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Harmonic Solutions of Transport Equations*

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1. INTRODUCTION

By definition, time-independent solutions of the *diffusion* equation with infinitesimal mean free path in a *homogeneous* medium, are *harmonic functions* (satisfy $\nabla^2 u = 0$). The primary purpose of this paper is to show that *transport equations* for monoenergetic neutrons also have harmonic solutions, including exact *harmonic polynomial* solutions. This result is shown to follow from Gauss' Theorem of the Arithmetic Mean.

More precisely, let monoenergetic neutrons be scattered in an infinite homogeneous medium, without absorption or fission.[1] For algebraic simplicity, we choose units of length and time such that the neutron speed $v = 1$ and the mean free path $\lambda = 1/\sigma = 1$. We shall consider mainly solutions defined in free **x**-space R^p of p dimensions, allowing p to be arbitrary; the (monoenergetic) neutron velocity **v** will then range over the unit sphere Ω in R^p.

We shall utilize below both the integrodifferential and (for the case of isotropic scattering) the integral forms of the transport equation. The p-dimensional forms of these equations for time-independent transport, with isotropic scattering in the laboratory frame, are[2]

$$\mathbf{v} \cdot \nabla\psi(\mathbf{x}, \mathbf{v}) + \psi(\mathbf{x}, \mathbf{v}) = \int_\Omega \psi(\mathbf{x}, \mathbf{v})\, dm(\mathbf{v}). \tag{1.1}$$

$$\Phi(\mathbf{x}) = \int_0^\infty dr\, e^{-r} \int_\Omega dm(\mathbf{v})\, \Phi(\mathbf{x} - r\mathbf{v}). \tag{1.2}$$

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[1] In a superficially more general form [4], the assumption is that the "mean number of secondaries per collision is one," $c = 1$. For the time dependent case, the transport equation with $c \neq 1$ can be reduced to the special case $c = 1$ ([4], p. 175).

[2] Equations (1.1) and (1.2) are straightforward generalization of the standard integral transport equation ([4], Sec. 3.6) to p dimensions.

In Eqs. (1.1), (1.2) and below, $dm(\mathbf{v})$ denotes the differential of solid angle on Ω divided by the total solid angle. Thus, if $p = 3$, $dm = d\Omega/4\pi$.

The essential equivalence of the integrodifferential and the integral forms (1.1) and (1.2) is well-known ([4], Section 3.6). However, this equivalence becomes mathematically rigorous (and Equation (1.2) meaningful) only under (absolute) *integrability* conditions which are usually omitted:

$$\int_0^\infty dr\, e^{-r} \int_\Omega dm(\mathbf{w}) \mid \Phi(r\mathbf{w}) \mid < +\infty \tag{1.3}$$

and

$$\int_0^\infty dr\, e^{-r} \int_{\Omega\times\Omega}\!\!\int \mid \psi(r\mathbf{w}, \mathbf{v}) \mid dm(\mathbf{w})\, dm(\mathbf{v}) < +\infty. \tag{1.3$'$}$$

For (1.3) and (1.3′) to hold, either of the mutually equivalent order-of-growth conditions, for large r

$$\Phi(\mathbf{x} - r\mathbf{v}) = O(r^{-1-\epsilon}e^r), \tag{1.4}$$

$$\psi(\mathbf{x} - r\mathbf{v}', \mathbf{v}) = O(r^{-1-\epsilon}e^r), \tag{1.4$'$}$$

is sufficient. We now formulate this equivalence in rigorous terms.

