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Characterization of H-Monotone Operators with Applications to Variational Inclusions

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Abstract—This paper establishes necessary and sufficient conditions for operators to be H-monotone. Based on these conditions, we introduce a new iterative algorithm for solving a class of variational inclusions. Strong convergence of this algorithm is established under appropriate assumptions on the parameters. Estimate of its convergence rate is also provided.

Keywords—H-monotone operator, Resolvent operator technique, Variational inclusion, Iterative algorithm.

1. INTRODUCTION

Variational inequality was initially studied by Stampacchia [1] in 1964. Since then, it has been extensively studied because it plays a crucial role in the study of mechanics, physics, optimization and control, economics and transportation equilibrium and engineering sciences, etc. Thanks to its wide applications, the classical variational inequality has been well studied and generalized in various directions. The reader is referred to [2–8] and the references therein. Among these generalizations, variational inclusion is of interest and importance. Recent development of the variational inequality is to design efficient iterative algorithms to compute approximate solutions for variational inequalities and their generalizations. Up to now, many authors have presented

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implementable and significant numerical methods such as projection method and its variant forms, linear approximation, descent method, Newton's method and the method based on auxiliary principle technique. In particular, the method based on the resolvent operator technique is a generalization of projection method and has been widely used to solve variational inclusions; see, e.g., [9–18].

In 2003, Fang and Huang [19] introduced a new class of monotone operators which were called H-monotone operators. For an H-monotone operator, they established the definition and Lipschitz continuity for its resolvent operator. Furthermore, based on the resolvent operator technique, they constructed an iterative algorithm for approximating the solution of a class of variational inclusions involving H-monotone operators. Their results improved and extended many known results in the literature.

In this paper, under the assumption that H is strongly monotone, continuous and single-valued, we first prove that a multivalued monotone operator is H-monotone if and only if it is maximal monotone. Subsequently, we define the resolvent operator associated with a strongly H-monotone operator, prove its Lipschitz continuity, and estimate its Lipschitz constant. Further, we study the variational inclusion introduced in [19] with strongly H-monotone operators. We construct a new algorithm for solving this class of variational inclusions by using the resolvent operator technique. Thanks to our estimate of Lipschitz constant of the resolvent operator, our convergence criteria for the algorithm are very different from corresponding ones in [19].

Throughout this paper, we suppose that $\mathcal{H}$ is a real Hilbert space endowed with a norm $\|\ \|$ and an inner product $\langle\cdot,\cdot\rangle$, respectively. Let $2^{\mathcal{H}}$ denote the family of all the nonempty subsets of $\mathcal{H}$. In what follows, we recall some concepts which will be used in the sequel.

DEFINITION 1.1. *Let $T, H : \mathcal{H} \to \mathcal{H}$ be two single-valued operators. T is said to be*

(i) *monotone if*

$$\langle Tx - Ty, x - y\rangle \geq 0, \qquad \forall x, y \in \mathcal{H};$$

(ii) *strictly monotone if T is monotone and*

$$\langle Tx - Ty, x - y\rangle = 0 \Leftrightarrow x = y;$$

(iii) *strongly monotone if there exists some constant $r > 0$, such that*

$$\langle Tx - Ty, x - y\rangle \geq r\|x - y\|^2, \qquad \forall x, y \in \mathcal{H};$$

(iv) *strongly monotone with respect to H if there exists some constant $\gamma > 0$, such that*

$$\langle Tx - Ty, Hx - Hy\rangle \geq \gamma\|x - y\|^2, \qquad \forall x, y \in \mathcal{H};$$

(v) *Lipschitz continuous if there exists some constant $s > 0$, such that*

$$\|Tx - Ty\| \leq s\|x - y\|, \qquad \forall x, y \in \mathcal{H}.$$

REMARK 1.1 (See [19].) If T and H are Lipschitz continuous with constants τ and s, respectively, and T is strongly monotone with respect to H with constant γ, then $\gamma \leq \tau s$.

