On projective embeddings of partial planes and rank-three matroids
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Abstract

Any finite partial plane $\mathcal{F}$, and thus any finite linear space and any (simple) rank-three matroid, can be embedded into a translation plane. It even turns out, that $\mathcal{F}$ is embeddable into a projective plane of Lenz class V, and that the characteristic of this plane can be chosen arbitrarily. In particular, any rank three matroid is realizable over a (not necessarily associative) division algebra.


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By M. Hall's fundamental paper on projective planes [1], we know that any partial plane $\mathcal{F}$ can be embedded into a projective plane, namely into its free plane extension. Unfortunately, in general such a free plane extension is a quite weak projective plane lying in the lowest Lenz–Barlotti class I.1. In face of the significance embeddings into 'nice structures' usually have (think of, say, realizable matroids), it is justified and promising to study the question to which extend partial planes can be embedded into better projective planes.

In this note we shall show that any finite partial plane — however wild it may be — can be embedded into a countable translation plane. It is even possible, to embed it into a plane of Lenz class V, and to choose the characteristic of the kernel of this plane arbitrarily. Furthermore, there does exist a translation plane of Lenz class V containing all finite partial planes and thus all finite linear spaces and all (simple) rank-three matroids. In particular, any rank-three matroid is realizable over a (not necessarily associative) division algebra of arbitrarily given characteristic.

Recall, that a (not necessarily associative) division algebra is a set $V$ endowed with two binary operations $\cdot$ and $\cdot$ and two distinguished elements $0, 1 \in V$, $0 \neq 1,$
such that \((V, +)\) is a group with neutral element 0, \(V^* := V \setminus \{0\}\) is a loop with respect to \(\cdot\) and with neutral element 1, and the distributive laws hold. For this and for further notions from geometric algebra, as for instance for the Lenz–Barlotti classification, the reader is referred to Pickert's book [3]. Since each division algebra \(V\) is a vector space over its center, a division algebra can also be characterized as a vector space \(V\) over some (commutative) field \(K\) together with a product \(\mu : V \times V \to V\) such that

- \(\mu\) is a bilinear mapping,
- there exists \(1 \in V\) such that \(\mu(1, v) = \mu(v, 1) = v\) for all \(v \in V\),
- for all \(v, w \in V\) \(\mu(v, w) = 0\) implies \(v = 0\) or \(w = 0\),
- for all \(v, w \in V^*\) there are \(x, y \in V\) such that \(\mu(x, v) = w = \mu(y, v)\).

Clearly, \(\mu\) is uniquely determined by its values \(\mu(a, b), a \in A, b \in B\), where \(A\) and \(B\) are bases of \(V\). Hence, one easily verifies that a division algebra \(V\) may also be defined in terms of its left multiplications \(L_c : x \mapsto \mu(c, x) = cx\) as a vector space \(V\) together with a family \((L_a)_{a \in A}\) of linear mappings \(L_a : V \to V\) fulfilling
- \(A\) is a basis of \(V\),
- there exists \(1 \in A\) such that \(L_1\) is the identity on \(V\),
- \(L_a(1) = a\) for all \(a \in A\),
- for each \(c = \sum_{a \in A} \gamma_a a \in V^*\), \(\gamma_a \in K\), the mapping \(L_c := \sum_{a \in A} \gamma_a L_a\) is bijective,
- for all \(v, w \in V^*\) there exists \(x \in V\) such that \(L_c(v) = w\).

These characterizations of division algebras motivate the following two definitions.

**Definition 1.** Let \(V\) be a vector space over some field \(K\) and \(1 \in V^*\) be a specified element. A mapping \(\mu : A \times B \to V\) defined on subsets \(A = \{a_1, \ldots, a_k\}\) and \(B = \{b_1, \ldots, b_m\}\) of \(V(k, m \in \mathbb{N})\) is called admissible, if

(a) \(1 = a_1 = b_1\) and \(\mu(a, 1) = a, \mu(1, b) = b\) for all \(a \in A, b \in B\),

(b) for all \(x_1, \ldots, x_k, \beta_1, \ldots, \beta_m \in K\) the equation

\[
\sum_{i=1}^{k} \sum_{j=1}^{m} x_i \beta_j \mu(a_i, b_j) = 0
\]

implies \(x_i \beta_j = 0\) for all \(i = 1, \ldots, k\) and all \(j = 1, \ldots, m\).

Note, that by (a) and (b), the sets \(A\) and \(B\) are independent. Hence, any admissible mapping \(\mu\) defines a unique bilinear mapping on \(\langle A \rangle \times \langle B \rangle\), also denoted by \(\mu\), where \(\langle X \rangle\) means the subvector space of \(V\) generated by \(X \subset V\). Since the condition \(x_i \beta_j = 0\) for all \(i = 1, \ldots, k\) and all \(j = 1, \ldots, m\) in (b) is equivalent to \(x_1 = \cdots = x_k = 0\) or \(\beta_1 = \cdots = \beta_m = 0\), (b) means that the induced product \(\mu : \langle A \rangle \times \langle B \rangle \to V\) has no zero divisors.

