Topology Vol. 10, pp. 133–147 Pergamon Press, 1971. Printed in Great Britain.

PARTIALLY HYPERBOLIC FIXED POINTS

FLORIS TAKENS†

(*Received* 25 *July* 1970)

§1. INTRODUCTION

WE CONSIDER a C^∞-diffeomorphism $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ with $\varphi(0) = 0$. The differential $d\varphi \mid T_0(\mathbb{R}^n)$ induces a splitting $T_0(\mathbb{R}^n) = T^c \oplus T^s \oplus T^u$, where T^c, T^s and T^u are invariant under $d\varphi$ and the eigenvalues of $d\varphi$, restricted to T^c, resp. T^s, resp. T^u, are, in absolute value, $=1$, resp. <1, resp. >1. The fixed point 0 of φ is called *hyperbolic* if $\dim(T^c) = 0$. We shall consider the *partially hyperbolic* case where $\dim(T^c) \neq 0$ and $\dim(T^s \oplus T^u) \neq 0$. Such partially hyperbolic fixed points arise for example as fixed points of the time t integral of a vectorfield with a generic closed orbit with period t.

A rather general example of a diffemorphism with a partially hyperbolic fixed point is the following:

$$\begin{aligned}\varphi(x_1, \ldots, x_c, y_1, \ldots, y_s, z_1, \ldots, z_u) \\ = (\varphi_1(x_1, \ldots, x_c), \ldots, \varphi_c(x_1, \ldots, x_c), \\ \sum_i a_{1i}(x_1, \ldots, x_c) \cdot y_i, \ldots, \sum_i a_{si}(x_1, \ldots, x_c) \cdot y_i, \\ \sum_j b_{1j}(x_1, \ldots, x_c) \cdot z_j, \ldots, \sum_j b_{uj}(x_1, \ldots, x_c) \cdot z_j)\end{aligned}$$

where:

(1) $x_1, \ldots, x_c, y_1, \ldots, y_s, z_1, \ldots, z_u$ are coordinate functions on $\mathbb{R}^n$; $n = c + s + u$;

(2) all eigenvalues of $\left(\dfrac{\partial \varphi_i}{\partial x_j}\right)$ in $(x_1 = \cdots = x_c = 0)$ have absolute value one;

(3) all eigenvalues of $(a_{ij}(0, \ldots, 0))$ have absolute value <1;

(4) all eigenvalues of $(b_{ij}(0, \ldots, 0))$ have absolute value >1.

Definition 1. If a diffeomorphism φ has the above form with respect to the coordinates $(x_1, \ldots, z_u)$ we say that $(\varphi; x_1, \ldots, z_u)$ is in *standard form*.

If φ has the above form, with respect to $(x_1, \ldots, z_u)$, only in a neighbourhood of the fixed point we say that $(\varphi; x_1, \ldots, z_u)$ is *locally* in *standard form*.

Definition 2. Let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism with $\varphi(0) = 0$ and let $T_0(\mathbb{R}^n) = T^c \oplus T^s \oplus T^u$ be the induced splitting. The eigenvalues of $d\varphi \mid T^s \oplus T^u$ are denoted by $\lambda_1, \ldots, \lambda_h$. We say that φ satisfies the *Sternberg k-condition* if

† Research supported by the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

$$|\lambda_j^{-1} \cdot \lambda_1^{\nu_1} \cdot \cdots \cdot \lambda_h^{\nu_h}| \neq 1$$

for all $(j; \nu_1, \ldots, \nu_h)$ with $1 \leq j \leq h$, $\nu_i \geq 0$ and $2 \leq \Sigma \nu_i \leq k$ and

$$|\lambda_1^{\nu_2} \cdot \cdots \cdot \lambda_h^{\nu_h}| \neq 1$$

for all $(\nu_1, \ldots, \nu_h)$ with $\nu_i \geq 0$ and $2 \leq \Sigma \nu_i \leq k$.

