left(\sum_{m=1}^k (k-m+1)\left(A_m^{p-1}a_m\right)^2\right)^{1/2} \times \left(\sum_{n=1}^r (r-n+1)\left(B_n^{q-1}b_n\right)^2\right)^{1/2}, \tag{3}$$

unless $\{a_m\}$ *or* $\{b_n\}$ *is null, where*

$$C(p,q,k,r) = \tfrac{1}{2}pq\sqrt{kr}\,. \tag{4}$$

Proof. By using the following inequality (see [1, 6]),

$$\left(\sum_{m=1}^{n} z_m\right)^{\alpha} \le \alpha \sum_{m=1}^{n} z_m \left(\sum_{k=1}^{m} z_k\right)^{\alpha-1},$$

where $\alpha \ge 1$ is a constant and $z_m \ge 0$, $(m = 1, 2, \ldots)$, it is easy to observe that

$$A_m^p \le p \sum_{s=1}^{m} a_s A_s^{p-1}, \qquad m = 1, 2, \ldots, k, \tag{5}$$

$$B_n^q \le q \sum_{t=1}^{n} b_t B_t^{q-1}, \qquad n = 1, 2, \ldots, r. \tag{6}$$

From (5) and (6) and using the Schwarz inequality and the elementary inequality $c^{1/2} d^{1/2} \le \frac{1}{2}(c + d)$, (for c, d nonnegative reals) we observe that

$$\begin{aligned} A_m^p B_n^q &\le pq \left(\sum_{s=1}^{m} a_s A_s^{p-1}\right)\left(\sum_{t=1}^{n} b_t B_t^{q-1}\right) \\ &\le pq(m)^{1/2} \left(\sum_{s=1}^{m} \left(a_s A_s^{p-1}\right)^2\right)^{1/2} (n)^{1/2} \left(\sum_{t=1}^{n} \left(b_t B_t^{q-1}\right)^2\right)^{1/2} \\ &\le \tfrac{1}{2} pq(m + n) \left(\sum_{s=1}^{m} \left(a_s A_s^{p-1}\right)^2\right)^{1/2} \left(\sum_{t=1}^{n} \left(b_t B_t^{q-1}\right)^2\right)^{1/2}. \end{aligned} \tag{7}$$

Dividing both sides of (7) by $m + n$ and then taking the sum over n from 1 to r first and then the sum over m from 1 to k and using the Schwarz inequality and then interchanging the order of the summations (see [6, 7]) we observe that

$$\begin{aligned} \sum_{m=1}^{k} \sum_{n=1}^{r} \frac{A_m^p B_n^q}{m + n} &\le \frac{1}{2} pq \left\{ \sum_{m=1}^{k} \left(\sum_{s=1}^{m} \left(a_s A_s^{p-1}\right)^2 \right)^{1/2} \right\} \\ &\quad \times \left\{ \sum_{n=1}^{r} \left(\sum_{t=1}^{n} \left(b_t B_t^{q-1}\right)^2 \right)^{1/2} \right\} \\ &\le \frac{1}{2} pq(k)^{1/2} \left\{ \sum_{m=1}^{k} \left(\sum_{s=1}^{m} \left(a_s A_s^{p-1}\right)^2 \right) \right\}^{1/2} \\ &\quad \times (r)^{1/2} \left\{ \sum_{n=1}^{r} \left(\sum_{t=1}^{n} \left(b_t B_t^{q-1}\right)^2 \right) \right\}^{1/2} \end{aligned}$$

$$= \frac{1}{2} pq\sqrt{kr} \left\{ \sum_{s=1}^{k} \left(a_s A_s^{p-1}\right)^2 \left(\sum_{m=s}^{k} 1 \right) \right\}^{1/2}$$

$$\times \left\{ \sum_{t=1}^{r} \left(b_t B_t^{q-1}\right)^2 \left(\sum_{n=t}^{r} 1 \right) \right\}^{1/2}$$

