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Sheaves and Localization

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In this note, we calculate projective limits of localization functors. We relate the results, thus obtained, to the construction of structure sheaves for noncommutative rings.

INTRODUCTION

Whereas in [1, 5, 9], the authors used compatibility results between torsion and localization functors in order to construct structure sheaves for suitable topologies on the spectrum of a noncommutative ring, here we do exactly the opposite, i.e., we investigate what happens in the case of non-compatibility. One of the main results in this direction states that two idempotent kernel functors σ and τ in R-mod, the category of left R-modules, are compatible, i.e., $\sigma Q_\tau = Q_\tau \sigma$ (cf. [3, 4, 7] for notations and definitions) if and only if the canonical sequence of functors

$$0 \to Q_{\sigma \wedge \tau} \to Q_\sigma \oplus Q_\tau \to Q_{\sigma \vee \tau}$$

is exact. This is the torsion-theoretic analogue of the well-known fact that for any quasi-coherent sheaf $\mathscr{E}$ on an algebraic variety X and any open subsets U, V, and W of X, there is an exact sequence

$$0 \to \mathscr{E}((U \cup V) \cap W) \to \mathscr{E}(U \cap W) \oplus \mathscr{E}(V \cap W) \to \mathscr{E}(U \cap V \cap W).$$

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In general, if σ and τ are not necessarily compatible, it appears that one may still show that the sequence (with obvious morphisms)

$$0 \to Q_{\sigma \wedge \tau} \to Q_\sigma \oplus Q_\tau \to Q_\sigma Q_\tau \oplus Q_\tau Q_\sigma$$

is exact. Actually, we will prove that for any finite family $\{\sigma_\alpha; \alpha \in A\}$ of idempotent kernel functors over R, the localization functor $Q_{\wedge \sigma_\alpha}$ is the projective limit of the family

$$\left\{Q_{\sigma_\alpha} \to Q_{\sigma_\alpha} Q_{\sigma_\beta}, Q_{\sigma_\beta} \to Q_{\sigma_\alpha} Q_{\sigma_\beta}; \alpha, \beta \in A\right\}.$$

This result is then strengthened to encompass projective limits of ''localization functors'' with respect to words in the free monoid over the set of all Gabriel filters for R.

1. PROJECTIVE LIMITS OF LOCALIZATION FUNCTORS

(1.1) As stated in the Introduction, two idempotent kernel functors σ and τ are mutually compatible if and only if the following sequence of functors is exact

$$0 \to Q_{\sigma \wedge \tau} \to Q_\sigma \oplus Q_\tau \to Q_{\sigma \vee \tau}.$$

Of course, the exactness of the latter sequence may also be formulated by saying that $Q_{\sigma \wedge \tau}$ is the limit (equalizer) of the projective system

$$\{Q_\sigma \to Q_{\sigma \vee \tau}, Q_\tau \to Q_{\sigma \vee \tau}\}.$$

In this section, we will take a look at this type of projective limits, when σ and τ are no longer assumed to be compatible.

Actually, one may prove:

(1.2) PROPOSITION. *Let $K = \{\sigma_\alpha; 1 \le \alpha \le n\}$ be a finite family of idempotent kernel functors over R and put $\sigma = \wedge_\alpha \sigma_\alpha$. Let M be a left R-module and denote by $j_{M,\alpha} : M \to Q_\alpha M$ the localization map with respect to σ_α. If $\mathscr{Q}_K M$ denotes the projective system*

$$\left\{j^\alpha_{\alpha\beta} : Q_\alpha M \to Q_\alpha Q_\beta M, j^\beta_{\alpha\beta} : Q_\beta M \to Q_\alpha Q_\beta M; 1 \le \alpha, \beta \le n\right\},$$

where $j^\alpha_{\alpha\beta} = Q_\alpha j_{M,\beta}$, and $j^\beta_{\alpha\beta} = j_{Q_\beta M,\alpha}$, then $\varprojlim \mathscr{Q}_K M = Q_\sigma M$.