Our main result can be stated as follows:

THEOREM. *Let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be a C^∞ diffeomorphism with $\varphi(0) = 0$. If φ satisfies the Sternberg $\alpha((\mathrm{d}\varphi)_0, k)$-condition, then there are C^k-coordinates $(x_1, \ldots, x_c, y_1, \ldots, y_s, z_1, \ldots, z_u)$ on $\mathbb{R}^n$ such that $(\varphi; x_1, \ldots, z_u)$ is locally in standard form; the function α is defined below.*

The functions α and β. We first have to introduce some notation:

Let $\lambda_1, \ldots, \lambda_h$ be the eigenvalues of $\mathrm{d}\varphi \mid T^s \oplus T^u$ and suppose $|\lambda_1| \leq |\lambda_2| \leq \cdots \leq |\lambda_s| < 1 < |\lambda_{s+1}| \leq \cdots \leq |\lambda_h|$; we define $\overline{M} = |\lambda_h|$, $\overline{N} = |\lambda_1|^{-1}$, $\overline{m} = |\lambda_{s+1}|$, $\bar{n} = |\lambda_s|^{-1}$.

The integer valued function β is the function which assigns to a pair $((\mathrm{d}\varphi)_0, k)$ the smallest integer $\beta((\mathrm{d}\varphi)_0, k)$ for which $\overline{N} \cdot \overline{M}^r \cdot \bar{n}^{r-\beta((\mathrm{d}\varphi)_0, k)} < 1$ for all $r \leq k$ ($\overline{N}$, $\overline{M}$ and $\bar{n}$ are functions of $(\mathrm{d}\varphi)_0$). Because $\bar{n} > 1$, $\beta((\mathrm{d}\varphi)_0, k)$ is always finite; also $\beta((\mathrm{d}\varphi)_0, k) > k$.

The function α assigns to a pair $((\mathrm{d}\varphi)_0, k)$ the smallest integer $\alpha((\mathrm{d}\varphi)_0, k)$ for which $\overline{M} \cdot \overline{N}^r \cdot \overline{m}^{r-\alpha((\mathrm{d}\varphi)_0, k)} < 1$ for all $r \leq \beta((\mathrm{d}\varphi)_0, k)$.

Remark. Suppose a splitting $T_0(\mathbb{R}^n) = T^c \oplus T^s \oplus T^u$ is given. Consider the set $L(T^c, T^s, T^u)$ of those linear automorphisms of $\mathbb{R}^n$ which leave T^c, T^s and T^u invariant and whose eigenvalues on T^c, T^s and T^u are, in absolute values, $= 1$, < 1 and > 1. For any given k, the set of elements $A \in L(T^c, T^s, T^u)$ which satisfy the Sternberg $\alpha((\mathrm{d}A)_0, k)$-condition is open and dense. If, for some diffeomorphism $\varphi: (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$, the derivative belongs to that open and dense subset of $L(T^c, T^s, T^u)$, then φ satisfies the assumptions of our theorem. Hence one can say that "generically" the assumptions in our theorem are satisfied.

Remark. For the case where $\dim T^c = 0$, i.e. in the hyperbolic case, we get a weakened form of Sternberg's theorem [5]. From Hartman's theorem, generalized by Hirsch, Pugh and Shub, it follows that for every partially hyperbolic fixed point there is a C^0-change of coordinates which brings it in normal form.

The main theorem will follow from the next three propositions:

PROPOSITION 1. *Let $\varphi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a C^∞ diffeomorphism satisfying the Sternberg l-condition. Then, for any integer N, there is a neighbourhood U of 0 in $\mathbb{R}^n$ and C^N coordinate functions $x_1', \ldots, x_i', y_1', \ldots, y_s', z_1', \ldots, z_u'$ such that $\varphi | U = S\varphi \mid U + R\varphi | U$ (addition with respect to $x_1', \ldots, z_n'$), where:*

1. *$(S\varphi; x_1', \ldots, z_n')$ is in standard form;*
2. *$R\varphi$, as well as its derivatives up to order l, are zero along $\{y_1' = \cdots = y_s' = z_s' = \cdots = z_u' = 0\}$;*
3. *the subspace $\{y_1' = \cdots = y_s' = 0\}$ is invariant under $R\varphi$.*

Remark. It is enough to prove the theorem for $S\varphi + R\varphi$ obtained in Proposition 1.