$$= C(p,q,k,r) \left(\sum_{s=1}^{k} \left(a_s A_s^{p-1}\right)^2 (k-s+1) \right)^{1/2}$$

$$\times \left(\sum_{t=1}^{r} \left(b_t B_t^{q-1}\right)^2 (r-t+1) \right)^{1/2}$$

$$= C(p,q,k,r) \left(\sum_{m=1}^{k} (k-m+1) \left(a_m A_m^{p-1}\right)^2 \right)^{1/2}$$

$$\times \left(\sum_{n=1}^{r} (r-n+1) \left(b_n B_n^{q-1}\right)^2 \right)^{1/2}.$$

This completes the proof.

Remark 1. If we take $p = q = 1$ in Theorem 1, then the inequality (3) reduces to the following inequality,

$$\sum_{m=1}^{k} \sum_{n=1}^{r} \frac{A_m B_n}{m+n} \le C(1,1,k,r) \left(\sum_{m=1}^{k} (k-m+1)(a_m)^2 \right)^{1/2}$$

$$\times \left(\sum_{n=1}^{r} (r-n+1)(b_n)^2 \right)^{1/2}, \tag{8}$$

where $C(1,1,k,r)$ is obtained by taking $p = q = 1$ in (4).

Our next result deals with the further generalization of the inequality obtained in (8).

THEOREM 2. *Let* $\{a_m\}, \{b_n\}, A_m, B_n$ *be as defined in Theorem* 1. *Let* $\{p_m\}$ *and* $\{q_n\}$ *be two positive sequences for* $m = 1, 2, \dots, k$ *and* $n = 1, 2, \dots, r$ *and define* $P_m = \sum_{s=1}^{m} p_s$ *and* $Q_n = \sum_{t=1}^{n} q_t$. *Let* ϕ *and* ψ *be two real-valued,* *nonnegative, convex, and submultiplicative functions defined on* $R_+ = [0, \infty)$. *Then*

$$\sum_{m=1}^{k} \sum_{n=1}^{r} \frac{\phi(A_m)\psi(B_n)}{m+n} \le M(k,r)\left(\sum_{m=1}^{k}(k-m+1)\left[p_m \phi\left(\frac{a_m}{p_m}\right)\right]^2\right)^{1/2} \times \left(\sum_{n=1}^{r}(r-n+1)\left[q_n \psi\left(\frac{b_n}{q_n}\right)\right]^2\right)^{1/2}, \tag{9}$$

where

$$M(k,r) = \frac{1}{2}\left(\sum_{m=1}^{k}\left[\frac{\phi(P_m)}{P_m}\right]^2\right)^{1/2}\left(\sum_{n=1}^{r}\left[\frac{\psi(Q_n)}{Q_n}\right]^2\right)^{1/2}. \tag{10}$$

Proof. From the hypotheses and by using Jensen's inequality and the Schwarz inequality (see [4]), it is easy to observe that

$$\begin{aligned} \phi(A_m) &= \phi\left(\frac{P_m \sum_{s=1}^{m} p_s a_s/p_s}{\sum_{s=1}^{m} p_s}\right) \\ &\le \phi(P_m)\phi\left(\frac{\sum_{s=1}^{m} p_s a_s/p_s}{\sum_{s=1}^{m} p_s}\right) \\ &\le \frac{\phi(P_m)}{P_m}\sum_{s=1}^{m} p_s \phi\left(\frac{a_s}{p_s}\right) \\ &\le \frac{\phi(P_m)}{P_m}(m)^{1/2}\left\{\sum_{s=1}^{m}\left[p_s \phi\left(\frac{a_s}{p_s}\right)\right]^2\right\}^{1/2}, \end{aligned} \tag{11}$$

and similarly,

$$\psi(B_n) \le \frac{\psi(Q_n)}{Q_n}(n)^{1/2}\left\{\sum_{t=1}^{n}\left[q_t \psi\left(\frac{b_t}{q_t}\right)\right]^2\right\}^{1/2}. \tag{12}$$