Proof. We write $Q_{\alpha\beta} = Q_\alpha Q_\beta$ resp. $Q_{\alpha\beta\gamma} = Q_\alpha Q_\beta Q_\gamma$. One easily reduces to $n = 2$, and in this case, one has to show that the sequence

$$0 \to Q_\sigma M \xrightarrow{i} Q_1 M \oplus Q_2 M \xrightarrow{\pi} Q_{12} M \oplus Q_{21} M$$

is exact, where i is the canonical map and where for any $(m_1, m_2) \in Q_1 M \oplus Q_2 M$, we put

$$\pi(m_1, m_2) = \left(j_{12}^1(m_1) - j_{12}^2(m_2), j_{21}^1(m_1) - j_{21}^2(m_2)\right).$$

Let

$$K_{\alpha\beta} = \operatorname{Ker}\left(\pi_{\alpha\beta} = j_{\alpha\beta}^1 - j_{\alpha\beta}^2 : Q_1 M \oplus Q_2 M \to Q_{\alpha\beta} M\right).$$

To prove the assertion, we have to verify that $K = \operatorname{Ker}(\pi) = K_{12} \cap K_{21}$ coincides with $Q_\sigma M$ within $Q_1 M \oplus Q_2 M$. Since $Q_\sigma M \subseteq K_{12} \cap K_{21} = K$ and all of these are σ-closed, it suffices to show that $K_{12} \cap K_{21}/M$ is σ-torsion.

Localizing at σ yields exact sequences

$$0 \to Q_1 K_{12} \to Q_1 M \oplus Q_{12} M \to Q_{12} M,$$

resp.

$$0 \to Q_1 K_{21} \to Q_1 M \oplus Q_{12} M \to Q_{121} M.$$

Since (y, z) belongs to $Q_1 K_{12}$ resp. $Q_1 K_{21}$ if and only if $j_{12}^1(y) = z$ resp. $j_{12}^1(y) - z \in \operatorname{Ker}(Q_{12} M \to Q_{121} M)$, it follows that

$$Q_1(K_{12} \cap K_{21}) = Q_1 K_{12} \cap Q_1 K_{21} = Q_1 K_{12} = Q_1 M.$$

Similarly, $Q_1(K_{12} \cap K_{21}) = Q_2 M$, so, as $\sigma = \sigma_1 \wedge \sigma_2$, we find that $K_{12} \cap K_{21}/M$ is σ-torsion, indeed. ∎

(1.3) If σ and τ are mutually compatible, then $Q_\sigma Q_\tau$ is $\sigma \vee \tau$-torsion free, so, in particular, the canonical map $Q_\sigma Q_\tau \to Q_{\sigma \vee \tau}$ is injective. Applying this to a family $\{\sigma_\alpha; 1 \leq \alpha \leq n\}$ of mutually compatible idempotent kernel functors over R, it then follows from the previous result that there is an exact sequence

$$0 \to Q_\sigma M \to \bigoplus_\alpha Q_\alpha M \to \bigoplus_{\alpha, \beta} Q_{\alpha \vee \beta} M,$$

where $Q_{\alpha \vee \beta}$ is the localization functor at $\sigma_\alpha \vee \sigma_\beta$.

We thus recover the results in [1, 5, 6, 10], to which we refer for the implications of the exactness of this sequence upon the construction of structure sheaves.

2. A GENERALIZATION

(2.1) Denote by $\mathbb{G}$ the free semigroup generated by all Gabriel filters over R, i.e., elements in $\mathbb{G}$ are *words* $\mathbf{L} = \mathscr{L}_1 \dots \mathscr{L}_n$, where all the $\mathscr{L}_i$ are Gabriel filters. Recall from [2] that two Gabriel filters $\mathscr{L}$ and $\mathscr{H}$ yield a new filter $\mathscr{L} \circ \mathscr{H}$, which consists of all left ideals L of R, with the property that there exists some $H \in \mathscr{H}$ containing L and such that $(L : U) \in \mathscr{L}$ for some finite subset U of H. In general, $\mathscr{L} \circ \mathscr{H}$ is not a Gabriel filter anymore; however, since it is *uniform* (in the sense of [2]), torsion with respect to it may still be defined. We will denote by $\epsilon(\mathbf{L})$ the uniform filter $\mathscr{L}_1 \circ \dots \circ \mathscr{L}_n$ and by $\sigma_{\mathbf{L}}$ the associated torsion functor $\sigma_{\epsilon(\mathbf{L})}$. We write $Q_{\mathbf{L}}$ for the composition $Q_{\mathscr{L}_n} \dots Q_{\mathscr{L}_1}$.