PROPOSITION 2. *Let $\varphi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a C^N diffeomorphism, $N \geq \alpha((d\varphi)_0, k)$. Suppose that $\varphi = S\varphi + R\varphi$ as in the conclusion of Proposition 1 (with respect to the coordinates $x_1, \ldots, z_u$) with $l = \alpha((d\varphi)_0, k)$. Then there is a $C^{\beta((d\varphi)_0, k)}$-coordinate system $(x_1', \ldots, z_u')$ on $\mathbb{R}^n$ such that, in a neighborhood of 0, we have $\varphi = S'\varphi + R'\varphi$ where:*

(1) *$(S'\varphi; x_1', \ldots, z_u')$ is in standard form;*
(2) *$R'\varphi$, as well as its derivatives up to order $\beta((d\varphi)_0, k)$, are zero along $\{y_1' = \cdots = y_s' = 0\}$.*

PROPOSITION 3. *Let $\varphi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a $C^{\beta((d\varphi)_0, k)}$-diffeomorphism. Suppose that, with respect to the coordinates $x_1, \ldots, z_u$, $\varphi = S'\varphi + R'\varphi$ as in the conclusion of Proposition 2. Then there are C^k-coordinates $x_1', \ldots, z_u'$ such that $(\varphi, x_1', \ldots, z_u')$ is locally in standard form.*

The Propositions 1, 2 and 3 are proved in Sections 2 and 3. In Section 4 we formulate the analogue of our main theorem for a partially hyperbolic zero point of a vectorfield and indicate how the proof for that case can be obtained from the proof for diffeomorphisms. In Section 5 we give an application to hyperbolic closed orbits. Our result there is that, in a neighbourhood of a generic closed orbit of a vectorfield, the vectorfield is "linear" with respect to suitable C^k-coordinates.

I would like to thank R. Thom for suggesting to me the problem treated in this paper. In conversations with C. C. Pugh and M. Shub I learned about techniques which they developed in treating similar problems [1], [2]; these techniques were basic for the proof of the Propositions 2 and 3.

§2. THE PROOF OF PROPOSITION 1

By the invariant manifold theorems [2], we can choose coordinates $x_1, \ldots, x_c, y_1, \ldots, y_s, z_1, \ldots, z_u$ on $\mathbb{R}^n$ such that the following submanifolds of $\mathbb{R}^n$ are locally invariant for φ:

$$\begin{aligned} W^s &= \{x = 0 \text{ and } z = 0\} \\ W^{cs} &= \{z = 0\} \\ W^c &= \{y = 0 \text{ and } z = 0\} \\ W^{cu} &= \{y = 0\} \\ W^u &= \{x = 0 \text{ and } y = 0\} \end{aligned}$$

(x, y, z stand for $(x_1, \ldots, x_c), (y_1, \ldots, y_s), (z_1, \ldots, z_u)$) and such that T^c, resp. T^s, resp. T^u are tangent to W^c, resp. W^s, resp. W^u. A submanifold, W, containing the origin, is *locally invariant* for φ if there is a neighbourhood U of the origin such that $\varphi(W) \cap U = W \cap U = \varphi^{-1}(W) \cap U$. All coordinate systems on $\mathbb{R}^n$ in this paragraph are supposed to have the above property.

Remark. We may assume that the above coordinate system is C^m for any $m < \infty$; here it is enough to assume that m is large compared with N.

Definition 2.1 (the spaces V_r). Let $\tilde{V}_r$ be the manifold of r-jets of embeddings $(\mathbb{R}^h, 0) \to (\mathbb{R}^n, 0)$; $h = \dim(T^s \oplus T^u)$. V_r is obtained from $\tilde{V}_r$ by the following identifications: α_1

and $\alpha_2 \in \tilde{V}_r$ are identified in V_r if there is a linear map $\alpha : (\mathbb{R}^h, 0) \to (\mathbb{R}^h, 0)$ such that $\alpha_1 \circ \alpha = \alpha_2$.

V_r is clearly a manifold. φ induces on each $\tilde{V}_r$ a transformation $\tilde{\varphi}_r : \tilde{V}_r \to \tilde{V}_r$, which assigns to the jet of $\alpha : (\mathbb{R}^h, 0) \to (\mathbb{R}^n, 0)$ the jet of $\varphi \circ \alpha$. Since $\tilde{\varphi}_r$ commutes with the identifications, there is an induced transformation $\varphi_r : V_r \to V_r$. There is also a natural projection $\pi_r : V_r \to V_{r-1}$; the following diagram commutes:

$$\begin{tikzcd} V_r \arrow[r, "\varphi_r"] \arrow[d, "\pi_r"] & V_r \arrow[d, "\pi_r"] \\ V_{r-1} \arrow[r, "\varphi_{r-1}"] & V_{r-1} \end{tikzcd}$$

LEMMA 2.2. *Under the assumption (as in Proposition* 1) *that φ satisfies the Sternberg l-condition, there is for each $1 \leq r \leq l$ a unique element $[\alpha]_r \in V_r$ such that:*

(i) *$[\alpha]_r$ can be represented by an embedding $(\mathbb{R}^h, 0) \to (\mathbb{R}^n, 0)$ with image tangent to $T^s \oplus T^u$;*

(ii) *$[\alpha]_r$ is a hyperbolic fixed point of φ_r.*

Proof. We shall prove the lemma by induction on r; first we do the induction step, then we give the proof for $r = 1$.