From (11) and (12) and using the elementary inequality $c^{1/2}d^{1/2} \leq \frac{1}{2}(c+d)$, (for c, d nonnegative reals) we observe that

$$\phi(A_m)\psi(B_n) \leq \frac{1}{2}(m+n)\left[\frac{\phi(P_m)}{P_m}\left\{\sum_{s=1}^{m}\left[p_s\phi\left(\frac{a_s}{p_s}\right)\right]^2\right\}^{1/2}\right]$$
$$\times\left[\frac{\psi(Q_n)}{Q_n}\left\{\sum_{t=1}^{n}\left[q_t\psi\left(\frac{b_t}{q_t}\right)\right]^2\right\}^{1/2}\right]. \tag{13}$$

Dividing both sides of (13) by $m+n$ and then taking the sum over n from 1 to r first and then the sum over m from 1 to k and using the Schwarz inequality and then interchanging the order of the summations we observe that

$$\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{\phi(A_m)\psi(B_n)}{m+n}$$
$$\leq \frac{1}{2}\left\{\sum_{m=1}^{k}\left[\frac{\phi(P_m)}{P_m}\left\{\sum_{s=1}^{m}\left[p_s\phi\left(\frac{a_s}{p_s}\right)\right]^2\right\}^{1/2}\right]\right\}$$
$$\times\left\{\sum_{n=1}^{r}\left[\frac{\psi(Q_n)}{Q_n}\left\{\sum_{t=1}^{n}\left[q_t\psi\left(\frac{b_t}{q_t}\right)\right]^2\right\}^{1/2}\right]\right\}$$
$$\leq \frac{1}{2}\left(\sum_{m=1}^{k}\left[\frac{\phi(P_m)}{P_m}\right]^2\right)^{1/2}$$
$$\times\left(\sum_{m=1}^{k}\left(\sum_{s=1}^{m}\left[p_s\phi\left(\frac{a_s}{p_s}\right)\right]^2\right)\right)^{1/2}$$
$$\times\left(\sum_{n=1}^{r}\left[\frac{\psi(Q_m)}{Q_m}\right]^2\right)^{1/2}$$
$$\times\left(\sum_{n=1}^{r}\left(\sum_{t=1}^{n}\left[q_t\psi\left(\frac{b_t}{q_t}\right)\right]^2\right)\right)^{1/2}$$

$$= M(k,r)\left(\sum_{s=1}^{k}\left[p_s\phi\left(\frac{a_s}{p_s}\right)\right]^2\left(\sum_{m=s}^{k}1\right)\right)^{1/2}$$

$$\times\left(\sum_{t=1}^{r}\left[q_t\psi\left(\frac{b_t}{q_t}\right)\right]^2\left(\sum_{n=t}^{r}1\right)\right)^{1/2}$$

$$= M(k,r)\left(\sum_{s=1}^{k}\left[p_s\phi\left(\frac{a_s}{p_s}\right)\right]^2(k-s+1)\right)^{1/2}$$

$$\times\left(\sum_{t=1}^{r}\left[q_t\psi\left(\frac{b_t}{q_t}\right)\right]^2(r-t+1)\right)^{1/2}$$

$$= M(k,r)\left(\sum_{m=1}^{k}(k-m+1)\left[p_m\phi\left(\frac{a_m}{p_m}\right)\right]^2\right)^{1/2}$$

$$\times\left(\sum_{n=1}^{r}(r-n+1)\left[q_n\psi\left(\frac{b_n}{q_n}\right)\right]^2\right)^{1/2}.$$

The proof is complete.