THEOREM A. *Let $\psi(\mathbf{x}, \mathbf{v})$ be any solution of* (1.1) *which satisfies* (1.3′). *Then the function $\Phi = L[\psi]$, defined by*

$$\Phi(\mathbf{x}) = \int_\Omega \psi(\mathbf{x}, \mathbf{v})\, dm(\mathbf{v}) \tag{1.5}$$

satisfies (1.2) *and* (1.3). *Conversely, let $\Phi(\mathbf{x})$ be any solution of* (1.2) *and* (1.3). *Then the function $\psi = M[\Phi]$, defined by*

$$\psi(\mathbf{x}, \mathbf{v}) = \int_0^\infty e^{-r}\Phi(\mathbf{x} - r\mathbf{v})\, dr \tag{1.6}$$

satisfies (1.1) *and* (1.3′). *Moreover $M[L[\psi]] = \psi$ and $L[M[\Phi]] = \Phi$: L and M are mutually inverse bijections.*

2. TIME-INDEPENDENT TRANSPORT WITH ISOTROPIC SCATTERING

The time-independent transport equation (1.2) with isotropic scattering (in the laboratory frame) will now be treated in detail. Our first main result is a rigorous proof of the existence of harmonic solutions:

THEOREM 1. *Any function* $\Phi(\mathbf{x})$ harmonic *in* R^p, *and satisfying the integrability condition* (1.3), *is a solution of the integral transport equation* (1.2).

PROOF. Let $\Phi(\mathbf{x})$ be harmonic in R^p. Then, by Gauss' Theorem of the Arithmetic Mean ([9], p. 223).

$$\Phi(\mathbf{x}) = \int_{\Omega} \Phi(\mathbf{x} - r\mathbf{v})\, dm(\mathbf{v}), \qquad 0 < r < +\infty. \tag{2.1}$$

Furthermore, for $\Phi(\mathbf{x} - r\mathbf{v})$ satisfying (1.4) (and hence (1.3)), Eq. (2.1) can be multiplied by e^{-r} and integrated over r from 0 to ∞. The left-hand side yields

$$\int_0^\infty dr\, e^{-r}\Phi(\mathbf{x}) = \Phi(\mathbf{x}), \tag{2.2}$$

and the right-hand side is the same as for Eq. (1.2). Q.E.D.

COROLLARY. *Any harmonic polynomial is a solution of the integral transport equation* (1.2).

For one dimension ($p = 1$; slab geometry), the Laplace equation reduces to $\Phi''(z) = 0$ and has only two linearly independent (harmonic) polynomial solutions, $\Phi = 1$ and $\Phi = z$.[3] The above corollary proves that for the interesting case of two or more dimensions ($p \geqslant 2$), the transport equation (1.2) has harmonic polynomial solutions of any degree n. For a given integer n these are, in three dimensions ($p = 3$), the $(2n + 1)$ linearly independent ordinary harmonics which are homogeneous polynomials of degree n in $\mathbf{x}$ ([7], Section 7.6). Examples are the three linearly independent harmonics, x, y, z of degree 1 and the five independent harmonics, $x^2 - y^2$, $y^2 - z^2$, xy, yz, zx of degree 2.

The correspondence between harmonic solutions $\Phi(\mathbf{x})$ of the scalar flux equation and those for the vector flux $\psi(\mathbf{x}, \mathbf{v})$ can now be described.

THEOREM 2. *The transformation* (1.6) *carries solutions* $\Phi(\mathbf{x})$ *of Eq.* (1.2) *in* R^p *which are harmonic and satisfy the integrability condition* (1.3), *into solutions* $\psi(\mathbf{x}, \mathbf{v})$ *of Eq.* (1.1) *which are harmonic in* $\mathbf{x}$ *for any fixed* $\mathbf{v}$.

PROOF. By hypothesis, we have as in (1.6)

$$\psi(\mathbf{x}, \mathbf{v}) = \int_0^\infty e^{-s}\Phi(\mathbf{x} - s\mathbf{v})\, ds, \qquad \text{all} \qquad \mathbf{x} \in R^p, \mathbf{v} \in \Omega.$$

[3] In view of Theorem 4 below, we obtain as a corollary the result of ([1], Part C): all polynomial solutions of (1.2) are linear of the form $\Phi = a + bz$. This result also holds for anisotropic scattering in plane geometry (see Sec. 3).