DEFINITION 1.2. *A multivalued operator $M : \mathcal{H} \to 2^{\mathcal{H}}$ is said to be*

(i) *monotone if*

$$\langle x - y, u - v\rangle \geq 0, \qquad \forall u, v \in \mathcal{H}, \quad x \in Mu, \quad y \in Mv;$$

(ii) *strongly monotone if there exists some constant $\eta > 0$, such that*

$$\langle x - y, u - v\rangle \geq \eta\|u - v\|^2, \qquad \forall u, v \in \mathcal{H}, \quad x \in Mu, \quad y \in Mv,$$

(iii) *maximal monotone if M is monotone and $(I + \lambda M)(\mathcal{H}) = \mathcal{H}$ for all $\lambda > 0$, where I denotes the identity mapping on $\mathcal{H}$;*

(iv) *maximal strongly monotone if M is strongly monotone and $(I + \lambda M)(\mathcal{H}) = \mathcal{H}$ for all $\lambda > 0$.*

REMARK 1.2. A multivalued operator M is maximal monotone if and only if M is monotone and there is no other monotone operator whose graph properly contains the graph $\mathrm{Gr}(M)$ of M where $\mathrm{Gr}(M) = \{(u, x) \in \mathcal{H} \times \mathcal{H} : x \in Mu\}$.

2. STRONGLY H-MONOTONE OPERATORS

Recently, Fang and Huang [19] have introduced a new class of monotone operators, i.e., H-monotone operators and discussed some properties of this class of operators.

DEFINITION 2.1. *(See [19].) Let $H : \mathcal{H} \to \mathcal{H}$ be a single-valued operator and $M : \mathcal{H} \to 2^{\mathcal{H}}$ be a multivalued operator. M is said to be*

(i) *H-monotone if M is monotone and $(H + \lambda M)(\mathcal{H}) = \mathcal{H}$ holds for every $\lambda > 0$;*
(i) *strongly H-monotone if M is strongly monotone and $(H + \lambda M)(\mathcal{H}) = \mathcal{H}$ holds for every $\lambda > 0$.*

REMARK 2.1. If $H = I$, then the definition of I-monotone operators reduces to that of maximal monotone operators. As a matter of fact, the class of H-monotone operators has close relation with that of maximal monotone operators

In order to give a characterization of H-monotone operators, we need the following propositions and lemmas.

PROPOSITION 2.1. *(See [19].) Let $H : \mathcal{H} \to \mathcal{H}$ be a single-valued strictly monotone operator and $M : \mathcal{H} \to 2^{\mathcal{H}}$ be an H-monotone operator. Then M is maximal monotone*

Now recall the notion of m-accretive operators. Let X be a real Banach space with a norm $\|\cdot\|$, X^* denote the dual space of X and let $\langle x, f\rangle$ denote the value of $f \in X^*$ at $x \in X$. For $k \in (-\infty, +\infty)$, a multivalued operator $A : D(A) \subset X \to 2^X$ is said to be k-accretive if for each $x, y \in D(A)$ there exists $j(u - v) \in J(u - v)$, such that

$$\langle x - y, j(u - v)\rangle \geq k\|u - v\|^2, \qquad \forall\, x \in Au, \quad y \in Av. \tag{2.1}$$

Here $J : X \to 2^{X^*}$ is the normalized duality mapping defined by

$$J(x) = \left\{ f \in X^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2 \right\},$$

where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing. It is an immediate consequence of the Hahn-Banach theorem that $J(x)$ is nonempty for each $x \in X$. Moreover, it is known that J is single-valued if and only if X is smooth. For $k > 0$ in inequality (2.1), we say that A is strongly accretive while for $k = 0$, A is simply called accretive. In addition, if the range of $I + \lambda A$ is precisely X for all $\lambda > 0$, where I is the identity mapping on X, then A is said to be m-accretive. In particular, if $X = \mathcal{H}$ a real Hilbert space, then the definitions of strong accretiveness, accretiveness and m-accretiveness reduce to the ones of strong monotonicity, monotonicity and maximal monotonicity, respectively. Recently, Jung and Morales [20] proved the following deep and important result.