Remark 2. By applying the elementary inequality $c^{1/2}d^{1/2} \le \frac{1}{2}(c+d)$, (for c, d nonnegative reals) on the right sides of (3) and (9), we get, respectively, the following inequalities,

$$\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{A_m^p B_n^q}{m+n} \le \frac{1}{2}C(p,q,k,r)\left[\sum_{m=1}^{k}(k-m+1)\left(A_m^{p-1}a_m\right)^2 + \sum_{n=1}^{r}(r-n+1)\left(B_n^{q-1}b_n\right)^2\right], \quad (14)$$

and

$$\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{\phi(A_m)\psi(B_n)}{m+n} \le \frac{1}{2}M(k,r)\left[\sum_{m=1}^{k}(k-m+1)\left[p_m\phi\left(\frac{a_m}{p_m}\right)\right]^2 + \sum_{n=1}^{r}(r-n+1)\left[q_n\psi\left(\frac{b_n}{q_n}\right)\right]^2\right], \quad (15)$$

which we believe are new to the literature.

The following theorems deal with slight variants of the inequality given in Theorem 2.

THEOREM 3. *Let* $\{a_m\}$ *and* $\{b_n\}$ *be as in Theorem* 1 *and define* $A_m = 1/m \sum_{s=1}^{m} a_s$ *and* $B_n = 1/n \sum_{t=1}^{n} b_t$, *for* $m = 1, 2, \dots, k$ *and* $n = 1, 2, \dots, r$, *where* k, r *are the natural numbers. Let* ϕ *and* ψ *be two real-valued, nonnegative, and convex functions defined on* R_+. *Then*

$$\sum_{m=1}^{k} \sum_{n=1}^{r} \frac{mn}{m+n} \phi(A_m) \psi(B_n) \leq C(1,1,k,r) \left(\sum_{m=1}^{k} (k-m+1) [\phi(a_m)]^2 \right)^{1/2} \times \left(\sum_{n=1}^{r} (r-n+1) [\psi(b_n)]^2 \right)^{1/2}, \tag{16}$$

where $C(1,1,k,r)$ *is defined by taking* $p = q = 1$ *in* (4).

Proof. From the hypotheses and by using Jensen's inequality and the Schwarz inequality, it is easy to observe that

$$\phi(A_m) = \phi\left(\frac{1}{m} \sum_{s=1}^{m} a_s \right) \leq \frac{1}{m} \sum_{s=1}^{m} \phi(a_s) \leq \frac{1}{m} (m)^{1/2} \left\{ \sum_{s=1}^{m} [\phi(a_s)]^2 \right\}^{1/2}, \tag{17}$$

$$\psi(B_n) = \psi\left(\frac{1}{n} \sum_{t=1}^{n} b_t \right) \leq \frac{1}{n} \sum_{t=1}^{n} \psi(b_t) \leq \frac{1}{n} (n)^{1/2} \left\{ \sum_{t=1}^{n} [\psi(b_t)]^2 \right\}^{1/2}. \tag{18}$$

The rest of the proof can be completed by following the same steps as in the proofs of Theorems 1 and 2 with suitable changes and hence we omit the details.

THEOREM 4. *Let* $\{a_m\}, \{b_n\}, \{p_m\}, \{q_n\}, P_m, Q_n$ *be as in Theorem* 2 *and define* $A_m = 1/P_m \sum_{s=1}^{m} p_s a_s$ *and* $B_n = 1/Q_n \sum_{t=1}^{n} q_t b_t$, *for* $m = 1, 2, \dots, k$

and $n = 1, 2, \ldots, r$, *where* k, r *are the natural numbers. Let* ϕ *and* ψ *be as defined in Theorem* 3. *Then*

$$\sum_{m=1}^{k} \sum_{n=1}^{r} \frac{P_m Q_n \phi(A_m) \psi(B_n)}{m+n}$$

$$\leq C(1,1,k,r) \left(\sum_{m=1}^{k} (k-m+1) [p_s \phi(a_s)]^2 \right)^{1/2}$$

$$\times \left(\sum_{n=1}^{r} (r-n+1) [q_t \psi(b_t)]^2 \right)^{1/2}, \tag{19}$$

where $C(1,1,k,r)$ *is defined by taking* $p = q = 1$ *in* (4).