**Definition 2.** Let \(V\) be a vector space over some field \(K\), and let \(1 \in V^*\) be a specified element. A family \((L_a)_{a \in A}\) of linear mappings \(L_a : V \to V\), indexed by a subset \(A\) of \(V\), is called admissible, if it fulfills

(a) \(A\) is independent,

(b) \(1 \in A\) and \(L_1(v) = v\) for all \(v \in V\),
Now let a finite partial plane $\mathcal{J} = (\mathcal{P}, \mathcal{L}, I)$ be given, i.e. $\mathcal{P}$ and $\mathcal{L}$ are finite, non-empty, disjoined sets and $I \subseteq \mathcal{P} \times \mathcal{L}$ is a relation such that

$$p_i \in \mathcal{P}, L_j \in \mathcal{L},$$

and $p_i I L_j$ for $i, j = 1, 2$ implies $p_1 = p_2$ or $L_1 = L_2$.

Assume for a moment, that we have already an embedding of $\mathcal{J}$ into the affine plane $\mathcal{A}(V)$ over some division algebra $V$, the points of which are the elements $(x, y) \in V \times V$, and the lines of which are the point sets (cf. [3, p. 31]).

$$[m,c] := \{(x,y) \mid y = mx + c\} \quad \text{with} \quad m,c \in V,$$

$$[d] := \{(x,y) \mid x = d\} \quad \text{with} \quad d \in V.$$

Additionally assuming that each line of $\mathcal{J}$ is mapped to a line of the first kind, then to each point $p$ and to each line $L$ of $\mathcal{J}$ there are associated coordinates $x_p, y_p, m_L, c_L \in V$ such that the following rule holds:

$$m_L \cdot x_p + c_L = y_p \iff p I L.$$

The main idea to achieve such an embedding is now as follows. We first associate to each point $p$ and to each line $L$ of $\mathcal{J}$ linearly independent vectors $x_p, y_p, m_L, c_L$ of some vector space $V$ over some field $K$. Secondly, we define an admissible partial product on $V$ which obeys the rule given above, and, thirdly, we extend this product to $V$ making $V$ a division algebra. We start with proving the necessary algebraic extension theorems.

For the following lemmata, let $V$ be a vector space of countably infinite dimension over some finite or countably infinite field $K$, and let $1$ be some arbitrary, but fixed element of $V^*$.\[ \]

**Lemma 3.** Let $(L_a)_{a \in A}$ be a finite, admissible family of linear mappings of $V$, $A \subset V$. Given some element $b \in V \setminus \langle A \rangle$ and a linear mapping $L_b : U \to V$ defined on some finitely dimensional subspace $U$ of $V$ such that

(a) $1 \in U$, $L_b(1) = b$ and

(b) for all $c \in \langle A \rangle$ the mapping $L_b + L_c : U \to V$ is injective,

$L_b$ can be extended to a linear mapping $L_b : V \to V$, such that the family $(L_a)_{a \in A \cup \{b\}}$ is admissible.

**Proof.** Let $1 = b_1, b_2, \ldots, b_m$ be a basis of $U$, and extend it to a basis $B = \{b_i \mid i \in \mathbb{N}\}$ of $V$. Since $V$ is countable, $\langle A \rangle \times V^*$ contains only countably many pairs. Hence, we may write

$$\langle A \rangle \times V^* = \{(c_n,d_n) \mid n \in \mathbb{N}, n > m\}.$$
Now let \( B_n := \{ b_1, \ldots, b_m \} \). By induction on \( n \in \mathbb{N} \), for each \( n > m \), we shall define a finite subset \( B_n \subset V \) with

1. \( B_{n-1} \subset B_n \),
2. \( B_n \) is independent,
3. \( b_n \in \langle B_n \rangle \)

and extend \( L_b \) to a linear mapping on \( \langle B_n \rangle \) such that

\( (d) \) \( L_b + L_c : \langle B_n \rangle \to V \) is injective for all \( c \in \langle A \rangle \), and

\( (e) \) \( (L_b + L_c)(x) = d_n \) for some \( x \in \langle B_n \rangle \).

So let us assume that for some \( n - 1 \geq m \) the mapping \( L_b : \langle B_{n-1} \rangle \to V \) is defined according to the settings above (recall \( U = \langle B_m \rangle \)). We proceed in two steps.

**Step 1:** If there is already an element \( x \) fulfilling \( (L_b + L_c)(x) = d_n \), we put \( B_n := B_{n-1} \) and go to step 2. Otherwise, consider the subspace

\[
W := \langle \bigcup \{ L_c(\langle B_{n-1} \rangle) \mid c \in \langle A \rangle \} \rangle + L_b(\langle B_{n-1} \rangle) + \langle d_n \rangle
\]