PROPOSITION 2.2. *(See [20].) Let X be a smooth Banach space, $A : D(A) \subset X \to 2^X$ be m-accretive, and $S : D(S) \subset X \to X$ be continuous and strongly accretive with $\overline{D(A)} \subset D(S)$. Then for each $z \in X$, the equation $z \in Sx + \lambda Ax$ has a unique solution x_λ for $\lambda > 0$*

COROLLARY 2.1. *Let $\mathcal{H}$ be a real Hilbert space. Let $M : \mathcal{H} \to 2^{\mathcal{H}}$ be a maximal monotone multivalued operator and $H : \mathcal{H} \to \mathcal{H}$ be a strongly monotone, continuous and single-valued operator Then for each $z \in \mathcal{H}$ the equation $z \in Hx + \lambda Mx$ has a unique solution x_λ for $\lambda > 0$*

REMARK 2.2. If $H : \mathcal{H} \to \mathcal{H}$ is a strongly monotone, continuous, single-valued operator and $M : \mathcal{H} \to 2^{\mathcal{H}}$ is a maximal monotone multivalued operator, then from Corollary 2.1 we know that the operator $(H + \lambda M)^{-1}$ is single-valued. Hence, we can define the resolvent operator $R^H_{M,\lambda} : \mathcal{H} \to \mathcal{H}$ as follows:

$$R^H_{M,\lambda}(u) = (H + \lambda M)^{-1}(u), \qquad \forall\, u \in \mathcal{H}. \tag{2.2}$$

We are ready to give a characterization for the class of H-monotone operators.

THEOREM 2.1 *Let H $\mathcal{H} \to \mathcal{H}$ be a strongly monotone, continuous and single-valued operator. Then a multivalued operator $M : \mathcal{H} \to 2^{\mathcal{H}}$ is H-monotone if and only if M is maximal monotone.*

PROOF. At first, let M $\mathcal{H} \to 2^{\mathcal{H}}$ be H-monotone. Since $H : \mathcal{H} \to \mathcal{H}$ is strongly monotone, H is strictly monotone. Thus, it follows from Proposition 2.1 that M is maximal monotone

Conversely, suppose that M is maximal monotone. Then M is monotone. Note that H is a strongly monotone, continuous and single-valued operator. Hence, it follows from Corollary 2.1 that for each $z \in \mathcal{H}$ the equation $z \in Hx + \lambda Mx$ has a unique solution x_λ for $\lambda > 0$. This implies that $(H + \lambda M)(\mathcal{H}) = \mathcal{H}$ holds for every $\lambda > 0$. Therefore, M is H-monotone. ∎

COROLLARY 2.2. *Let $H : \mathcal{H} \to \mathcal{H}$ be a strongly monotone, continuous and single-valued operator. Then a multivalued operator $M : \mathcal{H} \to 2^{\mathcal{H}}$ is strongly H-monotone if and only if M is maximal strongly monotone*

Let H be continuous and strongly monotone and M be maximal strongly monotone. Now we prove the Lipschitz continuity of the resolvent operator $R^H_{M,\lambda}$ defined by (2.2) and estimate its Lipschitz constant.

THEOREM 2.2. *Let $H : \mathcal{H} \to \mathcal{H}$ be continuous and strongly monotone with constant γ Let $M \cdot \mathcal{H} \to 2^{\mathcal{H}}$ be maximal strongly monotone with constant η. Then the resolvent operator $R^H_{M,\lambda} : \mathcal{H} \to \mathcal{H}$ is Lipschitz continuous with constant $1/(\gamma + \lambda\eta)$, i.e.,*

$$\left\|R^H_{M,\lambda}(u) - R^H_{M,\lambda}(v)\right\| \le \left(\frac{1}{(\gamma + \lambda\eta)}\right) \|u - v\|, \qquad \forall\, u, v \in \mathcal{H}.$$

PROOF Let u, v be any given points in $\mathcal{H}$. It follows from (2.2) that

$$R^H_{M,\lambda}(u) = (H + \lambda M)^{-1}(u) \qquad \text{and} \qquad R^H_{M,\lambda}(v) = (H + \lambda M)^{-1}(v).$$