Let us call a left R-module $\mathbf{L}$-*torsion*, if it is $\epsilon(\mathbf{L})$-torsion. Then one may show, by induction on the length of $\mathbf{L} = \mathscr{L}_1 \dots \mathscr{L}_n \in \mathbb{G}$, that for any $M \in R$-mod with associated canonical morphism $j_{\mathbf{L}} : M \to Q_{\mathbf{L}}(M)$, we have that $\operatorname{Ker}(j_{\mathbf{L}}) = \sigma_{\mathbf{L}} M$ and that $\operatorname{Coker}(j_{\mathbf{L}})$ is $\mathbf{L}$-torsion.

(2.2) Let $K = \{\mathbf{L}_\alpha; 1 \le \alpha \le n\}$ be a finite family of words in $\mathbb{G}$. Put $Q_{\mathbf{L}_\alpha} = Q_\alpha$ and consider the projective system

$$\mathscr{Q}_K M = \left\{ j^\alpha_{\alpha\beta} : Q_\alpha M \to Q_\alpha Q_\beta M, j^\beta_{\alpha\beta} : Q_\beta M \to Q_\alpha Q_\beta M; 1 \le \alpha, \beta \le n \right\},$$

where the maps are the obvious ones. Let $\mathscr{L} = \bigcap_{\alpha=1}^n \epsilon(\mathbf{L}_\alpha)$, then the previous remarks imply that

$$\sigma_{\mathscr{L}} M \subseteq \epsilon(\mathbf{L}_\alpha) M = \operatorname{Ker}(M \to Q_\alpha M)$$

for all $1 \le \alpha \le n$. The canonical map $M \to \varprojlim \mathscr{Q}_K M$ thus factorizes through

$$j : M / \sigma_{\mathscr{L}} M \to \varprojlim \mathscr{Q}_K M.$$

We then have:

(2.3) PROPOSITION. *With the previous notations*, $\operatorname{Ker}(j) = 0$ *and the cokernel* $\operatorname{Coker}(j)$ *is* $\mathscr{L} \circ \mathscr{L}$*-torsion.*

Proof. Let $\sigma_\alpha = \sigma_{\epsilon(\mathbf{L}_\alpha)}$ and consider the canonical map

$$j_\alpha : \overline{M} = M / \sigma_{\mathscr{L}} M \to M / \sigma_\alpha M \hookrightarrow Q_\alpha M.$$

Clearly, $\operatorname{Ker}(j_\alpha) = \sigma_\alpha \overline{M}$, and from this, it easily follows that j is injective. On the other hand, assume $q = (q_\alpha) \in \oplus_{\alpha=1}^n Q_\alpha M$ belongs to $\varprojlim \mathscr{Q}_K M$ and fix $1 \le \beta \le n$. Then, an easy induction argument (cf. [10] for details), shows that for any index α, there exists some $L_\alpha \in \epsilon(\mathbf{L}_\alpha)$ such that

$$L_\alpha q_\beta \subseteq \operatorname{Im}\big(M \to Q_\beta M\big) = \operatorname{Im}\big(\overline{M} \to Q_\beta M\big).$$

Clearly, $L = \sum_{\alpha=1}^{n} L_\alpha \in \bigcap_{\alpha=1}^{n} \epsilon(\mathbf{L}_\alpha) = \mathscr{L}$ and $Lq_\beta \subseteq \operatorname{Im}(\overline{M} \to Q_\beta M)$. Since the q_β are finite in number, we may find a single $L \in \mathscr{L}$, which works for all β, i.e., $Lq_\beta \subseteq \operatorname{Im}(\overline{M} \to Q_\beta M)$ for all $1 \leq \beta \leq n$.

Let $l \in L$, then $lq_\alpha = j_\alpha(x_\alpha)$ for some $x_\alpha \in \overline{M}$. Fix β for a moment and, for each α, consider the commutative diagram

$$\begin{array}{ccc} \overline{M} & \xrightarrow{\ j_\beta\ } & Q_\beta M \\ {\scriptstyle j_\alpha}\downarrow & & \downarrow{\scriptstyle j^{\beta}_{\beta\alpha}} \\ Q_\alpha M & \xrightarrow[\ j^{\alpha}_{\beta\alpha}\ ]{} & Q_\alpha Q_\beta M \end{array}$$