Since \( B_{n-1} \) and \( A \) are finite, \( W \) is finitely dimensional, and therefore has some cospace \( W' \) of finite codimension in \( V, W \oplus W' = V \). Since, for each \( a \in A \), the mapping \( L_a \) is bijective, each space \( L_a^{-1}(W') \) has a finite codimension, showing that also the finite intersection

\[
\bigcap_{a \in A} L_a^{-1}(W')
\]

has a finite codimension in \( V \). Thus, we may fix some \( x \in \bigcap_{a \in A} L_a^{-1}(W'), x \neq 0, 1 \). Then we have \( L_a(x) \in W \) for all \( a \in A \), which leads to \( L_c(x) \in W' \) for all \( c \in \langle A \rangle \), and thus to

\[
L_c(x) \notin W \quad \text{for all } c \in \langle A \rangle^*.
\]

By \( 1 \in A \), this means in particular \( x \notin \langle B_{n-1} \rangle \), i.e. the set \( B'_n := B_{n-1} \cup \{ x \} \) is independent. Further, since \( c_n \in \langle A \rangle \), from \( L_{c-c_n}(x) \notin W \) we obtain

\[
L_c(x) \notin W + L_c(x) \quad \text{for all } c \in \langle A \rangle \setminus \{ c_n \}.
\]

Now we extend \( L_b \) to \( \langle B'_n \rangle \) by choosing

\[
L_b(x) := d_n - L_{c_n}(x).
\]

We shall show that \( L_b + L_c : \langle B'_n \rangle \to V \) is injective for all \( c \in \langle A \rangle \). Since this is clear on \( \langle B_{n-1} \rangle \), we only have to check

\[
(L_b + L_c)(x) \notin (L_b + L_c)(\langle B_{n-1} \rangle),
\]

i.e. we have to verify for all \( c \in \langle A \rangle \)

\[
L_c(x) \notin (L_b + L_c)(\langle B_{n-1} \rangle) - d_n + L_c(x).
\]

In view of our choice of \( x \), this is clear for all \( c \in \langle A \rangle \setminus \{ c_n \} \), and for \( c = c_n \), it follows from our hypothesis that \( (L_b + L_{c_n})(x) = d_n \) has no solution \( x \) in \( \langle B_{n-1} \rangle \).
Step 2. From step 1 we have a finite, independent set \( B'_n \) with \( B_{n-1} \subseteq B'_n \subseteq V \), and on which \( L_b \) is defined such that (δ) and (ε) are fulfilled, i.e.

\[ L_b + L_c : \langle B'_n \rangle \rightarrow V \]

is injective for all \( c \in \langle A \rangle \), and

\[ (L_b + L_c)(x) = d_n \quad \text{for some } x \in \langle B'_n \rangle. \]

If \( b_n \) lies already in \( \langle B'_n \rangle \), we put \( B_n := B'_n \) and are done. Otherwise, we choose some

\[ y \in V \setminus \bigcup \{ (L_b + L_c)(\langle B'_n \rangle) - L_c(b_n) \mid c \in \langle A \rangle \}, \]

which is possible, since \( B'_n \) and \( A \) are finite but \( \dim V = \infty \). Put \( B_n := B'_n \cup \{ b_n \} \), and extend \( L_b \) to \( \langle B'_n \rangle \) by taking

\[ L_b(b_n) := y. \]

Then we have for all \( c \in \langle A \rangle \)

\[ (L_b + L_c)(b_n) \not\in (L_b + L_c)(\langle B'_n \rangle), \]

which proves that \( L_b + L_c \) is injective on \( \langle B'_n \rangle \). Hence (α) up to (ε) are fulfilled.

Clearly, by (α) and (γ), the common extension of all \( L_b \) to the union of all \( \langle B'_n \rangle \), \( n \geq m \), is a linear mapping on \( V \), also denoted by \( L_b \). By (δ) and (ε), the mapping \( L_b + L_c \) is bijective for all elements \( c \in \langle A \rangle \). Hence, \( L_c \) is bijective for all \( c \in \langle A \cup \{ b \} \rangle^* \), showing that the family \( (L_a)_{a \in A \cup \{ b \}} \) is admissible. \( \square \)

**Lemma 4.** Any finite admissible family \( (L_a)_{a \in A} \) of linear mappings \( L_a : V \rightarrow V \) can be enlarged to an admissible family \( (L_a')_{a \in A'} \) of linear mappings of \( V \), such that

(a) \( A' \) is a basis of \( V \) containing \( A \), and

(b) for all \( v, w \in V^* \) there exists \( x \in V \) such that \( L_x(v) = w \).

**Proof.** We extend \( A = \{ a_1, \ldots, a_t \} \) to a basis \( \{ a_n \mid n \in \mathbb{N} \} \) of \( V \). Since \( V \) is countable, there are only countably many pairs in \( V^* \times V^* \), i.e. we may write

\[ V^* \times V^* = \{ (v_n, w_n) \mid n \in \mathbb{N} \}. \]

Let \( A_0 := A \). By induction on \( n \in \mathbb{N} \), for each \( n > 0 \) we define a finite subset \( A_n \subseteq V \) with

(α) \( A_{n-1} \subseteq A_n \),

(β) \( A_n \) is independent,

(γ) \( a_n \in \langle A_n \rangle \),

and construct linear mappings \( L_a \) for each \( a \in A_n \setminus A_{n-1} \) such that

(δ) \( (L_a)_{a \in A_n} \) is an admissible family of linear mappings on \( V \), and

(ε) if \( v_n \not\in \langle 1 \rangle \), then \( L_x(v_n) = w_n \) for some \( x \in \langle A_n \rangle \).