This implies that

$$\frac{1}{\lambda}\left(u - H\left(R^H_{M,\lambda}(u)\right)\right) \in M\left(R^H_{M,\lambda}(u)\right) \qquad \text{and} \qquad \frac{1}{\lambda}\left(v - H\left(R^H_{M,\lambda}(v)\right)\right) \in M\left(R^H_{M,\lambda}(v)\right)$$

Since M is strongly monotone, we have

$$\begin{aligned}\eta\left\|R^H_{M,\lambda}(u) - R^H_{M,\lambda}(v)\right\|^2 &\le \frac{1}{\lambda}\left\langle u - H\left(R^H_{M,\lambda}(u)\right) - \left(v - H\left(R^H_{M,\lambda}(v)\right)\right), R^H_{M,\lambda}(u) - R^H_{M,\lambda}(v)\right\rangle \\ &= \frac{1}{\lambda}\left\langle u - v - \left(H\left(R^H_{M,\lambda}(u)\right) - H\left(R^H_{M,\lambda}(v)\right)\right), R^H_{M,\lambda}(u) - R^H_{M,\lambda}(v)\right\rangle\end{aligned}$$

It follows that

$$\begin{aligned}\|u - v\| \cdot \left\|R^H_{M,\lambda}(u) - R^H_{M,\lambda}(v)\right\| &\ge \left\langle u - v, R^H_{M,\lambda}(u) - R^H_{M,\lambda}(v)\right\rangle \\ &\ge \left\langle H\left(R^H_{M,\lambda}(u)\right) - H\left(R^H_{M,\lambda}(v)\right), R^H_{M,\lambda}(u) - R^H_{M,\lambda}(v)\right\rangle \\ &\quad + \lambda\eta\left\|R^H_{M,\lambda}(u) - R^H_{M,\lambda}(v)\right\|^2 \\ &\ge \gamma\left\|R^H_{M,\lambda}(u) - R^H_{M,\lambda}(v)\right\|^2 + \lambda\eta\left\|R^H_{M,\lambda}(u) - R^H_{M,\lambda}(v)\right\|^2 \\ &= (\gamma + \lambda\eta)\left\|R^H_{M,\lambda}(u) - R^H_{M,\lambda}(v)\right\|^2,\end{aligned}$$

and hence,

$$\left\|R^H_{M,\lambda}(u) - R^H_{M,\lambda}(v)\right\| \le \left(\frac{1}{(\gamma + \lambda\eta)}\right) \|u - v\|, \qquad \forall\, u, v \in \mathcal{H}$$

This completes the proof ∎

3. VARIATIONAL INCLUSIONS

In this section, we consider a class of variational inclusions involving strong H-monotone operators in Hilbert spaces. We construct a new iterative algorithm for approximating solutions of this class of variational inclusions by using the resolvent operator technique

Let $A, H : \mathcal{H} \to \mathcal{H}$ be two single-valued operators and $M : \mathcal{H} \to 2^{\mathcal{H}}$ be a multivalued operator. Consider the following variational inclusion: find $u \in \mathcal{H}$, such that

$$0 \in A(u) + M(u). \tag{3.1}$$

SPECIAL CASES.

(1) When M is maximal monotone and A is strongly monotone and Lipschitz continuous, problem (3.1) has been studied by Huang [15].
(2) If $M = \partial\varphi$ where $\partial\varphi$ denotes the subdifferential of a proper, convex and lower semicontinuous functional $\varphi : \mathcal{H} \to R \cup \{+\infty\}$, then problem (3.1) reduces to the following problem: find $u \in \mathcal{H}$, such that

$$\langle A(u), v - u\rangle + \varphi(v) - \varphi(u) \geq 0, \qquad \forall\, v \in \mathcal{H}, \tag{3.2}$$

which is called a nonlinear variational inequality and has been studied by many authors; see, for example, [2–5,18,21].
(3) If $M = \partial\delta_K$ where δ_K is the indicator function of a nonempty, closed and convex subset K of $\mathcal{H}$, then problem (3.1) reduces to the following problem: find $u \in K$, such that

$$\langle A(u), v - u\rangle \geq 0, \qquad \forall\, v \in K, \tag{3.3}$$

which is the classical variational inequality; see, e.g., [1,7,22].

