On the 3-kings and 4-kings in multipartite tournaments

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Abstract

Koh and Tan gave a sufficient condition for a 3-partite tournament to have at least one 3-king in [K.M. Koh, B.P. Tan, Kings in multipartite tournaments, Discrete Math. 147 (1995) 171–183, Theorem 2]. In Theorem 1 of this paper, we extend this result to \( n \)-partite tournaments, where \( n \geq 3 \). In \cite{Koh1996, Koh1997} Koh and Tan showed that in any \( n \)-partite tournament with no transmitters and 3-kings, where \( n \geq 2 \), the number of 4-kings is at least eight, and completely characterized all \( n \)-partite tournaments having exactly eight 4-kings and no 3-kings. Using Theorem 1, we strengthen substantially the above result for \( n \geq 3 \). Motivated by the strengthened result, we further show that in any \( n \)-partite tournament \( T \) with no transmitters and 3-kings, where \( n \geq 3 \), if there are \( r \) partite sets of \( T \) which contain 4-kings, where \( 3 \leq r \leq n \), then the number of 4-kings in \( T \) is at least \( r + 8 \). An example is given to justify that the lower bound is sharp.

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1. Introduction

Let \( D \) be a digraph with vertex set \( V(D) \). Given \( u, v \in V(D) \), the distance from \( u \) to \( v \) is denoted by \( d(u, v) \). The eccentricity \( e(v) \) of a vertex \( v \) in \( D \) is defined by \( e(v) = \max\{d(v, x) | x \in V(D)\} \). Following \cite{Landau1953}, call a vertex \( w \) in \( D \) an \( r \)-king, where \( r \) is a positive integer, if \( e(w) \leq r \). The set and the number of \( r \)-kings in \( D \) are, respectively, denoted by \( K_r(D) \) and \( k_r(D) \).

Given \( u, v \in V(D) \), we write \( u \rightarrow v \) if \( u \) is adjacent to \( v \) (i.e., \( u \) dominates \( v \)). For any two subsets \( A, B \) of \( V(D) \), we write \( A \rightarrow B \) to signify that \( a \rightarrow b \) for each \( a \in A \) and \( b \in B \). If \( A = \{a\} \), then \( A \rightarrow B \) is replaced by \( a \rightarrow B \). Likewise, if \( B = \{b\} \), then \( A \rightarrow B \) is replaced by \( A \rightarrow b \). For \( v \in V(D) \), let \( O(v) = \{x \in V(D) | x \rightarrow v\} \), \( I(v) = \{x \in V(D) | x \leftarrow v\} \), \( s(v) = |O(v)| \) and \( s^-(v) = |I(v)| \). We call \( s(v) \) and \( s^-(v) \) the outdegree and indegree of \( v \), respectively. A vertex \( u \) in \( D \) is called a transmitter if \( s^-(u) = 0 \) and \( s(u) > 0 \).

The study of the existence of kings was originated in the class of tournaments. The concept of a king was implicitly introduced in 1953 by Landau \cite{Landau1953}. Let \( H \) be a tournament. It is trivial that a vertex \( w \) is a 1-king of \( H \) if and only if \( w \) is a transmitter (and hence the only transmitter) of \( H \). Thus, \( k_1(H) \leq 1 \). Landau noted in \cite{Landau1953} that every vertex...
of maximum outdegree in $H$ is a 2-king, and so $k_2(H) \geqslant 1$. It is known [13] that if $H$ contains no transmitter, then $k_2(H) \geqslant 3$ (see also [1,12,16]).

Given a digraph $D$, a trivial necessary condition for the existence of an $r$-king in $D$ for some $r$ is that $D$ contains at most one transmitter. Let $T$ be an $n$-partite tournament with at most one transmitter, where $n \geqslant 2$. Gutin [3] (and, independently, Petrovic and Thomassen [15]) showed that $k_4(T) \geqslant 1$. Gutin [3] also proved that there exist infinitely many multipartite tournaments $T$ such that $k_3(T) = 0$ and $K_4(T) \neq \emptyset$. Thus, in the study of multipartite tournaments, 4-kings are of special interest. It is obvious that $k_4(T) = k_2(T) = 1$ if and only if $T$ contains a unique transmitter. By considering $T$ with no transmitters, Koh and Tan [7] showed that (1) $k_4(T) \geqslant 4$ if $n = 2$, (2) $k_4(T) \geqslant 3$ if $n \geqslant 3$, and (3) completely characterized all $T$ with no transmitters such that the equalities in (1) and (2) hold. All $T$ with no transmitters and $n \geqslant 3$ such that $k_4(T) = 4$ were characterized in [5]. In characterizing multipartite tournaments $T$ with least possible values of $k_4(T)$, it was found that all the existing 4-kings are actually 3-kings. Indeed, it was shown, respectively, in [7,5] that for $n \geqslant 3$, if $k_4(T) = 3$, then $K_4(T) = K_2(T)$; and if $k_4(T) = 4$, then $K_4(T) = K_3(T)$. The following problem thus arises naturally:

If an $n$-partite tournament $T$ contains no transmitters and $k_3(T) = 0$, what is the least possible value of $k_4(T)$?

The problem was solved completely by Koh and Tan in [8] (for $n = 2$) and [9] (for $n \geqslant 3$). It was shown in [8,9] that if $T$ contains no transmitters and $k_3(T) = 0$, then $k_4(T) \geqslant 8$. All $T$ with no transmitters such that $k_3(T) = 0$ and $k_4(T) = 8$ were also completely characterized in [8,9]. In Section 3 of this paper, we strengthen substantially the result obtained in [9] (for $n \geqslant 3$) to the above problem by establishing Theorem 2. This theorem then motivates us to consider the following problem:

In an $n$-partite tournament $T$ with no transmitters and 3-kings, where $n \geqslant 3$, if there are $r$ partite sets of $T$ which contain 4-kings, where $3 \leqslant r \leqslant n$, what is the least possible value of $k_4(T)$?

A complete solution to this problem is given in this paper. In the process of proving Theorem 2, we also obtain some sufficient conditions for an $n$-partite tournament $T$, where $n \geqslant 3$, to have $k_3(T) \geqslant 1$. Koh and Tan [7] gave a sufficient condition for a 3-partite tournament to have at least one 3-king. In Theorem 1 of this paper, we extend their result to $n$-partite tournaments, where $n \geqslant 3$.

More results on 4-kings in multipartite tournaments can be found in [6,10,14]. For information on kings in other families of digraphs, see [4,17] and the book [1, pp. 74–78].

2. Notation and basic lemmas

In this section, we shall state a series of basic results on tournaments and multipartite tournaments which will be used to prove our main results in the next section. First of all, we shall introduce some notation.

Throughout this paper, the $n$ partite sets of an $n$-partite tournament $T$, where $n \geqslant 2$, are denoted by $V_1, V_2, \ldots, V_n$. For $i = 1, 2, \ldots, n$, let

$$M_i = \{w \in V_i | s(w) \geqslant s(x) \text{ for each } x \in V_i\}.$$ 

For a digraph $D$ and $A \subseteq V(D)$, we denote by $D[A]$ the subdigraph of $D$ induced by $A$.

The following lemma on tournaments can be proved easily.

**Lemma 1.** Let $H$ be a tournament of order $n \geqslant 3$ with no transmitters. Then each vertex $u$ in $K_2(H)$ lies on some 3-cycle of $H$.

In the remaining lemmas of this section, we shall assume that $T$ is an $n$-partite tournament, where $n \geqslant 2$. Let $x_i \in M_i$, $i = 1, 2, \ldots, n$. It is clear that $H = T[\{x_1, x_2, \ldots, x_n\}]$ forms itself a tournament of order $n$. We shall call such a tournament $H$ a maximum-score-tournament (in short, an MS-tournament) of $T$.

**Lemma 2** (Petrovic and Thomassen [15]). Assume that $T$ contains at most one transmitter. Let $H$ be an MS-tournament of $T$. Then $K_2(H) \subseteq K_4(T)$, and so $k_4(T) \geqslant k_2(H) \geqslant 1$. 

Theorem 1. Let $T$ be an $n$-partite tournament, where $n \geq 3$, with no transmitters. If $T$ contains an MS-tournament $H = T[\{x_1, x_2, \ldots, x_n\}]$ such that $H$ itself has no transmitter, Koh and Tan showed in [9, Theorem 1] that if $k_3(T) = 0$, then $k_3(T) \geq 9$. Our Theorem 1 tells us that $k_3(T) \geq 1$ for such an $n$-partite tournament $T$, and so it is not possible to have $k_3(T) = 0$ for such a $T$.

**Theorem 1.** Let $T$ be an $n$-partite tournament, where $n \geq 3$, with no transmitters. If $T$ contains an MS-tournament $H = T[\{x_1, x_2, \ldots, x_n\}]$ such that $H$ itself has no transmitters, then $k_3(T) \geq 1$.