Averaging over a sphere with center $\mathbf{x}$ and radius r:

$$\int_\Omega \psi(\mathbf{x} + r\mathbf{v}', \mathbf{v})\, dm(\mathbf{v}') = \int_\Omega dm(\mathbf{v}') \int_0^\infty \Phi(x + r\mathbf{v}' - s\mathbf{v})\, e^{-s}\, ds$$

$$= \int_0^\infty e^{-s}\, ds \int_\Omega \Phi(\mathbf{x} + r\mathbf{v}' - s\mathbf{v})\, dm(\mathbf{v}')$$

$$= \int_0^\infty e^{-s}\, ds\, \Phi(\mathbf{x} - s\mathbf{v}) = \psi(\mathbf{x}, \mathbf{v}),$$

where: (i) the first equation follows from (1.6), (ii) the second by the assumed absolute integrability of Φ, which justifies changing the order of integration over $[0, \infty) \times \Omega$, (iii) the third equation because Φ satisfies the Theorem of the Arithmetic Mean, and the last by (1.6). This proves that ψ satisfies Gauss' Theorem of the Arithmetic Mean for any fixed $\mathbf{v}$, and so ([5], p. 277; [9], p. 224) is harmonic in $\mathbf{x}$. Q.E.D.

In the special case of (harmonic, see Theorem 4 below) polynomial solutions of the transport equation (1.1) [and (1.2)], the following result yields helpful formulas for obtaining the vector flux of (1.6) from the scalar flux.

THEOREM 3. *Let $H(\mathbf{x})$ be a homogeneous harmonic polynomial in R^p of degree n. Then the function*

$$U_{n-k,k}(\mathbf{x}, \mathbf{u}) = (\mathbf{u} \cdot \nabla)^k H(\mathbf{x}), \qquad k = 0, 1, \ldots, n, \tag{2.3}$$

is a homogeneous harmonic polynomial in $\mathbf{x}$ of degree $n - k$ for fixed $\mathbf{u}$, and a homogeneous harmonic polynomial in $\mathbf{u}$ of degree k for fixed $\mathbf{x}$.

PROOF. Since $H(\mathbf{x})$ is harmonic in $\mathbf{x}$, necessarily

$$\nabla^2 U_{n-k,k}(\mathbf{x}, \mathbf{u}) = (\mathbf{u} \cdot \nabla)^k \nabla^2 H(\mathbf{x}) = 0,$$

i.e., $U_{n-k,k}$ is harmonic in $\mathbf{x}$. Furthermore, since $H(\mathbf{x})$ is a homogeneous polynomial of degree n in $\mathbf{x}$ it follows that $(\mathbf{u} \cdot \nabla)^k H(\mathbf{x})$ is a homogeneous polynomial of degree $n - k$ in $\mathbf{x}$. Thus $U_{n-k,k}(\mathbf{x}, \mathbf{u})$ is a harmonic polynomial of degree $n - k$ in $\mathbf{x}$.

Moreover

$$\sum_{i=1}^n \frac{\partial^2}{\partial u_i^2} U_{n-k,k}(\mathbf{x}, \mathbf{u}) = \sum_{i=1}^n k(k-1)(\mathbf{u} \cdot \nabla)^{k-2} \frac{\partial^2}{\partial x_i^2} H(\mathbf{x})$$

$$= k(k-1)(\mathbf{u} \cdot \nabla)^{k-2} \nabla^2 H(\mathbf{x}) = 0.$$

Thus, $U_{n-k,k}(\mathbf{x}, \mathbf{u})$ is harmonic in $\mathbf{u}$. Furthermore $(\mathbf{u} \cdot \nabla)^k H(\mathbf{x})$ is a homogeneous polynomial of degree k in $\mathbf{u}$. Hence $U_{n-k,k}(\mathbf{x}, \mathbf{u})$ is a harmonic polynomial of degree k in $\mathbf{u}$. Q.E.D.