From the definition of $R^H_{M,\lambda}$, we have the following result

LEMMA 3.1. *Let $H : \mathcal{H} \to \mathcal{H}$ be a strongly monotone and continuous operator and $M : \mathcal{H} \to 2^{\mathcal{H}}$ be maximal monotone. Then $u \in \mathcal{H}$ is a solution of problem (3.1) if and only if*

$$u = R^H_{M,\lambda}[H(u) - \lambda A(u)],$$

for some $\lambda > 0$.

Based on Lemma 3.1, we construct the following iterative algorithm for solving problem (3.1).

ALGORITHM 3.1. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $[0,1]$ For any $u_0 \in \mathcal{H}$, the iterative sequence $\{u_n\} \subset \mathcal{H}$ is defined by

$$\begin{aligned} u_{n+1} &= (1 - \alpha_n)u_n + \alpha_n R^H_{M,\lambda}[H(v_n) - \lambda A(v_n)], \\ v_n &= (1 - \beta_n)u_n + \beta_n R^H_{M,\lambda}[H(u_n) - \lambda A(u_n)], \qquad n = 0, 1, \dots \end{aligned} \tag{3.4}$$

When $\alpha_n = 1$, $\beta_n = 0$, $\forall\, n \geq 0$, Algorithm 3 1 reduces immediately to Algorithm 3 1 in [19]. For easy reference, we present it here as follows

ALGORITHM 3.2. (See [19].) For any $u_0 \in \mathcal{H}$, the iterative sequence $\{u_n\} \subset \mathcal{H}$ is defined by

$$u_{n+1} = R^H_{M,\lambda}[H(u_n) - \lambda A(u_n)], \qquad n = 0, 1, \dots \tag{3.5}$$

THEOREM 3.1 *Let $\{\alpha_n\}$ and $\{\beta_n\}$ be employed by Algorithm 3.1 with $\sum_{n=0}^{\infty} \alpha_n = \infty$. Let $H : \mathcal{H} \to \mathcal{H}$ be a strongly monotone and Lipschitz continuous operator with constants γ and τ, respectively. Let $A : \mathcal{H} \to \mathcal{H}$ be Lipschitz continuous and strongly monotone with respect to H with constants s and r, respectively. Assume that $M : \mathcal{H} \to 2^{\mathcal{H}}$ is a strongly H-monotone operator with constant η and that there exists some constant $\lambda > 0$, such that*

$$\left|\lambda - \frac{r+\gamma\eta}{s^2-\eta^2}\right| < \frac{\sqrt{(r+\gamma\eta)^2 - (s^2-\eta^2)(\tau^2-\gamma^2)}}{s^2-\eta^2}, \qquad (*)$$
$$(r+\gamma\eta)^2 > \left(s^2-\eta^2\right)\left(\tau^2-\gamma^2\right), \qquad s > \eta.$$

Then the iterative sequence $\{u_n\}$ generated by Algorithm 3.1 converges strongly to the unique solution $u^* \in \mathcal{H}$ of problem (3.1). Moreover, we have for all $n \geq 0$

$$\|u_{n+1} - u^*\| \leq \prod_{j=0}^{n} (1 - (1-k)\alpha_j)\, \|u_0 - u^*\|,$$

where $k = (1/(\gamma+\lambda\eta))\sqrt{\tau^2 - 2\lambda r + \lambda^2 s^2}$. In particular, if we take $\alpha_n = \hat{l}(\hat{l}+n)/(\hat{l}+n+1)^2$, $\forall\, n \geq 0$, where $\hat{l} = 2/(1-k)$, then we have the following convergence rate estimate:

$$\|u_m - u^*\| = O\left(\frac{1}{m}\right).$$

PROOF. First note that H is strongly monotone and Lipschitz continuous. Since M is strongly H-monotone, it follows from Corollary 2.2 that M is maximal strongly monotone. Thus, from Remark 2.2 we know that the resolvent operator $R^H_{M,\lambda} = (H + \lambda M)^{-1}$ is well defined.