**Proof.** Suppose $k_3(T) = 0$. By assumption, $k_2(H) \geq 3$. We may assume $x_n \in K_2(H)$. By Lemma 2, $x_n \in K_4(T)$. Since $k_3(T) = 0$, there exists $z \in V_i$, $i \in \{1, 2, \ldots, n\}$ such that $d(x_n, z) = 4$. By Lemma 1, $x_n$ lies on some 3-cycle of $T$. By Lemma 4, $d(x_n, x) \leq 3$ for all $x \in V_n$. Thus $i \neq n$. As $x_n \in K_2(H)$, $d(x_n, x_i) \leq 2$. Hence $z \neq x_i$. Note that $x_i \to x_n$; otherwise, by Lemma 5, $d(x_n, z) \leq 3$. Observe also that $z \to x_j$ for all $j \neq i$; otherwise, $d(x_n, z) \leq d(x_n, x_j) + d(x_j, z) \leq 2 + 1 = 3$. We may assume $i = 1$. Thus $z \in V_1, z \to x_j$ for all $j \neq 1$ and $x_1 \to x_n$.

Among the vertices $z$ in $V_1$ such that $z \to x_j$ for all $j \neq 1$, let $y$ have maximum outdegree. Since $y \to x_j$ for all $j \neq 1$, by Lemma 5, $d(y, x) \leq 3$ for all $x \in V(T) \setminus V_1$. By Lemma 7, $y \in K_4(T)$. As $k_3(T) = 0$, there exists $u \in V_1 \setminus \{y\}$ such that $d(y, u) = 4$. Note that $O(y) \subseteq O(u)$. Thus, $u \to x_j$ for all $j \neq 1$. It follows from our choice of $y$ that $O(y) = O(u)$. By Lemma 4, $y$ lies on no 3-cycles in $T$.

**Claim 1.** $I(y) \subseteq K_4(T)$.

As $T$ has no transmitters, $I(y) \neq \emptyset$. Let $v \in I(y)$. Then $v \to y \to x_j$ for all $j \neq 1$. Thus, $d(v, x_j) \leq 2$ for all $j \neq 1$. By Lemma 6, $d(v, x) \leq 4$ for all $x \in V(T) \setminus V_1$. Suppose $v \not\in K_4(T)$. Then there exists $x \in V_1 \setminus \{y\}$ such that
\[d(v, x) \geq 5.\] Observe that \(x \to O(y) \cup \{v\};\) otherwise, \(d(v, x) \leq 3.\) Thus \(x \to x_j\) for all \(j \neq 1\) and \(s(x) \geq s(y) + 1,\) a contradiction to the choice of \(y.\) Hence \(v \in K_4(T),\) and so \(I(y) \subseteq K_4(T).\) This proves Claim 1.

Since \(I(y) \neq \emptyset,\) there exists \(k \in \{2, 3, \ldots, n\}\) such that \(I(y) \cap V_k \neq \emptyset.\) Among the vertices in \(I(y) \cap V_k,\) let \(v\) have maximum outdegree. By Claim 1, \(v \in K_4(T).\) Since \(y\) is not on any 3-cycle, \(v \to x_j\) for all \(j \neq 1, k.\) By Lemma 5,

(a) \(d(v, x) \leq 3\) for all \(x \in V(T) \setminus (V_1 \cup V_k).\)

We now show that \(d(v, x) \leq 3\) for all \(x \in V_1.\) Suppose \(d(v, x) = 4\) for some \(x \in V_1 \setminus \{y, u\}.\) Then \(x \to O(y) \cup \{v\};\) otherwise, \(d(v, x) \leq 3.\) Thus \(x \to x_j\) for all \(j \neq 1\) and \(s(x) \geq s(y) + 1,\) a contradiction to the choice of \(y.\) Hence

(b) \(d(v, x) \leq 3\) for all \(x \in V_1.\)

Now as \(k_3(T) = 0,\) there exists \(w \in V_k \setminus \{v\}\) such that \(d(v, w) = 4.\) Note that \(O(v) \subseteq O(w).\) Thus \(w \in I(y) \cap V_k.\) It follows from our choice of \(v\) that \(O(v) = O(w).\) By Lemma 4, \(v\) lies on no 3-cycles in \(T.\)

Claim 2. \(I(v) \subseteq V_1.\)