NOTE. The function $U_{l,k} = V(\mathbf{x}, \mathbf{u})$ of Eq. (2.3) is defined for all $(\mathbf{x}, \mathbf{u}) \in R^p \times R^p$, and $V(\mathbf{x}, \mathbf{u})$ is doubly harmonic, i.e.,

$$\sum_i \frac{\partial^2}{\partial x_i^2} V = \sum_i \frac{\partial^2}{\partial u_i^2} V = 0.$$

We recall that if a function $V(\mathbf{x}, \mathbf{u})$ is homogeneous of degree l in $\mathbf{x}$, then

$$V(\lambda\mathbf{x}, \mathbf{u}) = \lambda^l V(\mathbf{x}, \mathbf{u}).$$

If $V(\mathbf{x}, \mathbf{u})$ is also homogeneous of degree k in $\mathbf{u}$, then

$$V(\mathbf{x}, \lambda\mathbf{u}) = \lambda^k V(\mathbf{x}, \mathbf{u}).$$

Furthermore, its restriction $V(\mathbf{x}, \boldsymbol{\theta})$ to the unit sphere $|\boldsymbol{\theta}| = 1$ is by definition a spherical harmonic of degree k in $\boldsymbol{\theta}$.

COROLLARY 1. *For $\Phi(\mathbf{x})$ a harmonic polynomial of degree n, the vector flux*

$$\psi(\mathbf{x}, \mathbf{v}) = \sum_{k=0}^{n} (-1)^k (\mathbf{v} \cdot \nabla)^k \Phi(\mathbf{x}) \tag{2.4}$$

is a solution of (1.1) *with scalar flux Φ. Each term*

$$V_{n-k,k}(\mathbf{x}, \mathbf{v}) = (-1)^k (\mathbf{v} \cdot \nabla)^k \Phi(\mathbf{x})$$

in (2.4) *is a harmonic polynomial in* $\mathbf{x}$ *of degree $n - k$ and a spherical harmonic in* $\mathbf{v}$ *of degree k.*

The detailed proof will be omitted, since a more general result will be proved as Theorem 5 below, *without* the assumption of isotropic scattering.

COROLLARY 2. *The vector flux $\psi(\mathbf{x}, \mathbf{v})$ given by* (2.4) *is also given as $\psi = M[\Phi]$ by* (1.6).

PROOF. By definition, the scalar flux $\Phi(\mathbf{x})$ in Corollary 1 is given by (1.5) as $\Phi = L[\psi]$. This fact can also be verified directly from Eq. (2.4), since by Theorem 3 and the orthogonality properties of harmonic polynomials ([9], p. 252) indeed

$$\int_\Omega \psi(\mathbf{x}, \mathbf{v})\, dm(\mathbf{v}) = \Phi(x). \tag{2.5}$$

Hence, Corollary 2 follows from Theorem A. Q.E.D.

Using Corollary 1, it is easy to compute exact polynomial solutions $\psi(\mathbf{x}, \mathbf{v})$ of the integrodifferential transport equation (1.1). Thus let $\mathbf{x} = (x, y)$, $\mathbf{v} = (\xi, \eta)$ for $p = 2$, and let $\mathbf{x} = (x, y, z)$, $\mathbf{v} = (\xi, \eta, \zeta)$ for $p = 3$.

Then the solutions of (1.1) corresponding to the homogeneous harmonic polynomial scalar fluxes $\Phi = xy$, $x^2 - y^2$ and $x^3 - 3xy^2$ for $p = 2$ are:

$$\psi = xy - x\eta - y\xi + 2\xi\eta \tag{2.6a}$$

$$\psi = x^2 - y^2 - 2\{x\xi - y\eta\} + 2\{\xi^2 - \eta^2\} \tag{2.6b}$$

$$\psi = x^3 - 3xy^2 - 3\{(x^2 - y^2)\,\xi - 2xy\eta\} + 6\{x(\xi^2 - \eta^2) - 2y\xi\eta\} - 6\{\xi^3 - 3\xi\eta^2\}. \tag{2.6c}$$