So let us assume that for some \( n - 1 \geq 0 \) the family \( (L_a)_{a \in A_n} \) is already defined according to the setting above. We proceed in two steps.
Step 1: If \( v_n \in \langle 1 \rangle \) or if there is already an element \( x \in \langle A_{n-1} \rangle \) fulfilling \( L_x(v_n) = w_n \), we put \( A_n := A_{n-1} \) and go to step 2. Otherwise, we fix an arbitrary element
\[
x \in V \setminus \langle c, w_n + L_c(v_n) \mid c \in \langle A_{n-1} \rangle \rangle,
\]
which is possible, since \( A_{n-1} \) is finite and \( \dim V = \infty \). We define a linear mapping
\[
L_x : U \to V \quad \text{on } U := \langle \{1, v_n\} \rangle \text{ by } L_x(1) := x \text{ and } L_x(v_n) := w_n.
\]
Clearly, \( x \in V \setminus \langle A_{n-1} \rangle \), and \( L_x \) fulfills (3a). To show (3b), i.e. to check that the mapping \( L_x + L_c : U \to V \) is injective for each \( c \in \langle A_{n-1} \rangle \), we compute
\[
(L_x + L_c)(1) = x + c,
\]
\[
(L_x + L_c)(v_n) = w_n + L_c(v_n).
\]
By our choice of \( x \), these two vectors are linearly independent (note that \( w_n + L_c(v_n) = w_n - L_c(v_n) \neq 0 \), since \( L_c(v_n) = w_n \) has no solution \( x \) in \( \langle A_{n-1} \rangle \)). Hence, in view of Lemma (3), \( L_x \) may be extended to \( V \) such that \( (L_a)_{a \in A_n} \) with \( A_n := A_{n-1} \cup \{x\} \) is admissible.

Step 2: From step 1 we have a finite, admissible family \( (L_a)_{a \in A_n} \) of linear mappings on \( V \), which, in the case \( v_n \notin \langle 1 \rangle \), fulfills \( L_x(v_n) = w_n \) for some \( x \in \langle A_{n-1} \rangle \). If \( a_n \in \langle A_n \rangle \), we put \( A_n := A_n \) and are done. Otherwise, we define
\[
L_{a_n} : U \to V \quad \text{on } U := \langle 1 \rangle \text{ by } L_{a_n}(1) := a_n.
\]
Clearly, \( L_{a_n} \) satisfies (3a) and (3b) with respect to the given family \( (L_a)_{a \in A_n} \). Hence, in view of (3), it may be extended to \( V \) such that \( (L_a)_{a \in A_n} \) with \( A_n := A_n \cup \{a_n\} \) is an admissible family of linear mappings on \( V \), and (a) up to (e) are fulfilled.

Clearly, by (a), (b) and (d), the family \( (L_a)_{a \in A'} \) with \( A' := \cup \{A_n \mid n \geq 0\} \) is admissible, and by (f) \( A' \) is a basis of \( V \). Finally, for all \( v, w \in V^* \) with \( v \notin \langle 1 \rangle \), in view of (e) there exists an \( x \in V \) such that \( L_x(v) = w \). And for all \( v = \lambda w \), \( w \in V^* \) with \( \lambda \in K \setminus \{0\} \), we plainly have \( L_x(v) = \lambda \cdot L_x(1) = \lambda x = w \) with \( x := \lambda^{-1} w \).

Lemma 5. For any admissible product \( : A \times B \to V \) there exists an admissible family \( (L_a)_{a \in A} \) of linear mappings \( L_a : V \to V \), such that \( L_a(b) = a \cdot b \) for all \( a \in A, b \in B \).

Proof. By definition, \( A = \{a_1, \ldots, a_k\} \) and \( B = \{b_1, \ldots, b_m\} \) are independent subsets of \( V \) with \( 1 = a_1 = b_1 \) \((k, m \in \N)\). Note, that for each \( i = 1, \ldots, k \) the given product provides us with a linear mapping from \( U := \langle B \rangle \) into \( V \), namely with
\[
L_i = L_{a_i} : U \to V \quad \text{defined by } L_i(b_j) := a_i \cdot b_j \text{ for } j = 1, \ldots, m.
\]
We shall show, that each \( L_i \) can be extended to \( V \) such that \( (L_i)_{i=1}^{k} \) forms an admissible family of linear mappings on \( V \). We proceed by induction on \( k \).