As \(T\) has no transmitters, \(I(v) \neq \emptyset.\) Let \(z \in I(v).\) Suppose \(z \notin V_1.\) Then \(z \in V_j \setminus \{x_j\}\) for some \(j \neq 1, k.\) Observe that \(z \to y;\) otherwise, \(v_3zv\) is a 3-cycle. Since \(z \to v \to x_j\) and \(x_j \in M_j,\) there exists \(t \in V_i, i \neq j,\) such that \(x_j \to t \to z.\) Assume \(i \neq 1.\) If \(y \to t,\) then \(yztv\) is a 3-cycle. If \(t \to y,\) then \(tyztv\) is a 3-cycle. In either case, \(y\) lies on a 3-cycle, a contradiction. Assume now \(i = 1.\) If \(v \to t,\) then \(vztv\) is a 3-cycle. If \(t \to v,\) then \(vxtj\) is a 3-cycle. In either case, \(v\) lies on a 3-cycle, again a contradiction. Thus \(z \in V_1.\) Hence \(I(v) \subseteq V_1.\)

Claim 3. \(x_k \to V_j\) for all \(j \neq 1, k.\)

By Claim 2, \(v \to V_j\) for all \(j \neq 1, k.\) Since \(v \to y \to x_k\) and \(x_k \in M_k,\) there exists \(z \in V(T) \setminus V_k\) such that \(x_k \to z \to v.\) By Claim 2, \(z \in V_1.\) Also, since \(v\) lies on no 3-cycles, \(z \to V_j\) for all \(j \neq 1, k.\) Note that \(x_k \to z \to V_j\) for all \(j \neq 1, k.\) Thus

(c) \(d(x_k, x) \leq 2\) for all \(x \in V(T) \setminus (V_1 \cup V_k).\)

Suppose there exists \(t \in V_j, j \neq 1, k\) such that \(t \to x_k.\) Then \(txkzt\) is a 3-cycle containing \(x_k.\) By Lemma 4,

(d) \(d(x_k, x) \leq 3\) for all \(x \in V_k.\)

Assume \(x_k \to x_1.\) By Lemma 5,

(e) \(d(x_k, x) \leq 3\) for all \(x \in V_1.\)

By (c)–(e), \(x_k \in K_3(T),\) a contradiction. Thus \(x_1 \to x_k.\) By Lemma 5,

(f) \(d(x_1, x) \leq 3\) for all \(x \in V_k.\)

Also, by (c),

(g) \(d(x_1, x) \leq 3\) for all \(x \in V(T) \setminus (V_1 \cup V_k).\)

Since \(x_1 \to x_n\) and \(x_n \in K_2(H),\) there exists \(x_s\) such that \(x_n \to x_s \to x_1;\) otherwise, \(d(x_n, x_1) \geq 3\) in \(H.\) Observe that \(x_1x_nx_sx_1\) is a 3-cycle containing \(x_1.\) By Lemma 4,

(h) \(d(x_1, x) \leq 3\) for all \(x \in V_1.\)
Theorem 2. Let $T$ be an $n$-partite tournament $\{T\}$, where $n \geq 3$, with at most one transmitter. Suppose $|M_i| = 1$ for each $i = 1, 2, \ldots, n$, then $k_3(T) \geq 1$.

**Proof.** The result is obvious if $T$ contains a transmitter. Assume now $T$ has no transmitters. Let $M_i = \{x_i\}$ for each $i = 1, 2, \ldots, n$. Consider $H = T[\{x_1, x_2, \ldots, x_n\}]$. If $H$ contains no transmitter, then the result follows from Theorem 1. Suppose now $H$ contains a transmitter. We may assume $x_1$ is the transmitter of $H$. By Lemma 3, $d(x_1, x) = 2$ for all $x \in V_1 \setminus \{x_1\}$. By Lemma 5, $d(x_1, x) \leq 3$ for all $x \in V_i$ for each $i = 2, 3, \ldots, n$. Thus $x_1 \in K_3(T)$, and so $k_3(T) \geq 1$. \qed

We shall now use Theorem 1 to strengthen substantially the result obtained by Koh and Tan in [9, Theorem 2] by establishing Theorem 2. Note that Theorem 2 is also a direct extension of [8, Theorems 1 and 2].