The induction step. Take $l \geq r > 1$ and assume that there is a unique $[\alpha]_{r-1} \in V_{r-1}$ satisfying the conditions (i) and (ii). If there is an element $[\alpha]_r \in V_r$ satisfying the same two conditions, it must lie in $\pi_r^{-1}([\alpha]_{r-1})$. We may, and do, assume that our coordinates $(x_1, \ldots, z_u)$ in $\mathbb{R}^n$ are such that $[\alpha]_{r-1}$ can be represented by the linear embedding $\vartheta : (\mathbb{R}^h, 0) \to (\mathbb{R}^n, 0)$ given by $\vartheta(w_1, \ldots, w_h) = (0, \ldots, 0, w_1, \ldots, w_h)$. Using this coordinate system, every element in $\pi_r^{-1}([\alpha]_{r-1})$ can be uniquely represented by a map $(\mathbb{R}^h, 0) \to (\mathbb{R}^n, 0)$ of the form

$$(w_1, \ldots, w_h) \mapsto (p_1, \ldots p_c, w_1 + q_1, \ldots, w_h + q_h)$$

where $p_1, \ldots, p_c$ and $q_1, \ldots, q_h$ are homogeneous polynomials of degree r in $w_1, \ldots, w_h$. Because the set of homogeneous polynomials is a vector space we can give $\pi_r^{-1}([\alpha]_{r-1})$ the structure of a vector space. We take $[\vartheta]_r$ as the origin in $\pi_r^{-1}([\alpha]_{r-1})$ (ϑ is the linear embedding).

φ_r maps $\pi_r^{-1}([\alpha]_{r-1})$ to itself; so we can define $\bar{\varphi}_r : \pi_r^{-1}([\alpha]_{r-1}) \to \pi_r^{-1}([\alpha]_{r-1})$ by $\bar{\varphi}_r[\beta] = \varphi_r[\beta] - \varphi_r[\vartheta]_r$ (" $-$ " is defined by means of the above vector space structure in $\pi_r^{-1}([\alpha]_{r-1})$). We now only have *to show that $\bar{\varphi}_r$ is linear and hyperbolic*, because from that it follows immediately that φ_r has exactly one hyperbolic fixed point in $\pi_r^{-1}([\alpha]_{r-1})$.

First we introduce some notation: for $[\beta] \in \pi_r^{-1}([\alpha]_{r-1})$ the corresponding map of the form $(w_1, \ldots, w_h) \mapsto (p_1, \ldots, p_c, w_1 + q_1, \ldots, w_h + q_h)$ is denoted by β (p_i and q_j are homogeneous polynomials of degree r). The linear map $A : (\mathbb{R}^h, 0) \to (\mathbb{R}^h, 0)$ is such that the $(r-1)$-jet of $\varphi \circ \vartheta \circ A$ and ϑ are equal.

For any $[\beta] \in \pi_r^{-1}([\alpha]_{r-1})$ the jet $\varphi_r[\beta]$ is represented by $\varphi \circ \beta \circ A$; $\bar{\varphi}_r[\beta]$ is represented by $\varphi \circ \beta \circ A - \varphi \circ \vartheta \circ A + \vartheta$ ("$+$ and $-$" refer here to the vector space structure in $\mathbb{R}^n$ corresponding to the coordinates $x_1, \ldots, z_u$). Because the $(r$-$1)$-jets of $\beta \circ A$ and $\vartheta \circ A$ are equal, the r-jet of $\varphi \circ \beta \circ A - \varphi \circ \vartheta \circ A + \vartheta$ depends linearly on the r-jet of β, so $\bar{\varphi}_r$ is linear. We now come to the hyperbolicity:

(*a*) *Diagonal case*. We first assume that, with respect to our coordinate system $x_1, \ldots, z_u$, $(d\varphi)_0$ is in diagonal form. This means that φ has the form

$$\varphi(x_1, \ldots, z_u) = (\mu_1 z_1, \ldots, \mu_c z_c, \lambda_1 x_1, \ldots, \lambda_h z_u) + \text{terms of order} \geq 2,$$

where $|\mu_i| = 1$ and $\lambda_1, \ldots, \lambda_h$ are the hyperbolic eigenvalues of $(d\varphi)_0$ which occur in the Sternberg condition. In this case $A : (\mathbb{R}^h, 0) \to (\mathbb{R}^h, 0)$ is given by $A(w_1, \ldots, w_h) = (\lambda_1^{-1} w_1, \ldots, \lambda_h^{-1} w_h)$. $\bar{\varphi}_r$ can now be computed; by straightforward calculation it follows that $\bar{\varphi}_r$ is in diagonal form, i.e. we can find a basis of $\pi_r^{-1}([\alpha]_{r-1})$ consisting of eigenvectors of $\bar{\varphi}_r$. The elements of this basis are denoted by

$$[i; i_1, \ldots, i_h] \quad i = 1, \ldots, c \quad i_1, \ldots, i_v \geq 0 \quad \sum_{v=1}^{h} i_v = r$$

and

$$\{j; i_1, \ldots, i_h\} \quad j = 1, \ldots, h \quad i_1, \ldots, i_v \geq 0 \quad \sum_{v=1}^{h} i_v = r.$$

$[i; i_1, \ldots, i_h]$ is represented by $(w_1, \ldots, w_h) \mapsto (p_1, \ldots, p_c, w_1, \ldots, w_h)$ with $p_i = w_1^{i_1} \cdot \cdots \cdot w_h^{i_h}$ and $p_{i'} = 0$ for $i' \neq i$. $\{j; i_1, \ldots, i_h\}$ is represented by $(w_1, \ldots, w_h) \mapsto (0, \ldots, 0, w_1 + q_1, \ldots, w_h + q_h)$ with $q_j = w_1^{i_1}, \ldots, w_h^{i_h}$ and $q_{j'} = 0$ for $j' \neq j$. The eigenvalue corresponding to $[i; i_1, \ldots, i_h]$ is $\mu_i \cdot \lambda_1^{-i_1} \cdot \cdots \cdot \lambda_h^{-i_h}$, the eigenvalue corresponding to $\{j; i_1, \ldots, i_h\}$ is $\lambda_j \cdot \lambda_1^{-i_1} \cdot \cdots \cdot \lambda_1^{-i_h}$. None of these eigenvalues has absolute value one because of the Sternberg condition, hence $\bar{\varphi}_r$ is hyperbolic.

(*b*) *General case*. We reduce the general case to the diagonal case by "complexifying". We first remark that almost everything which has been done in the proof of the induction step up to now also makes sense if we replace the reals everywhere by the complex numbers (because we mainly worked with polynomials). The only thing which must be changed is φ: we replace it by a polynomial map which has the right r-jet. The eigenvalues of $\bar{\varphi}_k$ are the same for the two cases (real and complex), so it is enough to compute them for the complex case. By the Jordan normal form theorem, there are, for every $\varepsilon > 0$, linear coordinate transformations

$$\bar{x}_i = \sum_{j=1}^{c} X_{ij} x_j, \quad \bar{y}_i = \sum_{j=1}^{s} Y_{ij} y_j, \quad \bar{z}_i = \sum_{j=1}^{u} Z_{ij} z_j$$

(X_{ij} Y_{ij} and $Z_{ij} \in \mathbb{C}$), such that, with respect to $\bar{x}_1, \ldots, \bar{z}_u$, $(d\varphi)_0$ is in Jordan normal form with $\mu_1, \ldots, \mu_c, \lambda_1, \ldots, \lambda_h$ on the diagonal and with off diagonal terms 0 and ε.

The eigenvalues of $\bar{\varphi}_r$ are independent of such coordinate changes, so they are independent of the above ε; so we may assume $\varepsilon = 0$ (the eigenvalues of $\bar{\varphi}_k$ depend continuously on $(d\varphi)_0$). Now we are back in the "diagonal case" for which we proved hyperbolicity.

Proof for $r = 1$. There is only one element $[\alpha]_1 \in V_1$ which can be represented by an embedding with image tangent to $T^s \oplus T^u$. The hyperbolicity of $[\alpha]_1$ as a fixed point of φ_1 is the only thing we have to prove.