Next we divide the proof into three steps.

STEP 1. We claim that problem (3.1) has a unique solution $u^* \in \mathcal{H}$. Indeed, we define the mapping $F : \mathcal{H} \to \mathcal{H}$ as follows:

$$F(u) = R^H_{M,\lambda}[H(u) - \lambda A(u)], \qquad \forall\, u \in \mathcal{H}.$$

Then it follows from Theorem 2.2 that for each $u, v \in \mathcal{H}$,

$$\begin{aligned}\|F(u) - F(v)\| &= \left\|R^H_{M,\lambda}(H(u) - \lambda A(u)) - R^H_{M,\lambda}(H(v) - \lambda A(v))\right\| \\ &\leq \left(\frac{1}{(\gamma+\lambda\eta)}\right) \|H(u) - H(v) - \lambda(A(u) - A(v))\|.\end{aligned} \qquad (3.6)$$

By assumptions,

$$\begin{aligned}\|H(u) - H(v) - \lambda(A(u) - A(v))\|^2 &= \|H(u) - H(v)\|^2 - 2\lambda\langle A(u) - A(v), H(u) - H(v)\rangle \\ &\quad + \lambda^2 \|A(u) - A(v)\|^2 \\ &\leq \left(\tau^2 - 2\lambda r + \lambda^2 s^2\right) \|u - v\|^2.\end{aligned} \qquad (3.7)$$

Combining (3.6) with (3.7) yields

$$\|F(u) - F(v)\| \leq k\|u - v\|, \qquad (3.8)$$

where $k = (1/(\gamma+\lambda\eta))\sqrt{\tau^2 - 2\lambda r + \lambda^2 s^2}$.

From $(*)$ and (3.8), we know that $0 \leq k < 1$, and thus, F is a contraction. By the Banach Contraction Principle, we conclude that F has a unique fixed point $u^* \in \mathcal{H}$. This implies that

$$u^* = F(u^*) = R^H_{M,\lambda}[H(u^*) - \lambda A(u^*)],$$

and hence, u^* is a unique solution of the equation $u = R^H_{M,\lambda}[H(u) - \lambda A(u)]$. Therefore, from Lemma 3.1 it follows that u^* is the unique solution of problem (3.1).

STEP 2. We claim that the sequence $\{u_n\}$ converges strongly to the unique solution u^* of problem (3.1). Indeed from (3.4), Lemma 3.1 and Theorem 2.2, we obtain

$$\begin{aligned}\|u_{n+1} - u^*\| &= \left\|(1-\alpha_n)(u_n - u^*) + \alpha_n \left(R^H_{M,\lambda}[H(v_n) - \lambda A(v_n)] - u^*\right)\right\| \\ &\le (1-\alpha_n)\|u_n - u^*\| + \alpha_n \left\|R^H_{M,\lambda}[H(v_n) - \lambda A(v_n)] - u^*\right\| \\ &= (1-\alpha_n)\|u_n - u^*\| \\ &\quad + \alpha_n \left\|R^H_{M,\lambda}[H(v_n) - \lambda A(v_n)] - R^H_{M,\lambda}[H(u^*) - \lambda A(u^*)]\right\| \\ &\le (1-\alpha_n)\|u_n - u^*\| \\ &\quad + \alpha_n \left(\frac{1}{(\gamma + \lambda\eta)}\right) \|H(v_n) - H(u^*) - \lambda(A(v_n) - A(u^*))\| \\ &\le (1-\alpha_n)\|u_n - u^*\| + \alpha_n k \|v_n - u^*\|,\end{aligned} \tag{3.9}$$

where $k = (1/(\gamma + \lambda\eta))\sqrt{\tau^2 - 2\lambda r + \lambda^2 s^2}$.