**Theorem 2.** Let $T$ be an $n$-partite tournament, where $n \geq 3$, with no transmitters and $k_3(T) = 0$. Then:

(i) there exist $p, q \in \{1, 2, \ldots, n\}$, $p \neq q$, such that $|K_4(T) \cap V_p| \geq 4$ and $|K_4(T) \cap V_q| \geq 4$;
(ii) $T$ contains the digraph of Fig. 1 as a subdigraph;
(iii) $k_4(T) = 8$ if and only if $T$ is isomorphic to a multipartite tournament of Fig. 2, where $T[V'_1 \cup V'_2], T[V_i \cup V_j]$ for $i, j \in \{3, 4, \ldots, n\}, i \neq j$, and $T[V'_r \cup V_s]$ for $r = 1, 2$ and $s = 3, 4, \ldots, n$, are arbitrary bipartite tournaments.

**Proof.** (i) and (ii): By Theorem 1, we may assume that every $MS$-tournament of $T$ has a transmitter. Let $H = T[\{x_1, x_2, \ldots, x_n\}]$ be an $MS$-tournament of $T$. We may assume $x_1$ is a transmitter of $H$. Then $x_1 \in K_2(H)$. By Lemma 2, $x_1 \in K_4(T)$. Since $x_1 \rightarrow x_j$ for all $j \geq 2$, by Lemma 5, $d(x_1, x) \leq 3$ for all $x \in V(T) \setminus V_1$. As $k_3(T) = 0$, there exists $u \in V_1$ such that $d(x_1, u) = 4$. Since $x_1 \in M_1$, by Lemma 3, $O(u) = O(x_1)$. By Lemma 9, $u \in K_4(T)$. Fig. 1.
Since \( k_3(T) = 0 \), by Lemma 8, \( u \) and \( x_1 \) lie on no 3-cycles in \( T \). As \( T \) has no transmitters, \( I(x_1) \neq \emptyset \). Let \( y \in I(x_1) \). Then \( d(y, x_i) \leq 2 \) for each \( i = 1, 2, \ldots, n \). By Lemma 6, \( y \in K_4(T) \), and so \( I(x_1) \subseteq K_4(T) \).

We may assume \( I(x_1) \cap V_2 \setminus \{x_2\} \neq \emptyset \). Among the vertices in \( I(x_1) \cap V_2 \setminus \{x_2\} \), let \( v \) have maximum outdegree. Since \( x_1 \) is not on any 3-cycle, \( v \rightarrow x_i \) for all \( i \geq 3 \). Thus, \( v \rightarrow x_i \) for all \( i \neq 2 \). By Lemma 5, \( d(v, x) \leq 3 \) for all \( x \in V(T) \setminus V_2 \).

Now as \( k_3(T) = 0 \), there exists \( w \in V_2 \setminus \{v\} \) such that \( d(v, w) = 4 \). It follows that \( O(v) \subseteq O(w) \). Hence \( w \in I(x_1) \). In addition, from the choice of \( v \), we have \( O(v) = O(w) \). By Lemma 4, \( v \) lies on no 3-cycles in \( T \); otherwise, \( d(v, w) \leq 3 \).

**Claim 1.** \( I(v) \subseteq V_1 \).

This follows from the same arguments as in the proof of Claim 2 of Theorem 1.

**Claim 2.** \( I(v) \subseteq K_4(T) \).

By Claim 1, \( v \rightarrow V_j \) for all \( j \geq 3 \). Let \( a \in I(v) \). Since \( v \) lies on no 3-cycles in \( T \), \( a \rightarrow V_j \) for all \( j \geq 3 \). Thus \( I(a) \subseteq V_2 \). Let \( z \in V_2 \setminus \{v\} \). If \( d(v, z) = 2 \), then \( d(a, z) \leq d(a, v) + d(v, z) = 1 + 2 = 3 \). If \( O(v) \subseteq O(z) \), then \( z \in I(x_1) \). By the choice of \( v \), we have \( O(v) = O(z) \). Thus \( I(z) = I(v) \) and so \( a \rightarrow z \). In either case, \( d(a, z) \leq 3 \).

Hence

(a) \( d(a, x) \leq 3 \) for all \( x \in V(T) \setminus V_1 \).

By Lemma 7, \( a \in K_4(T) \), and so \( I(v) \subseteq K_4(T) \). This proves Claim 2.

By Claims 1 and 2, we have \( I(v) \subseteq K_4(T) \cap V_1 \). Among the vertices in \( I(v) \), let \( a \) have maximum outdegree. As \( k_3(T) = 0 \), by (a), there exists \( b \in V_1 \setminus \{a\} \) such that \( d(a, b) = 4 \). It follows that \( O(a) \subseteq O(b) \). Thus \( b \in I(v) \). From the choice of \( a \), we have \( O(a) = O(b) \). By Lemma 4, \( a \) lies on no 3-cycles in \( T \); otherwise, \( d(a, b) \leq 3 \).