The elements of V_1 near $[\alpha]_1$ can be represented in a unique way by the following type of linear embeddings:

$$(w_1, \ldots, w_h) \mapsto (p_1, \ldots, p_c, w_1, \ldots, w_h)$$

where $p_1, \ldots, p_c$ are linear functions of $w_1, \ldots, w_h$. The hyperbolicity now follows just as in the proof of the induction step.

Definition 2.3 (*the space* V_r^*). Let $\tilde{V}_r^*$ be the manifold of r-jets of embeddings $(\mathbb{R}^h, 0) \to (\mathbb{R}^n, W^c)$; h is the dimension of $T^s \oplus T^u$. V_r^* is obtained from $\tilde{V}_r^*$ by the identifications: α_1 and $\alpha_2 \in \tilde{V}_r^*$ are identified in V_r^* if there is a linear map $\alpha : (\mathbb{R}^h, 0) \to (\mathbb{R}^h, 0)$ such that $\alpha_1 \circ \alpha = \alpha_2$.

Remarks. It is clear that $V_r \subset V_r^*$; there is a natural projection $p : V_r^* \to W^c$ which assigns to each element $\alpha \in V_r^*$ the "image of 0". $p^{-1}(0) = V_r$.

Because W^c is locally invariant, the map $\varphi_r : V_r \to V_r$ extends to a map φ_r^* defined on a neighbourhood of V_r in V_r^*; $\varphi_r^*(\alpha)$ is the jet of $\varphi \circ \alpha$.

Construction of the required coordinate system $(x_1', \ldots, z_u')$. By Lemma 2.2 we have a hyperbolic fixed point $[\alpha]_l$ of φ_l in V_l. $[\alpha]_l$ is of course also a fixed point for φ_l^*, but not a hyperbolic fixed point (the set of eigenvalues of $(d\varphi_l^*)_{[\alpha]l}$ is the union of the set of eigenvalues of $(d\varphi_l)_{[\alpha]_l}$ and the set of eigenvalues of $d(\varphi \mid W^c)_0$). Let $W^{*c} \subset V_l^*$ be a center manifold for $[\alpha]_l$ in V_l^*. We may assume that W^{*c} is as differentiable as φ_l^*; so we may assume that the class of differentiability of W^{*c} is large, compared with N. $p : V_r^* \to W^c$ restricts to a map $p_c : W^{*c} \to W^c$ which is, restricted to a small neighbourhood of $[\alpha]_l$, a diffeomorphism. This means that for every P in W^c, sufficiently close to the origin, we have a class of l-jets of embeddings $(\mathbb{R}^h, 0) \to (\mathbb{R}^n, P)$ representing $p_c^{-1}(P)$. Now we choose the coordinate system $x_1', \ldots, z_u'$ such that:

(1) It has the property described in the introduction of this paragraph (in particular $\{y_1' = \cdots = y_s' = 0\}$ is locally invariant under φ);
(2) for each $P \in W^c$, close enough to the origin, $p_c^{-1}(P)$ is represented by the affine embedding (affine with respect to $x_1', \ldots, z_u'$):

$$(y_1', \ldots, y_s', z_1', \ldots, z_u') \mapsto (x_1', \ldots, x_c', y_1', \ldots, y_s', z_1', \ldots, z_u')$$

where

$$P = (x_1', \ldots, x_c', 0, \ldots, 0).$$

Because W^{*c} is very differentiable we may assume that $x_1', \ldots, z_u'$ is C^N.

By the local invariance of W^{*c} it follows that, for some neighborhood U_1 of the origin, $\varphi \mid U_1 = S\varphi \mid U_1 + R\varphi \mid U_1$ where $(S\varphi; x_1', \ldots, z_u')$ is in standard form, and $R\varphi$, as well as its derivatives up to order l, are zero along W^c. This proves Proposition 1.

§3. THE PROOF OF PROPOSITIONS 2 AND 3

We assume that the map φ, the coordinates $x_1, \ldots, z_u$ and the "splitting $\varphi = S\varphi + R\varphi$" are as in the assumptions of Proposition 2.

Definition 3.1 (*the transformation* Φ_r *on* $J^r(\mathbb{R}^n, \mathbb{R}^n)$. The elements of $J^r(\mathbb{R}^n, \mathbb{R}^n)$ can be represented by pairs $[p, \sigma]_r$, where $p \in \mathbb{R}^n$ and σ is a C^r map from a neighbourhood of p to $\mathbb{R}^n$. The transformation $\Phi_r: J^r(\mathbb{R}^n, \mathbb{R}^n) \to J^r(\mathbb{R}^n, \mathbb{R}^n)$ maps the jet, represented by $[p, \sigma]_r$ to the jet represented by $[S\varphi(p), \varphi \circ \sigma \circ (S\varphi)^{-1}]_r$, (we assume in this definition that φ is C^r).