Since

$$j^{\alpha}_{\beta\alpha} j_\alpha(x_\beta) = j^{\beta}_{\beta\alpha} j_\beta(x_\beta) = l j^{\beta}_{\beta\alpha}(q_\beta) = l j^{\alpha}_{\beta\alpha}(q_\alpha) = j^{\alpha}_{\beta\alpha} j_\alpha(x_\alpha),$$

there exists some $J_\beta \in \epsilon(\mathbf{L}_\beta)$ with $J_\beta(j_\alpha(x_\beta) - j_\alpha(x_\alpha)) = 0$, for all $1 \leq \alpha \leq n$ and it easily follows that $J_\beta lq \subseteq j(\overline{M})$. We thus obtain that $Jlq \subseteq j(\overline{M})$, with $J = \sum_{\beta=1}^{n} J_\beta \in \bigcap_{\beta=1}^{n} \epsilon(\mathbf{L}_\beta) = \mathscr{L}$, so $Kq \subseteq j(\overline{M})$ for some $K \in \mathscr{L} \circ \mathscr{L}$, which proves the assertion. ∎

As a consequence, let us mention:

(2.4) COROLLARY. *With the previous notations, assume that* $\mathbf{L}_\alpha = \mathscr{L}_1^\alpha \ldots \mathscr{L}_{n_\alpha}^\alpha$ *satisfy the following properties*:

(1) $\mathscr{L} = \bigcap_{\alpha=1}^{n} \epsilon(\mathbf{L}_\alpha)$ *is a Gabriel filter*;

(2) $\mathscr{L} \subseteq \mathscr{L}_1^\alpha$ *for all* $1 \leq \alpha \leq n$ *and* $1 \leq i \leq n_\alpha$.

Then $\varprojlim \mathscr{Q}_K M = Q_{\mathscr{L}} M$.

Proof. Let $j: M/\sigma_{\mathscr{L}} M \to \varprojlim \mathscr{Q}_K M$, as before. Since $\mathscr{L}$ is a Gabriel filter, we obtain an exact sequence

$$0 \to M \xrightarrow{j} \varprojlim \mathscr{Q}_K M \to T \to 0,$$

where T is $\mathscr{L}$-torsion. Now,

$$\varprojlim \mathscr{Q}_K M \subseteq \bigoplus_{\alpha=1}^{n} Q_\alpha M = \bigoplus_{\alpha=1}^{n} Q_{\mathscr{L}^\alpha_{n_\alpha}} \ldots Q_{\mathscr{L}^\alpha_1} M.$$

As $\mathscr{L} \subseteq \mathscr{L}_i^\alpha$ for all $1 \leq \alpha \leq n$ and $1 \leq i \leq n_\alpha$, it follows that $Q_{\mathscr{L}}$ commutes with all of the $Q_{\mathscr{L}_i^\alpha}$, so $\varinjlim \mathscr{Q}_K M$ is $\mathscr{L}$-closed. Applying $Q_{\mathscr{L}}$ to the above exact sequence thus easily yields the result. ∎

Note that (1.2) may be recovered from (2.4) by assuming for each $1 \leq \alpha \leq n$ that $\mathbf{L}_\alpha$ is the Gabriel filter associated to some idempotent kernel functor σ_α over R. On the other hand, this result strengthens [8], where only Gabriel filters associated to Ore sets are considered.

REFERENCES

1. J. L. Bueso, P. Jara, and A. Verschoren, "Compatibility, Stability and Sheaves," Dekker, New York, 1995.
2. P. Gabriel, Des Catégories Abéliennes, *Bull. Soc. Math. France* **90** (1962), 323–448.
3. J. Golan, "Torsion Theories," Longman, Harlow/New York, 1986.
4. O. Goldman, Rings and modules of quotients, *J. Algebra* **13** (1969), 10–47.
5. J. Mulet and A. Verschoren, On compatibility, II, *Comm. Algebra* **20** (1992), 1879–1905.
6. A. L. Rosenberg, Noncommutative affine semi-schemes and schemes, *in* "Seminar on Supermanifolds," Vol. 26, pp. 1–317, Reports of the Department of Mathematics of the University of Stockholm, 1988.
7. B. Stenström, "Rings of Quotients," Springer-Verlag, Berlin, 1975.
8. F. Van Oystaeyen and L. Willaert, Grothendieck topology, coherent sheaves and Serre's theorem for schematic algebras, *J. Pure Appl. Algebra*, in press.
9. A. Verschoren, Compatibility and stability, *in* "Notas de Matematica," Vol. 3, Univ. Murcia, Murcia, 1990.
10. A. Verschoren, On compatibility, III, preprint, University of Antwerp, RUCA, 1994.