Furthermore, observe that

$$\begin{aligned}\|v_n - u^*\| &= \left\|(1-\beta_n)(u_n - u^*) + \beta_n \left(R^H_{M,\lambda}[H(u_n) - \lambda A(u_n)] - u^*\right)\right\| \\ &\le (1-\beta_n)\|u_n - u^*\| + \beta_n \left\|R^H_{M,\lambda}[H(u_n) - \lambda A(u_n)] - u^*\right\| \\ &= (1-\beta_n)\|u_n - u^*\| + \beta_n \left\|R^H_{M,\lambda}[H(u_n) - \lambda A(u_n)] \right. \\ &\quad \left. - R^H_{M,\lambda}[H(u^*) - \lambda A(u^*)]\right\| \\ &\le (1-\beta_n)\|u_n - u^*\| + \beta_n \left(\frac{1}{(\gamma + \lambda\eta)}\right) \|H(u_n) - H(u^*) \\ &\quad - \lambda(A(u_n) - A(u^*))\| \\ &\le (1-\beta_n)\|u_n - u^*\| + \beta_n k \|u_n - u^*\| \\ &= (1 - (1-k)\beta_n)\|u_n - u^*\| \\ &\le \|u_n - u^*\|.\end{aligned} \tag{3.10}$$

Hence, it follows from (3.9) and (3.10) that

$$\begin{aligned}\|u_{n+1} - u^*\| &\le (1-\alpha_n)\|u_n - u^*\| + \alpha_n k \|v_n - u^*\| \\ &\le (1-\alpha_n)\|u_n - u^*\| + \alpha_n k \|u_n - u^*\| \\ &= (1 - (1-k)\alpha_n)\|u_n - u^*\| \\ &\le (1 - (1-k)\alpha_n) \cdots (1 - (1-k)\alpha_0)\|u_0 - u^*\| \\ &= \prod_{j=0}^{n} (1 - (1-k)\alpha_j) \ \|u_0 - u^*\|.\end{aligned} \tag{3.11}$$

Since $0 \le k < 1$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, we have

$$\prod_{n=0}^{\infty} (1 - (1-k)\alpha_n) = \lim_{n\to\infty} \prod_{j=0}^{n} (1 - (1-k)\alpha_j) = 0,$$

which, hence, implies that $\{u_n\}$ converges strongly to u^*.

STEP 3. We claim that there holds the convergence rate estimate $\|u_m - u^*\| = O(1/m)$ for $\alpha_n = \hat{l}(\hat{l} + n)/(\hat{l} + n + 1)^2$, $\forall n \ge 0$, where $\hat{l} = 2/(1-k)$. Indeed, by making use of (3.11), we derive for all $n \geq 0$

$$\begin{aligned}\|u_{n+1} - u^*\| &\leq (1-(1-k)\alpha_n)\|u_n - u^*\| \\ &= \left(1-(1-k)\cdot\left(\frac{2}{1-k}\right)\frac{\hat{l}+n}{(\hat{l}+n+1)^2}\right)\|u_n - u^*\| \\ &= \left(1-\frac{2(\hat{l}+n)}{(\hat{l}+n+1)^2}\right)\|u_n - u^*\| = \left(\frac{(\hat{l}+n)^2+1}{(\hat{l}+n+1)^2}\right)\|u_n - u^*\|,\end{aligned}$$

and hence,

$$\left(\hat{l}+n+1\right)^2\|u_{n+1} - u^*\| - \left(\hat{l}+n\right)^2\|u_n - u^*\| \leq \|u_n - u^*\|.$$

Summing this inequality from $n = 0$ to $m-1$ $(m \geq 1)$, we obtain

$$\left(\hat{l}+m\right)^2\|u_m - u^*\| - \hat{l}^2\|u_0 - u^*\| \leq m\|u_0 - u^*\|.$$

Thus,

$$\|u_m - u^*\| \leq \frac{m+\hat{l}^2}{(m+\hat{l})^2}\|u_0 - u^*\|,$$

from which it follows that

$$\|u_m - u^*\| = O\left(\frac{1}{m}\right).$$

This completes the proof. ∎

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