\((k = 1)\): Plainly, by taking \( L_1(v) := v \) for all \( v \in V \), we may extend \( L_1 \) to a linear mapping on \( V \) such that \( (L_i)_{i=1}^{1} \) is admissible.
Now assume, that \( L_1, \ldots, L_{k-1} \) are extended to \( V \) such that \((L_i)_{i=1}^{k-1}\) is admissible. Obviously, the linear mapping \( L_k: U \rightarrow V \) fulfills (3a). To show (3b), i.e., to check that

\[
L_k + L_c: U \rightarrow V
\]
is injective for all \( c \in \langle \{a_1, \ldots, a_{k-1}\} \rangle \),

let \( u = \beta_1 b_1 + \cdots + \beta_m b_m \in U \) and \( c = \alpha_1 a_1 + \cdots + \alpha_{k-1} a_{k-1} (\alpha_1, \ldots, \alpha_{k-1}, \beta_1, \ldots, \beta_m \in K) \) be given such that \((L_k + L_c)(u) = 0\), i.e.

\[
0 = \sum_{i=1}^{k} \sum_{j=1}^{m} \alpha_i \beta_j \ a_i \cdot b_j \quad \text{with } \alpha_k = 1.
\]

In view of (1b), we infer \( \beta_1 = \cdots = \beta_m = 0 \) and so \( u = 0 \). Thus, (3b) is satisfied, and by Lemma (3), \( L_k \) can be extended to \( V \) such that \((L_i)_{i=1}^{k-1}\) is admissible. \( \square \)

The preceding lemmata immediately imply the following extension result.

**Proposition 6.** Let \( V \) be a vector space of countably infinite dimension over a finite or countably infinite field \( K \), and let \( A = \{a_1, \ldots, a_k\} \) and \( B = \{b_1, \ldots, b_m\} \) be two finite, independent subsets of \( V \) with \( 1 := a_1 = b_1 \) \((k, m \in \mathbb{N})\). Then any admissible product \( ' \cdot ' : A \times B \rightarrow V \) can be extended to a product on \( V \) making \((V, +, ' \cdot ' )\) a (not necessarily associative) division algebra, which contains \( K \) in its center.

Now let \( \mathcal{F} = (\mathcal{P}, \mathcal{L}, I) \) be a finite partial plane. We consider the finite sets \( \mathcal{P} \times \{1, 2\}, \mathcal{L} \times \{1, 2\}, \mathcal{L} \times \mathcal{P} \) and simply write

\[
\begin{align*}
x_p &:= (p, 1), \quad y_p := (p, 2), \quad u_{L, p} := (L, p), \\
m_L &:= (L, 1), \quad c_L := (L, 2)
\end{align*}
\]

for all \( p \in \mathcal{P} \) and all \( L \in \mathcal{L} \) (of course, we assume that \( 1, 2 \notin \mathcal{P}, \mathcal{L} \)). Additionally, we take infinitely many, mutually distinct elements \( b_i, i \in \mathbb{N} \), not contained in the sets mentioned above. Let \( K \) be an arbitrary finite or countably infinite field. We consider the vector space \( V \) over \( K \) freely generated by the elements of

\[
E := \{x_p, y_p, m_L, c_L, u_{L, p} | p \in \mathcal{P}, L \in \mathcal{L} \} \cup \{b_i | i \in \mathbb{N}\},
\]

i.e. \( V = \bigoplus_{b \in E} Kb \), a vector space with the countably infinite basis \( E \). Put \( 1 := b_1 \), \( A := \{m_L | L \in \mathcal{L} \} \cup \{1\}, B := \{x_p | p \in \mathcal{P} \} \cup \{1\}, \) and define a product on \( A \times B \) by

\[
\begin{align*}
a \cdot a &:= a \quad \text{for all } a \in A, \\
1 \cdot b &:= b \quad \text{for all } b \in B, \\
m_L \cdot x_p &:= y_p - c_L \quad \text{for all } p \in \mathcal{P}, L \in \mathcal{L} \text{ with } p \not\in L, \text{ and}
\end{align*}
\]

\[
\begin{align*}
m_L \cdot x_p &:= u_{L, p} \quad \text{for all } p \in \mathcal{P}, L \in \mathcal{L} \text{ with } p \not\in L.
\end{align*}
\]
We shall show, that this product is admissible. Clearly, $A$ and $B$ are independent and $(1a)$ is fulfilled. To check $(1b)$, let $\alpha_1, \alpha_L, \beta_1, \beta_p \in K$ for all $p \in \mathcal{P}, L \in \mathcal{L}$, and consider the equation

$$0 = \alpha_1\beta_1 + \sum_{L \in \mathcal{L}} \alpha_L\beta_1 m_L + \sum_{p \in \mathcal{P}} \alpha_1\beta_p x_p + \sum_{p \in \mathcal{P}, L \in \mathcal{L}} \alpha_L\beta_p (y_p - c_L) + \sum_{p \in \mathcal{P}, L \in \mathcal{L}} \alpha_L\beta_p u_{L,p}.$$ 

Since 1 and all $m_L, x_p, y_p, c_L$, and $u_{L,p}$ are independent, we infer

$$\alpha_1\beta_1 = \alpha_L\beta_1 = \alpha_1\beta_p = 0 \quad \text{for all} \quad p \in \mathcal{P}, L \in \mathcal{L},$$

$$\alpha_L\beta_p = 0 \quad \text{for all} \quad p \in \mathcal{P}, L \in \mathcal{L} \quad \text{with} \quad p \neq L,$$

and

$$0 = \sum_{p \in \mathcal{P}, L \in \mathcal{L}} \alpha_L\beta_p (y_p - c_L).$$

If there is a non-zero coefficient $\alpha_L\beta_p$ in the latter sum, then, checking the coefficients for $c_L$, we observe that there must be at least one another coefficient $\alpha_L\beta_q \neq 0, q \neq p$. Similarly, checking for $y_p$, there must be a coefficient $\alpha_G\beta_p \neq 0$ with $G \neq L$. But this implies $\alpha_G\beta_q \neq 0$, and so $\alpha_G\beta_q \neq 0$. Hence, also $\alpha_G\beta_q$ occurs in the sum under consideration, and we infer $p, q \neq L, G$ contradicting $\mathcal{F}$ being a partial plane. So we finally have, that all coefficients in the equation above vanish.