**Claim 3.** \( I(a) \subseteq K_4(T) \cap V_2 \).

Since \( a \rightarrow V_j \) for all \( j \geq 3 \), we have \( I(a) \subseteq V_2 \). Let \( c \in I(a) \). As \( a \) lies on no 3-cycles in \( T \), \( c \rightarrow V_j \) for all \( j \geq 3 \).

Let \( z \in V_1 \setminus \{a\} \). If \( d(a, z) = 2 \), then \( d(c, z) \leq d(c, a) + d(a, z) = 1 + 2 = 3 \). If \( O(a) \subseteq O(z) \), then \( z \in I(v) \). By the choice of \( a \), we have \( O(a) = O(z) \). Thus \( I(z) = I(a) \) and so \( c \rightarrow z \). In either case, \( d(c, z) \leq 3 \).

Hence

(b) \( d(c, x) \leq 3 \) for all \( x \in V_1 \).

Thus \( d(c, x) \leq 3 \) for all \( x \in V(T) \setminus V_2 \). By Lemma 7, \( c \in K_4(T) \), and so \( I(a) \subseteq K_4(T) \). Hence \( I(a) \subseteq K_4(T) \cap V_2 \).

This proves Claim 3.
Since $a \rightarrow \{v, w\} \rightarrow x_1$ and $x_1 \in M_1$, there exist $c, e \in V(T) \setminus V_1$ such that $x_1 \rightarrow \{c, e\} \rightarrow a$. Note that by Claim 3 we have $\{c, e\} \subseteq K_4(T) \cap V_2$. Thus, $\{x_1, u, a, b\} \subseteq K_4(T) \cap V_1$ and $\{v, w, c, e\} \subseteq K_4(T) \cap V_2$. Since $a \rightarrow V_j$ for all $j \geq 3$ and $a$ lies on no 3-cycles in $T$, we have $\{c, e\} \rightarrow V_j$ for all $j \geq 3$. Also, as $x_1 \rightarrow \{c, e\}$ and $x_1$ lies on no 3-cycles in $T$, we have $x_1 \rightarrow V_j$ for all $j \geq 3$. Hence $T$ contains the digraph of Fig. 1 as a subdigraph. The proof of (i) and (ii) is now complete.

(iii) The proof of (iii) is similar to the proof of Theorem 2(ii) in [9] and shall be omitted. □

Corollary 1 extends a result of Gutin [3].

**Corollary 1.** Let $T$ be an $n$-partite tournament, where $n \geq 3$, with at most one transmitter. If $|K_4(T) \cap V_i| \leq 3$ for each $i = 1, 2, \ldots, n$, except possibly for at most one integer $j \in \{1, 2, \ldots, n\}$, then $k_3(T) \geq 1$.

**Corollary 2.** Let $T$ be an $n$-partite tournament, where $n \geq 3$, with no transmitters. Assume $|V_1| \leq |V_2| \leq |V_3| \leq \cdots \leq |V_n|$. Let $v$ be a vertex with maximum outdegree in $T$. Suppose $s(v) < |V_1| + |V_2| + \cdots + |V_{n-2}| + 2$. Then $k_3(T) \geq 1$.

**Corollary 3.** Let $T$ be an $n$-partite tournament, where $n \geq 3$. Suppose $|\{i \mid I(u) \cap V_i \neq \emptyset\}| \geq 2$ for each $u \in V(T)$. Then $k_3(T) \geq 1$.

The result of Theorem 2 tells us that if $k_3(T) = 0$ and $k_4(T) = 8$, then $|\{i \mid K_4(T) \cap V_i \neq \emptyset\}| = 2$. What can we say about the value of $k_4(T)$ if $k_3(T) = 0$ and $|\{i \mid K_4(T) \cap V_i \neq \emptyset\}| = r$, where $3 \leq r \leq n$? The next result provides a sharp lower bound for the value of $k_4(T)$.

**Theorem 3.** Let $T$ be an $n$-partite tournament, where $n \geq 3$, with no transmitters and $k_3(T) = 0$. Suppose $|\{i \mid K_4(T) \cap V_i \neq \emptyset\}| = r$, where $3 \leq r \leq n$. Then $k_4(T) \geq r + 8$.