Likewise for $p = 3$, the solutions corresponding to $\Phi = xyz$ and $\Phi = x^3 - \frac{3}{2}\,x(y^2 + z^2)$ are:

$$\psi = xyz - \{xy\zeta + yz\xi + zx\eta\} + 2\{x\eta\zeta + y\zeta\xi + z\xi\eta\} - 6\xi\eta\zeta. \tag{2.6d}$$

$$\psi = x^3 - \tfrac{3}{2}\,x(y^2 + z^2) - \tfrac{3}{2}\,\{2x^2 - y^2 - z^2)\,\xi - 2xy\eta - 2xz\zeta\} + 3\{x(2\xi^2 - \eta^2 - \zeta^2) - 2y\xi\eta - 2z\xi\zeta\} - 6\{\xi^3 - \tfrac{3}{2}\,\xi(\eta^2 + \zeta^2)\}. \tag{2.6e}$$

We will now prove a converse to the Corollary of Theorem 1.

THEOREM 4. *Any polynomial solution $\Phi(\mathbf{x})$ of the integral transport equation (1.2) for the scalar flux is harmonic.*

PROOF. Let $\Phi(\mathbf{x})$ be any polynomial solution of the transport equation (1.2) of degree n so that

$$\nabla^{2s}\Phi = 0 \qquad \text{for all} \qquad s \geqslant m + 1, \qquad m = \left[\frac{n}{2}\right]. \tag{2.7}$$

Here, $m = [n/2]$ is the largest integer with $2m \leqslant n$. It is known (see [10], Eq. (4), and [5], pp. 287-8) that if (2.7) holds, then Φ has the spherical mean

$$\int_\Omega dm(\mathbf{v})\,\Phi(\mathbf{x} - r\mathbf{v}) = \Phi(\mathbf{x}) + \sum_{i=1}^{m} a_{pi} r^{2i} \nabla^{2i}\Phi(\mathbf{x})/(2i)! \tag{2.8}$$

where

$$a_{pi} = (2i)!/2^i i!\,p(p + 2) \cdots (p + 2i - 2).$$

Now substituting Eq. (2.8) into the right-hand side of Eq. (1.2), integrating the finite sum term-by-term, and simplifying, we get

$$[a_{p,1}\nabla^2 + a_{p,2}\nabla^4 + \cdots + a_{p,m}\nabla^{2m}]\,\Phi(\mathbf{x}) = 0. \tag{2.9}$$

The key point for the rest of the proof is that $a_{p,1} \neq 0$. From (2.7), clearly $\nabla^{2m+2}\Phi(\mathbf{x}) = 0$, $m = [n/2]$. Consequently, multiplying (2.9) by

$(1/a_{p,1})\, \nabla^{2(m-1)}$, all terms except the first term on the left-hand side of (2.9) vanish and the result is $\nabla^{2m}\Phi = 0$. Analogously, it follows upon multiplying Eq. (2.9) by $(1/a_{p,1})\, \nabla^{2(k-1)}$, that $\nabla^{2(k+1)}\Phi = 0$ implies $\nabla^{2k}\Phi = 0$ $(k \geqslant 1)$. Hence, it follows by mathematical induction that indeed

$$\nabla^2\Phi = 0, \tag{2.10}$$

i.e., $\Phi(\mathbf{x})$ is a harmonic polynomial in $\mathbf{x}$. Q.E.D.

COROLLARY. *Any solution* $\psi(\mathbf{x}, \mathbf{v})$ *of the integrodifferential transport equation* (1.1) *which is a polynomial in* $\mathbf{x}$ *in* R^p *is harmonic in* $\mathbf{x}$ *for any fixed* $\mathbf{v}$ *and a spherical harmonic in* $\mathbf{v}$ *for any fixed* $\mathbf{x}$.