In light of Proposition 6, the product on $A \times B$ can be extended to a product on $V$ making $(V, +, \cdot)$ a division algebra. Let $\mathcal{A} = \mathcal{A}(V)$ be the affine plane over $V$ (as introduced above). We define a mapping $\varphi$ from the point and line set of $\mathcal{F}$ into that of $\mathcal{A}$ by

$$\varphi(p) := (x_p, y_p) \quad \text{for all} \quad p \in \mathcal{P},$$

$$\varphi(L) := [m_L, c_L] \quad \text{for all} \quad L \in \mathcal{L}.$$ 

Obviously, $\varphi$ is well-defined and injective. Further we have $u_{L,p} \neq y_p - c_L$ for all $p \in \mathcal{P}$ and $L \in \mathcal{L}$, and thus

$$p \parallel L \iff m_L \cdot x_p + c_L = y_p \iff (x_p, y_p) \in [m_L, c_L].$$

Hence $\varphi$ is an embedding, and we have proved

**Theorem 7.** Any finite partial plane $\mathcal{F} = (\mathcal{P}, \mathcal{L}, 1)$ can be embedded into the affine plane and thus also into the projective plane over a countable division algebra $V$, such that $V$ contains a finite or countably infinite, arbitrarily given field $K$ in its center.

Since the characteristic of a division algebra $V$ equals the characteristic of the kernel of the projective plane $\Pi = \Pi(V)$ over $V$, it is a geometric invariant not depending on the special choice of a coordinatizing division algebra (or quasifield).
Hence, we may refer to char(V) as the characteristic of Π(V) and of φ(V). As to the Lenz-Barlotti classification of projective planes, our theorem immediately implies (cf. [3, p. 334]).

Corollary 8. Any finite partial plane can be embedded into a countable translation plane of arbitrary characteristic lying in a Lenz class at least V.

In view of the fact that any (simple, finite) rank-three matroid may be regarded as a special finite partial plane (cf. [5]), we also have

Corollary 9. Any rank-three matroid can be realized over a (not necessarily associative) division algebra of arbitrary characteristic.

Given a (projective) translation plane Π(V) over some quasifield or division algebra V, up to now only the subplanes which arise from the subalgebras of V were known explicitly. They all have a Lenz-Barlotti class higher than or equal to that of Π(V), and they all have the same characteristic as Π(V). In contrast to this experience, our theorem allows the construction of a translation plane containing finitely many Desarguesian subplanes Π(GF(p1)), ..., Π(GF(pn)) of arbitrarily given characteristics p1, ..., pn ≠ 0 (recall that the union of partial planes is again a partial plane). Whereas such a phenomenon is only new for better planes (clearly Π(GF(p1)), ..., Π(GF(pn)) embed into the free plane extension of their union), up to now, no projective plane Π seemed to be known having a subplane of a Lenz-Barlotti class lower than that of Π. Note that our theorem also answers the associated question (firstly brought to my knowledge by S. Prieß-Crampe in 1985) in the positive.

Corollary 10. Given any finite projective plane J of Lenz class I, II or IV there exists a projective plane of class V containing J as a subplane.

We are now going to refine our construction to allow embeddings of certain infinite structures. We first need

Lemma 11. Let V be a vector space with basis {bi | i ∈ N} over some finite or countably infinite field K, let 1 := b1, B := {b1, ..., bn}, n ∈ N, and let μ be an admissible product on B × B. Then for any finite partial plane J = (P, L, I) there exists a number n' > n, B' = {b1, ..., bn'}, and an extension of μ to an admissible product μ' on B' × B', such that J can be embedded into φ(V') for every extension of μ' to a product μ'' on V making V'' = (V, +, μ'') a division algebra.