**Proof.** By Theorem 1, we may assume that every $MS$-tournament of $T$ has a transmitter. Let $H = T[\{x_1, x_2, \ldots, x_n\}]$ be an $MS$-tournament of $T$. Let $x_1, x_2, u, v, w, a, b, c, e$ be the vertices as described in the proof of (i) and (ii) of Theorem 2. Then $\{x_1, u, a, b\} \subseteq K_4(T) \cap V_1$ and $\{v, w, c, e\} \subseteq K_4(T) \cap V_2$.

**Case 1.** $\{a, b\} \rightarrow x_2$.

Observe that $x_2 \notin \{c, e\}$ since $\{c, e\} \rightarrow \{a, b\}$. As $v \rightarrow \{x_1, u\} \rightarrow x_2$ and $x_2 \in M_2$, there exist vertices $f$ and $g$ such that $x_2 \rightarrow \{f, g\} \rightarrow v$. By Claims 1 and 2 as in the proof of Theorem 2, we have $I(v) \subseteq K_4(T) \cap V_1$. Thus $\{f, g\} \subseteq K_4(T) \cap V_1$. Hence $\{x_1, u, a, b, f, g\} \subseteq K_4(T) \cap V_1$. Thus

$$k_4(T) = |K_4(T) \cap V_1| + |K_4(T) \cap V_2| + \sum_{i=3}^{n} |K_4(T) \cap V_i|$$

$$\geq 6 + 4 + (r - 2)$$

$$= r + 8.$$ 

**Case 2.** $x_2 \rightarrow \{a, b\}$.

Since $x_2 \in I(a)$, by Claim 3 as in the proof of Theorem 2, $x_2 \in K_4(T)$. Also, as in the proof of Claim 3 of Theorem 2, we have $x_2 \rightarrow V_j$ for all $j \geq 3$ and $d(x_2, x) \leq 3$ for all $x \in V_1$. As $k_3(T) = 0$, there exists $g \in V_2$ such that $d(x_2, g) = 4$. Since $x_2 \in M_2$, by Lemma 3, $O(g) = O(x_2)$. By Lemma 9, $g \in K_4(T)$. Since $k_3(T) = 0$, by Lemma 4, $x_2$ and $g$ lie on no 3-cycles in $T$. As $x_2 \rightarrow V_j$ for all $j \geq 3$, $I(x_2) \subseteq V_1$. Let $y \in I(x_2)$. Since $y \rightarrow x_2$ and $x_2 \in M_2$, by Lemma 5, $d(y, x) \leq 3$ for all $x \in V_2$. Note that $y \rightarrow x_2 \rightarrow V_j$ for all $j \geq 3$. Thus, $d(y, x) \leq 3$ for all $x \in V(T) \setminus V_1$. By Lemma 7, $y \in K_4(T)$. Hence $I(x_2) \subseteq K_4(T) \cap V_1$.

As $|\{i \mid K_4(T) \cap V_i \neq \emptyset\}| = r \geq 3$, we may assume $K_4(T) \cap V_3 \neq \emptyset$. Let $f \in K_4(T) \cap V_3$. Since $f \in K_4(T)$, $d(f, x_1) \leq 4$. Also, as $x_1 \rightarrow f$ and $x_1$ lies on no 3-cycles in $T$, we have $d(f, x_1) \geq 3$. Thus $3 \leq d(f, x_1) \leq 4$. Similarly, we have $3 \leq d(f, v) \leq 4$, $3 \leq d(f, a) \leq 4$ and $3 \leq d(f, x_2) \leq 4$.

**Case 2-1:** $s^{-1}(y) \geq 3$ for all $y \in \{x_1, a, u, v, x_2\}$.

Observe that in this case, since $s^{-1}(y) \geq 3$ for all $y \in \{x_1, a, u, x_2\}$, we have $|I(v) \cup I(x_2)| \geq 1$ and $|(I(a) \cup I(x_1)) \setminus \{v, w, x_2, g\}| \geq 1$. Now as $I(v) \cup I(x_2) \subseteq K_4(T) \cap V_1$ and $I(a) \cup I(x_1) \subseteq K_4(T) \cap V_2$, we...
we have

\[ k_4(T) = |K_4(T) \cap V_1| + |K_4(T) \cap V_2| + \sum_{i=3}^{n} |K_4(T) \cap V_i| \]

\[ \geq 5 + 5 + (r - 2) \]

\[ = r + 8. \]