Proof. From the hypotheses and by using Jensen's inequality and the Schwarz inequality, it is easy to observe that

$$\phi(A_m) = \phi\left(\frac{1}{P_m} \sum_{s=1}^{m} p_s a_s \right) \leq \frac{1}{P_m} \sum_{s=1}^{m} p_s \phi(a_s)$$

$$\leq \frac{1}{P_m} (m)^{1/2} \left\{ \sum_{s=1}^{m} [p_s \phi(a_s)]^2 \right\}^{1/2}, \tag{20}$$

$$\psi(B_n) = \psi\left(\frac{1}{Q_n} \sum_{t=1}^{n} q_t b_t \right) \leq \frac{1}{Q_n} \sum_{t=1}^{n} q_t \psi(b_t)$$

$$\leq \frac{1}{Q_n} (n)^{1/2} \left\{ \sum_{t=1}^{n} [q_t \psi(b_t)]^2 \right\}^{1/2}. \tag{21}$$

Proceeding now much as in the proof of Theorems 1 and 2 given in the preceding text with suitable modifications we get the required inequality in (19), so we leave out the details.

3. INTEGRAL ANALOGUES

In this section we present the integral analogues of the inequalities given in Theorems 1–4, which in fact are motivated by the integral analogue of the Hilbert's inequailty given in Theorem B.

An integral analogue of Theorem 1 is given in the following theorem.

THEOREM 5. *Let* $p \geq 1$, $q \geq 1$ *and* $f(\sigma) \geq 0$, $g(\tau) \geq 0$ *for* $\sigma \in (0, x)$, $\tau \in (0, y)$, *where* x, y *are positive real numbers and define* $F(s) = \int_0^s f(\sigma)\, d\sigma$

and $G(t) = \int_0^t g(\tau)\,d\tau$, *for* $s \in (0, x)$, $t \in (0, y)$. *Then*

$$\int_0^x \int_0^y \frac{F^p(s)G^q(t)}{s+t}\,ds\,dt \le D(p,q,x,y)\left(\int_0^x (x-s)\left(F^{p-1}(s)f(s)\right)^2 ds\right)^{1/2} \times \left(\int_0^y (y-t)\left(G^{q-1}(t)g(t)\right)^2 dt\right)^{1/2}, \tag{22}$$

unless $f \equiv 0$ *or* $g \equiv 0$, *where*

$$D(p,q,x,y) = \tfrac{1}{2}pq\sqrt{xy}\,. \tag{23}$$

Proof. From the hypotheses, it is easy to observe that

$$F^p(s) = p\int_0^s F^{p-1}(\sigma)f(\sigma)\,d\sigma, \qquad s \in (0, x), \tag{24}$$

$$G^q(t) = q\int_0^t G^{q-1}(\tau)g(\tau)\,d\tau, \qquad t \in (0, y). \tag{25}$$

From (24) and (25) and using the Schwarz inequality and the elementary inequality $c^{1/2}d^{1/2} \le \frac{1}{2}(c+d)$, (for c, d nonnegative reals) we observe that

$$\begin{aligned} F^p(s)G^q(t) &= pq\left(\int_0^s F^{p-1}(\sigma)f(\sigma)\,d\sigma\right)\left(\int_0^t G^{q-1}(\tau)g(\tau)\,d\tau\right) \\ &\le pq(s)^{1/2}\left(\int_0^s \left(F^{p-1}(\sigma)f(\sigma)\right)^2 d\sigma\right)^{1/2} \\ &\quad \times (t)^{1/2}\left(\int_0^t \left(G^{q-1}(\tau)g(\tau)\right)^2 d\tau\right)^{1/2} \\ &\le \frac{1}{2}pq(s+t)\left(\int_0^s \left(F^{p-1}(\sigma)f(\sigma)\right)^2 d\sigma\right)^{1/2} \\ &\quad \times \left(\int_0^t \left(G^{q-1}(\tau)g(\tau)\right)^2 d\tau\right)^{1/2}. \end{aligned} \tag{26}$$