Proof. Put U := ⟨B⟩ and fix some m ∈ N, m ≥ n, with μ(U, U) ⊂ ⟨b1, ..., bm⟩. Since J is finite, we may write P = {p1, ..., ps} and L = {L1, ..., Lr}. Now we rename the
\[ m' := 2r + 2s + sr + ms + rm \]
\[ \text{basis elements } b_{m+1}, b_{m+2}, \ldots, b_{m+m} \] arbitrarily (but one to one) by
\[ m_i, c_i, x_j, y_j, u_{i,j}, v_{k,j}, w_{i,k}, \]
where \( i \) runs over \( \{1, \ldots, r\} \), \( j \) over \( \{1, \ldots, s\} \) and \( k \) over \( \{1, \ldots, m\} \). We consider the sets \( A := \{b_1, \ldots, b_m, m_1, \ldots, m_s\} \) and \( C := \{b_1, \ldots, b_m, x_1, \ldots, x_s\} \), and define a product \( \cdot \cdot \cdot \) on \( A \times C \) by the rules
\[ b_k \cdot b_h := \mu(b_k, b_h), \]
\[ m_i \cdot 1 := m_i, \]
\[ m_i \cdot b_h := w_{i,h} \quad \text{for } h \neq 1, \]
\[ m_i \cdot x_j := \begin{cases} y_j - c_i & \text{if } p_j \not\mid L_i, \\ u_{i,j} & \text{if } p_j \mid L_i, \end{cases} \]
\[ b_k \cdot x_j := v_{k,j} \quad \text{for } k \neq 1, \]
\[ 1 \cdot x_j := x_j, \]
for all \( k, h \in \{1, \ldots, m\}, i \in \{1, \ldots, r\} \) and \( j \in \{1, \ldots, s\} \). We shall show, that this product is admissible. Clearly, \( A \) and \( C \) are independent and (1a) is fulfilled. To check (1b), for all \( k, h = 1, \ldots, m, i = 1, \ldots, r \) and \( j = 1, \ldots, s \), let \( \alpha_k, \beta_h, \lambda_i, \xi_j \in K \) be given such that the equation
\[ 0 = \sum_{k,h=1, \ldots, m} \alpha_k \beta_h \mu(b_k, b_h) + \sum_{i=1, \ldots, r} \lambda_i \beta_1 m_i + \sum_{i=1, \ldots, r, h=2, \ldots, m} \lambda_i \beta_h w_{i,h} \]
\[ + \sum_{i=1, \ldots, r, j=1, \ldots, s} \lambda_i \xi_j (y_j - c_i) + \sum_{j=1, \ldots, s} \lambda_i \xi_j u_{i,j} \]
\[ + \sum_{k=2, \ldots, m, j=1, \ldots, s} \alpha_k \xi_j v_{k,j} + \sum_{j=1, \ldots, s} \alpha_1 \xi_j x_j \]
is fulfilled. Since all \( \mu(b_k, b_h) \) lie in \( \langle b_1, \ldots, b_m \rangle \), and since \( b_1, \ldots, b_m \) and all the occurring \( m_i, w_{i,h}, y_j, c_i, u_{i,j}, v_{k,j}, \) and \( x_j \) are linearly independent, we immediately infer
\[ 0 = \lambda_i \beta_1 = \lambda_i \beta_h = \alpha_k \xi_j = \alpha_1 \xi_j \]
and
\[ 0 = \lambda_i \xi_j \quad \text{if } p_j \not\mid L_i, \]
for all \( k, h = 1, \ldots, m, i = 1, \ldots, r \) and \( j = 1, \ldots, s \). Hence, we are left with
\[ 0 = \sum_{k,h=1, \ldots, m} \alpha_k \beta_h \mu(b_k, b_h) + \sum_{i=1, \ldots, r, j=1, \ldots, s} \lambda_i \xi_j (y_j - c_i). \]
In view of \( \langle y_1, \ldots, y_s, c_1, \ldots, c_r \rangle \cap \langle b_1, \ldots, b_m \rangle = \{0\} \), we obtain

\[
0 = \sum_{k, h = 1, \ldots, m} \alpha_k \beta_h \mu(b_k, b_h)
\]

and

\[
0 = \sum_{i = 1, \ldots, s} \sum_{j = 1, \ldots, s} \lambda_i \xi_j (y_j - c_i).
\]

From the first equation, we infer \( \alpha_k \beta_h = 0 \) for all \( k, h = 1, \ldots, m \), since \( \mu \) is admissible. And in the very same way as in the proof of Theorem 7, we get \( \lambda_i \xi_j = 0 \) for all coefficients in the last sum. Hence, all products of coefficients in the equation considered above vanish, showing that our product on \( A \times C \) is admissible. So, in light of Proposition 6, it can be extended to a product \( \mu^* \) on \( V \) making \( V \) a division algebra. Let \( n' := m + m', B' := \{b_1, \ldots, b_{m + m'}\} \), and let \( \mu' \) be the restriction of \( \mu^* \) to \( B' \times B' \). By definition, \( \mu' \) extends \( \mu \) and is admissible. Now let \( \mu'' \) be any extension of \( \mu' \) to \( V \times V \) making \( V'' = (V, +, \mu'') \) a division algebra. In the very same way as in the proof of Theorem 7, the mapping

\[
\varphi(p_j) := (x_j, y_j) \quad \text{for all } j = 1, \ldots, s,
\]

\[
\varphi(L_i) := [m_i, c_i] \quad \text{for all } i = 1, \ldots, r
\]

can be shown to be an embedding of \( \mathcal{F} \) into \( \mathfrak{A}(V'') \). \( \square \)

**Theorem 12.** Let \( (\mathcal{F}_n)_{n \in \mathbb{N}} \) be a countable family of finite partial planes. Then there exists a countable translation plane of Lenz class \( V \) with an arbitrary characteristic \( p \neq 0 \) which contains each \( \mathcal{F}_n, n \in \mathbb{N} \).