**Case 2-2:** \( s^-(y) = 2 \) for some \( y \in \{x_1, a, v, x_2\} \).

Suppose \( s^-(a) = 2 \). Since \( x_2, g, c, e \in I(a) \), we have \( \{x_2, g\} = \{c, e\} \); otherwise, \( s^-(a) \geq 3 \). As \( I(a) = \{x_2, g\} \), every path from the vertex \( f \) to the vertex \( a \) must contain the arc \( x_2a \) or \( ga \). Thus \( d(f, a) = d(f, x_2) + d(x_2, a) \geq 3 + 1 = 4 \) (or \( d(f, a) = d(f, g) + d(g, a) \geq 3 + 1 = 4 \)). Hence \( d(f, a) = 4 \) (and so \( d(f, b) = 4 \)). Let \( fstx_2a \) (or \( fstga \)) be a path of length 4 from \( f \) to \( a \). Since \( d(f, x_1) \geq 3 \), \( t \notin \{x_1, u\} \). Also, as \( t \in I(x_2) \), \( t \in K_4(T) \cap V_1 \).

Observe that \( s^-(v) \geq 3 \); otherwise, every path from the vertex \( f \) to the vertex \( v \) must contain the arc \( av \) or \( bv \), and so, \( d(f, v) = d(f, a) + d(a, v) = 4 + 1 = 5 \) (or \( d(f, v) = d(f, b) + d(b, v) = 4 + 1 = 5 \)). Let \( h \in I(v) \setminus \{a, b\} \). By Claims 1 and 2 as in the proof of Theorem 2, we have \( I(v) \subseteq K_4(T) \cap V_1 \). Thus \( h \in K_4(T) \cap (V_1 \setminus \{a, b\}) \). Hence
\[ \{a, b, x_1, u, h, t\} \subseteq K_4(T) \cap V_1 \text{ and } \{v, w, x_2, g\} \subseteq K_4(T) \cap V_2. \text{ Thus if } h \neq t, \text{ then } k_4(T) \geq r + 8. \text{ Assume now } h = t. \text{ Note that} \\
\text{(a) } h \to \{x_2, g\} \to \{a, b\} \to V(T) \setminus (V_1 \cup \{x_2, g\}); \text{ and} \\
\text{(b) } h \to \{v, w\} \to \{x_1, u\}. \]

By (a) and (b), \(d(h, x) \leq 3\) for all \(x \in (V(T) \setminus V_1) \cup \{x_1, u, a, b\}. \text{ As } k_3(T) = 0, \text{ there exists } p \in V_1 \setminus \{x_1, u, a, b\} \text{ such that } d(h, p) = 4. \text{ It follows that } O(h) \subseteq O(p). \text{ By Lemma 9, } p \in K_4(T). \text{ Thus, } \{x_1, u, a, b, h, p\} \subseteq K_4(T) \cap V_1 \text{ and } \{v, w, x_2, g\} \subseteq K_4(T) \cap V_2. \text{ Hence } k_4(T) \geq r + 8. \]

Likewise, if \(s^{-}(x_1) = 2 \) or \(s^{-}(v) = 2 \) or \(s^{-}(x_2) = 2\), then \(k_4(T) \geq r + 8. \text{ The proof of the theorem is now complete.} \]

To end this paper, we now construct the \(n\)-partite tournament \(T\), where \(n \geq 3\), of Fig. 3 to justify that the lower bound for \(k_4(T)\) given in Theorem 3 is sharp. Note that all arcs not shown are of arbitrary direction. Below are some guidance to help the readers check that \(K_3(T) = \emptyset \) and \(K_4(T) = \{u_1, v_1, w_1, y_1, u_2, v_2, w_2, y_2, a_2, b_2, u_3, u_4, \ldots, u_r\}\):

\[ \text{(a) } d(u_i, v_i) = d(v_i, u_i) = 4 \text{ for } i = 1, 2; \]
\[ \text{(b) } d(w_i, y_i) = d(y_i, w_i) = 4 \text{ for } i = 1, 2; \]
\[ \text{(c) } d(a_1, u_2) = 5; \]
\[ \text{(d) } d(b_1, w_2) = 5; \]
\[ \text{(e) } d(u_2, w_2) = 4; \]
\[ \text{(f) } d(b_2, u_2) = 4; \text{ and} \]
\[ \text{(g) } d(u_2, u_i) = 4 \text{ for } i = 3, 4, \ldots, r. \]

References