PROOF. Let $\psi(\mathbf{x}, \mathbf{v})$ be any such solution. Then the corresponding scalar flux of (1.2) $\Phi = L[\psi]$ defined by (1.5) is also a polynomial in $\mathbf{x}$ and hence, by Theorem 4, is harmonic in $\mathbf{x}$. Therefore, by Corollaries 1 and 2 of Theorem 3, $M[\Phi]$ defined by (1.6) is harmonic in $\mathbf{x}$ for any fixed $\mathbf{v}$ and a spherical harmonic in $\mathbf{v}$ for any fixed $\mathbf{x}$. But by Theorem A

$$\psi(\mathbf{x}, \mathbf{v}) = M[L[\psi(\mathbf{x}, \mathbf{v})]] = M[\Phi].$$

Hence, the given polynomial solution $\psi(\mathbf{x}, \mathbf{v})$ is harmonic in $\mathbf{x}$ for any fixed $\mathbf{v}$ and a spherical harmonic in $\mathbf{v}$ for any fixed $\mathbf{x}$. Q.E.D.

3. Generalization to Anisotropic Scattering

We now consider the time-independent integrodifferential transport equation with *anisotropic* scattering, still monoenergetic and with pure diffusion $(c = 1)$

$$\mathbf{v} \cdot \nabla\psi(\mathbf{x}, \mathbf{v}) + \psi(\mathbf{x}, \mathbf{v}) = \int_\Omega dm(\mathbf{v}')\, f(\mathbf{v} \cdot \mathbf{v}')\, \psi(\mathbf{x}, \mathbf{v}') \tag{3.1}$$

where $f(\mathbf{v} \cdot \mathbf{v}')$ is the scattering function

$$f(\mathbf{v} \cdot \mathbf{v}') = \sum_{l=0}^{L} (2l + 1)\, \alpha_l P_l(\mathbf{v} \cdot \mathbf{v}'), \qquad \alpha_0 = 1 \tag{3.2}$$

and the $P_l(\mu)$ are Legendre polynomials. Physically, $f(\mathbf{v} \cdot \mathbf{v}') = f(\mu) \geqslant 0$, and $\int_{-1}^{1} f(\mu)\, d\mu = 2$ so that $\alpha_0 = 1$ and $|\, \alpha_l \,| \leqslant 1/(2l + 1)$.

No analogue of Theorem A and no integral equation for the scalar flux $\Phi(\mathbf{x})$ seem to be known. Hence we shall treat the integrodifferential transport equation (3.1) directly. One result of this section is to derive an expression for the vector flux in terms of the scalar flux which is valid at least when the scalar flux is a harmonic polynomial.

The following generalization of Corollary 1 of Theorem 3 enables one to compute a vector flux solution of (3.1) having any preassigned harmonic polynomial for its scalar flux.

THEOREM 5. *Corresponding to any homogeneous harmonic polynomial* $H(\mathbf{x})$ *of degree* n *in* R^p, *the transport equation* (3.1) *has a solution* $\psi(\mathbf{x}, \mathbf{v})$ *with scalar flux* $H(\mathbf{x})$. *This solution is*

$$\psi(\mathbf{x}, \mathbf{v}) = \sum_{k=0}^{n} (-1)^k \beta_k (\mathbf{v} \cdot \nabla)^k H(\mathbf{x}) \tag{3.3}$$

where

$$\beta_0 = 1, \qquad \beta_k = \frac{1}{\{(1-\alpha_1)(1-\alpha_2)\cdots(1-\alpha_k)\}}, \qquad k > 0. \tag{3.4}$$

Two cases of special interest are $f(\mu) = 1$ (isotropic scattering) and $f(\mu) = 1 + b\mu$, $|b| \leqslant 1$. For these cases we have $\beta_k = 1$ and $\beta_k = 1/(1 - b/3)$ respectively for all $k > 0$.