Dividing both sides of (26) by $s + t$ and then integrating over t from 0 to y first and then integrating the resulting inequality over s from 0 to x and using the Schwarz inequality we observe that

$$\begin{aligned}
&\int_0^x\int_0^y \frac{F^p(s)G^q(t)}{s+t}\,ds\,dt\\
&\quad\le \frac{1}{2}pq\left\{\int_0^x\left(\int_0^s \left(F^{p-1}(\sigma)f(\sigma)\right)^2 d\sigma\right)^{1/2}\right\}\\
&\qquad\times\left\{\int_0^y\left(\int_0^t \left(G^{q-1}(\tau)g(\tau)\right)^2 d\tau\right)^{1/2}\right\}\\
&\quad\le \frac{1}{2}pq(x)^{1/2}\left\{\int_0^x\left(\int_0^s \left(F^{p-1}(\sigma)f(\sigma)\right)^2 d\sigma\right) ds\right\}^{1/2}\\
&\qquad\times(y)^{1/2}\left\{\int_0^y\left(\int_0^t \left(G^{q-1}(\tau)g(\tau)\right)^2 d\tau\right) dt\right\}^{1/2}\\
&\quad= D(p,q,x,y)\left(\int_0^x (x-s)\left(F^{p-1}(s)f(s)\right)^2 ds\right)^{1/2}\\
&\qquad\times\left(\int_0^y (y-t)\left(G^{q-1}(\tau)g(\tau)\right)^2 d\tau\right)^{1/2}.
\end{aligned}$$

This completes the proof.

Remark 3. In the special case when $p = q = 1$, the inequality (22) reduces to the following inequality,

$$\begin{aligned}
\int_0^x\int_0^y \frac{F(s)G(t)}{s+t}\,ds\,dt \le D(1,1,x,y)\left(\int_0^x (x-s)f^2(s)\,ds\right)^{1/2}&\\
\times\left(\int_0^y (y-t)g^2(t)\,dt\right)^{1/2},& \qquad (27)
\end{aligned}$$

where $D(1, 1, x, y)$ is obtained by taking $p = q = 1$ in (23).

The integral analogues of the inequalities in Theorems 2–4 are established in the following theorems.

THEOREM 6. *Let f, g, F, G be as in Theorem* 5. *Let $p(\sigma)$ and $q(\tau)$ be two positive functions defined for $\sigma \in (0, x)$, $\tau \in (0, y)$ and define $P(s) = \int_0^s p(\sigma)\,d\sigma$ and $Q(t) = \int_0^t q(\tau)\,d\tau$ for $s \in (0, x)$, $t \in (0, y)$, where x, y are* *positive real numbers. Let ϕ and ψ be as in Theorem 2. Then*

$$\int_0^x \int_0^y \frac{\phi(F(s))\psi(G(t))}{s+t}\,ds\,dt$$
$$\leq L(x,y)\left(\int_0^x (x-s)\left[p(s)\phi\left(\frac{f(s)}{p(s)}\right)\right]^2 ds\right)^{1/2}$$
$$\times\left(\int_0^y (y-t)\left[q(t)\psi\left(\frac{g(t)}{q(t)}\right)\right]^2 dt\right)^{1/2}, \tag{28}$$

where

$$L(x,y) = \frac{1}{2}\left(\int_0^x \left[\frac{\phi(P(s))}{P(s)}\right]^2 ds\right)^{1/2}\left(\int_0^y \left[\frac{\psi(Q(t))}{Q(t)}\right]^2 dt\right)^{1/2}. \tag{29}$$