**Proof.** Let \( V \) be a vector space with countably infinite dimension over some finite field \( K \) with \( \text{char}(K) = p \), and let \( B = \{b_i | i \in \mathbb{N}\} \) be a basis of \( V, 1 := b_1 \). We put \( B_1 := \{b_1\} \) and define an obviously admissible product \( \mu_1 \) on \( B_1 \times B_1 \) by \( \mu_1(1, 1) := 1 \).

By induction on \( n \), for any \( n > 1 \) we specify a finite subset \( B_n = \{b_1, b_2, \ldots, b_m\} \) of \( B \) and define an admissible product \( \mu_n : B_n \times B_n \to V \), such that

- (a) \( B_{n-1} \cup \{b_n\} \subset B_n \),
- (b) \( \mu_n \) extends \( \mu_{n-1} \),
- (c) for any extension of \( \mu_n \) to a product \( \mu'' \) on \( V \) making \( V'' = (V, +, \mu'') \) a division algebra, \( \mathcal{F}_{n-1} \) can be embedded in \( \Pi(V'') \),
- (d) for all \( v, w \in \langle B_{n-1} \rangle^* \) there are \( x, y \in \langle B_n \rangle \) with \( \mu_n(v, x) = \mu_n(y, v) = w \).

So let us assume that for some \( n-1 \geq 1 \) the product \( \mu_{n-1} \) is already defined according to the setting above. By the preceding lemma, there exists a subset \( B_n = \{b_1, \ldots, b_m\} \) of \( B \) and an extension of \( \mu_{n-1} \) to an admissible product \( \mu_n : B_n \times B_n \to V \) such that \( \mathcal{F}_{n-1} \) is embeddable into \( \mathfrak{A}(V, +, \mu'') \) for each extension \( \mu'' \) of \( \mu_n \) making \( V \) a division algebra. Fix such an extension \( \mu'' \). Then for all
If \( v, w \in \langle B_{n-1} \rangle^* \) there are \( x, y \in V \) with \( \mu''(v, x) = \mu''(y, v) = w \). Since \( \langle B_{n-1} \rangle \) is finite, the space

\[
W := \langle \{ x \in V \mid \text{there are } v, w \in \langle B_{n-1} \rangle^* \text{ with } \mu''(v, x) = w \text{ or } \mu''(x, v) = w \} \rangle
\]
is finite, and therefore has a finite dimension. Hence, there exists an \( m_n \in \mathbb{N} \) such that \( m_n \geq m'_n, m_n \geq n \) and

\[
W \subset \langle b_1, \ldots, b_{m_n} \rangle.
\]

Then \( B_n := \{ b_1, \ldots, b_{m_n} \} \) and the restriction \( \mu_n \) of \( \mu'' \) to \( B_n \times B_n \) obviously fulfills (a) up to (δ).

Since each \( \mu_n \) extends its predecessor, there exists a common extension \( \mu \) of all \( \mu_n \). In view of (α), \( \mu \) is defined on \( B \times B \), and thus yields a product on \( V \). Since each \( \mu_n \) is admissible, \( V \) has no zero divisors with respect to \( \mu \), and in view of (δ), \( (V, +, \mu) \) is a division algebra. By (γ), each \( \mathscr{S}_n, n \in \mathbb{N} \), is embeddable into the affine plane \( \mathcal{A} \) over \( (V, +, \mu) \).

Since the collection of all finite partial planes forms a countable family, we immediately infer

**Corollary 13.** There exists a countable translation plane of Lenz class V containing all finite partial planes and thus all finite projective planes and all (simple) rank-three matroids.

**Remark 14.** (a) Although we actually embed each finite partial plane \( \mathscr{S} \) into the affine plane over some division algebra, our embedding is not an 'affine embedding', since we ignore any parallelism that might be defined on \( \mathscr{S} \). Indeed, no two lines of \( \mathscr{S} \) are mapped onto parallel lines by our construction.

(b) In [4], Radó has dealt with the question which partial planes can be extended to projective planes fulfilling certain general closure conditions. Note, that the closure conditions describing the Lenz classes I–V are not general in Radó's sense, since they rely on certain fixed elements.

(c) Making use of the mechanism described by Hughes in [2, Section 3], it is possible to extend the product on \( A \times B \) given in the proof of Theorem 7 to \( V \) such that \( (V, +, \cdot) \) becomes a proper Cartesian field. This should yield an embedding of \( \mathcal{A} \) into a projective plane of Lenz class II. Of course, such an embedding can also be achieved by firstly embedding \( \mathscr{S} \) into a translation plane \( \Pi \) of class V and then changing \( \Pi \) into a plane of class II using some standard constructions. Similarly, it is also possible, to embed \( \mathcal{A} \) into a translation plane of Lenz class IV. We will not pursue this theme here.

(d) In contrast to this, it is impossible to embed each finite partial plane or each rank-three matroid \( \mathscr{S} \) into a projective plane of a Lenz class higher than V. Such an embedding requires that the special form of the axiom of Desargues where center and axis are incident is not violated in \( \mathscr{S} \).
References