PROOF. By Theorem 3 each term $V_{n-k,k}(\mathbf{x}, \mathbf{v}) = (-1)^k \beta_k(\mathbf{v} \cdot \nabla)^k H(\mathbf{x})$ in (3.3) is a harmonic polynomial in $\mathbf{x}$ of degree $n - k$ and is a spherical harmonic in $\mathbf{v}$ of degree k. Hence, integrating $\psi(\mathbf{x}, \mathbf{v})$ of (3.3) over all $\mathbf{v} \in \Omega$ and using the orthogonality properties of harmonic polynomials ([9], p. 252), we get

$$\int_\Omega \psi(\mathbf{x}, \mathbf{v})\, dm(\mathbf{v}) = H(\mathbf{x}). \tag{3.5}$$

Furthermore, we observe from (3.4) that

$$\beta_0 = \alpha_0 = 1, \qquad \alpha_k\beta_k = \beta_k - (1 - \alpha_k)\beta_k = \beta_k - \beta_{k-1}, \qquad k > 0. \tag{3.6}$$

Now, substituting $\psi(\mathbf{x}, \mathbf{v})$ from (3.3) into the right hand side of the transport equation (3.1), interchanging integration and finite summations and again using Theorem 3, the orthogonality properties of spherical harmonics ([7], Section 95) and Eq. (3.6) yields

$$\begin{aligned}
\int_\Omega dm(\mathbf{v}') \sum_{l=0}^{\infty} (2l+1)\,\alpha_l P_l(\mathbf{v} \cdot \mathbf{v}')\, \psi(\mathbf{x}, \mathbf{v}') \\
&= \sum_{k=0}^{n} (-1)^k \alpha_k\beta_k (\mathbf{v} \cdot \nabla)^k H(\mathbf{x}) \\
&= \sum_{k=0}^{n} (-1)^k \beta_k (\mathbf{v} \cdot \nabla)^k H(x) + \mathbf{v} \cdot \nabla \sum_{k'=0}^{n} (-1)^{k'} \beta_{k'} (\mathbf{v} \cdot \nabla)^{k'} H(\mathbf{x}) \\
&= \psi(\mathbf{x}, \mathbf{v}) + \mathbf{v} \cdot \nabla\psi(\mathbf{x}, \mathbf{v}).
\end{aligned}$$

Here L appearing in (3.2) has been replaced by infinity; this is permissible because $\alpha_l = 0$ for $l > L$. We observe that, since $H(\mathbf{x})$ is a harmonic polynomial of degree n, $(\mathbf{v} \cdot \nabla)^{n+1} H(\mathbf{x}) = 0$. This fact has been used above to establish that $\psi(\mathbf{x}, \mathbf{v})$ of Eq. (3.3) is indeed a solution of the transport equation (3.1). Q.E.D.

A few samples of harmonic polynomial solutions of Eq. (3.1) for anisotropic scattering, in the notation of Sec. 2, are for $p = 2$

$$\psi = xy - \beta_1\{x\eta + y\xi\} + 2\beta_2\xi\eta$$

$$\psi = x^2 - y^2 - 2\beta_1\{x\xi - y\eta\} + 2\beta_2\{\xi^2 - \eta^2\}$$

and for $p = 3$

$$\psi = xyz - \beta_1\{xy\zeta + yz\xi + zx\eta\} + 2\beta_2\{x\eta\zeta + y\zeta\xi + z\xi\eta\} - 6\beta_3\xi\eta\zeta.$$

The scalar fluxes corresponding to the above solutions are respectively $\Phi = xy$, $\Phi = x^2 - y^2$ for $p = 2$ and $\Phi = xyz$ for $p = 3$.

In one dimension ($p = 1$; slab geometry), there are again only two linearly independent polynomial solutions (for $\Phi''(z) = 0$) for the scalar flux, $\Phi = 1$ and $\Phi = z$; the corresponding vector fluxes are $\psi = 1$ and $\psi = z - \mu/(1 - \alpha_1)$.