Proof. From the hypotheses and by using Jensen's inequality and the Schwarz inequality, it is easy to observe that

$$\phi(F(s)) = \phi\left(\frac{P(s)\int_0^s p(s)\frac{f(\sigma)}{p(\sigma)}\,d\sigma}{\int_0^s p(\sigma)\,d\sigma}\right)$$
$$\leq \frac{\phi(P(s))}{P(s)}\int_0^s p(\sigma)\phi\left(\frac{f(\sigma)}{p(\sigma)}\right)d\sigma$$
$$\leq \left[\frac{\phi(P(s))}{P(s)}\right](s)^{1/2}\left\{\int_0^s \left[p(\sigma)\phi\left(\frac{f(\sigma)}{p(\sigma)}\right)\right]^2 d\sigma\right\}^{1/2}, \tag{30}$$

and similarly,

$$\psi(G(t)) \leq \left[\frac{\psi(Q(t))}{Q(t)}\right](t)^{1/2}\left\{\int_0^t \left[q(\tau)\psi\left(\frac{g(\tau)}{q(\tau)}\right)\right]^2 d\tau\right\}^{1/2}. \tag{31}$$

From (30) and (31) and using the elementary inequality $c^{1/2}d^{1/2} \le \frac{1}{2}(c + d)$, (for c, d nonnegative reals) we observe that

$$\phi(F(s))\psi(G(t)) \le \frac{1}{2}(s+t)\left[\frac{\phi(P(s))}{P(s)}\left\{\int_0^s\left[p(\sigma)\phi\left(\frac{f(\sigma)}{p(\sigma)}\right)\right]^2 d\sigma\right\}^{1/2}\right] \times\left[\frac{\psi(Q(t))}{Q(t)}\left\{\int_0^t\left[q(\tau)\psi\left(\frac{g(\tau)}{q(\tau)}\right)\right]^2 d\tau\right\}^{1/2}\right]. \tag{32}$$

The rest of the proof can be completed by following the same steps as in the proof of Theorem 5 and closely looking at the proof of Theorem 2 and hence we omit the details.

THEOREM 7. *Let f, g be as in Theorem 5 and define $F(s) = 1/s\int_0^s f(\sigma)\,d\sigma$ and $G(t) = 1/t\int_0^t g(\tau)\,d\tau$, for $s \in (0, x)$, $t \in (0, y)$, where x, y are positive real numbers. Let ϕ and ψ be as in Theorem 3. Then*

$$\int_0^x\int_0^y \frac{st}{s+t}\phi(F(s))\psi(G(t))\,ds\,dt \le D(1,1,x,y)\left(\int_0^x (x-s)[\phi(f(\sigma))]^2\,ds\right)^{1/2} \times\left(\int_0^y (y-t)[\psi(g(t))]^2\,dt\right)^{1/2}, \tag{33}$$

where $D(1,1,x,y)$ is obtained by taking $p = q = 1$ in (23).

THEOREM 8. *Let f, g, p, q, P, Q be as in Theorem 6 and define $F(s) = 1/P(s)\int_0^s p(\sigma)f(\sigma)\,d\sigma$ and $G(t) = 1/Q(t)\int_0^t q(\tau)g(\tau)\,d\tau$ for $s \in (0, x)$, $t \in (0, y)$, where x, y are positive real numbers. Let ϕ and ψ be as defined in Theorem 3. Then*

$$\int_0^x\int_0^y \frac{P(s)Q(t)\phi(F(s))\psi(G(t))}{s+t}\,ds\,dt \le D(1,1,x,y)\left(\int_0^x (x-s)[p(s)\phi(f(s))]^2\,ds\right)^{1/2} \times\left(\int_0^y (y-t)[q(t)\psi(g(t))]^2\,dt\right)^{1/2}, \tag{34}$$

where $D(1,1,x,y)$ is defined by taking $p = q = 1$ in (23).

The proofs of Theorems 7 and 8 can be completed by following the proof of Theorem 6 and by closely looking at the proofs of Theorems 3 and 4 and by making use of the integral versions of Jensen's and the Schwarz inequalities. Here, we omit the details.

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