The form of the explicit solutions presented above and the form of the series in (3.3) suggests relating the vector flux to the scalar flux in the following way:

$$\psi(\mathbf{x}, \mathbf{v}) = \int_0^\infty dr\, e^{-r} \sum_{k=0}^\infty (2k + 1)\, \beta_k \int_\Omega dm(\mathbf{v}')\, \Phi(\mathbf{x} - r\mathbf{v}')\, P_k(\mathbf{v} \cdot \mathbf{v}') \tag{3.7}$$

where $\Phi(\mathbf{x} - r\mathbf{v})$ satisfies the integrability condition (1.4). The relation (3.7) holds whenever the scalar flux $\Phi(\mathbf{x})$ is a harmonic polynomial. To demonstrate this fact we need the following result.

Lemma 1. *Let $H(\mathbf{x})$ be a homogeneous harmonic polynomial of degree n in R^p. Then the identity*

$$(-1)^k (\mathbf{v} \cdot \nabla)^k H(\mathbf{x}) = \int_0^\infty dr\, e^{-r} \int_\Omega dm(\mathbf{v}')\, H(\mathbf{x} - r\mathbf{v}')\, (2k + 1)\, P_k(\mathbf{v} \cdot \mathbf{v}') \tag{3.8}$$

holds for all integers k.

Proof. Corollaries 1 and 2 of Theorem 3 imply that

$$\sum_{l=0}^n (-1)^l (\mathbf{v} \cdot \nabla)^l H(\mathbf{x}) = \int_0^\infty dr e^{-r} H(\mathbf{x} - r\mathbf{v}). \tag{3.9}$$

Since obviously $(\mathbf{v} \cdot \nabla)^l H(\mathbf{x}) = 0$ for all $l > n$, the summation on the left-hand side of (3.9) can be extended to infinity. Now we get (3.8) by using Theorem 3, and equating spherical harmonics of the same order in $\mathbf{v}$ in (3.9)—or, equivalently, by multiplying (3.9) by $(2k+1) P_k(\mathbf{v} \cdot \mathbf{v}')$, integrating over all $\mathbf{v}' \in \Omega$ and using orthogonality properties of spherical harmonics ([9], p. 252). Q.E.D.

THEOREM 6. *For $H(\mathbf{x})$ any homogeneous harmonic polynomial of degree n in R^p, the vector flux given by* (3.3) *is also given by* (3.7) *with scalar flux $\Phi(\mathbf{x}) = H(\mathbf{x})$.*

PROOF. Multiply Eq. (3.8) by β_k, sum over k, and interchange summation and integration over r (allowed because the absolute integrability condition (1.3) is satisfied). We obtain

$$\sum_{k=0}^{\infty} (-1)^k \beta_k (\mathbf{v} \cdot \nabla)^k H(\mathbf{x}) = \int_0^{\infty} dr\, e^{-r} \sum_{k=0}^{\infty} (2k+1) \beta_k \times \int_{\Omega} dm(\mathbf{v}') H(\mathbf{x} - r\mathbf{v}') P_k(\mathbf{v} \cdot \mathbf{v}'). \tag{3.10}$$

The right-hand side of (3.10) is the same as in (3.7) with scalar flux $\Phi = H$. Furthermore, since $(\mathbf{v} \cdot \nabla)^k H(\mathbf{x}) = 0$ if $k > n$, the left-hand side of (3.10) is identical to the vector flux in (3.3). Hence, for a given $\Phi = H(\mathbf{x})$, Eqs. (3.3) and (3.7) yield the same vector flux $\psi(\mathbf{x}, \mathbf{v})$. Q.E.D.

We conjecture that Eq. (3.7) relating the vector flux to the scalar flux for the transport equation (3.1) for anisotropic scattering holds in general, provided of course that the series term in (3.7) converges and the integrability condition (1.3) is satisfied.

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