Remark 3.2. $J^r(\mathbb{R}^n, \mathbb{R}^n)$ is fibered over $J^{r-1}(\mathbb{R}^n, \mathbb{R}^n)$. $\pi_r : J^r(\mathbb{R}^n, \mathbb{R}^n) \to J^{r-1}(\mathbb{R}^n, \mathbb{R}^n)$ is the projection. We clearly have $\Phi_{r-1} \circ \pi_r = \pi_r \circ \Phi_r$. Each fiber of π_r is an affine space; if $\alpha \in J^{r-1}(\mathbb{R}^n, \mathbb{R}^n)$ then the map of $\pi_r^{-1}(\alpha)$ to $\pi_r^{-1}(\Phi_{r-1}(\alpha))$, induced by Φ_r, is an affine map (see also the proof of Lemma 2.2). One also has a projection $\pi^0 : J^0(\mathbb{R}^n, \mathbb{R}^n) \to \mathbb{R}^n$ which assigns to $[p, \sigma]_0$ the point p (Φ_0 is not an affine map on fibers of π_0).

Definition 3.3 (*the fiber metrics*). We want to define in each fiber of π_r $r \geq 1$ a metric. For this purpose we first choose a (Euclidean) metric on $\mathbb{R}^n$ (this metric will be specified later). The distance $\rho_r([p, \sigma_1]_r, [p, \sigma_2]_r)$ between two jets, represented by $[p, \sigma_1]_r$ and $[p, \sigma_2]_r$, in the same fiber of π_r is then defined as follows:

$$(p, \sigma_1, \sigma_2) \text{ defines a map } \hat{\sigma}_{1,2} : T_p(\mathbb{R}^n) \to T_{\sigma(p)}(\mathbb{R}^n)$$

$(\sigma(p) = \sigma_1(p) = \sigma_2(p))$:

$$\hat{\sigma}_{1,2}(X) = \mathrm{Exp}_{\sigma(p)}^{-1} \circ \sigma_1 \circ \mathrm{Exp}_p(X) - \mathrm{Exp}_{\sigma(p)}^{-1} \circ \sigma_2 \circ \mathrm{Exp}_p(X)$$

for $X \in T_p(\mathbb{R}^n)$, where $\mathrm{Exp}_q : T_q(\mathbb{R}^n) \to \mathbb{R}^n$ is the usual exponential map. We take

$$\rho_r([p, \sigma_1]_r, [p, \sigma_2]_r) = \lim_{a \to 0}\left(\sup_{|X|=a}\left(\frac{|\hat{\sigma}_{1,2}(X)|}{|X|^r} \right)\right)$$

where $|\ \ |$ is the norm of a vector with respect to our Euclidean metric on $\mathbb{R}^n$. The limit is finite because $[p, \sigma_1]_r$ and $[p, \sigma_2]_r$ are in the same fiber of π_r and hence the $(r-1)$-jet of $\hat{\sigma}_{1,2}$ is zero. The limit is determined by the r-jet of $\hat{\sigma}_{1,2}$ and hence determined by the r-jets of σ_1 and σ_2 in p. The proof that ρ_r is really a metric (triangle inequality etc.) is left to the reader.

The metric ρ_0 on fibers of π_0 is defined by $\rho_0([p, \sigma_1]_0, [p, \sigma_2]_0) = \rho(p(\sigma_1), \rho(\sigma_2))$; ρ is the distance in $\mathbb{R}^n$ defined by the Euclidean metric.

LEMMA 3.4. *Let* $[p, \sigma_1]_r$ *and* $[p, \sigma_2]_r$ *represent two jets which are in the same fiber of* π_r, $r \geq 1$, *and let* $\sigma_1(p) = \sigma_2(p) = q$. *Then* $\rho_r(\Phi_r[p, \sigma_1]_r, \Phi_r[p, \sigma_2]_r \leq \| \mathrm{d}((S\varphi)^{-1})_{S\varphi(p)} \|^r \cdot \|\mathrm{d}\varphi_q\| \cdot \rho_r([p, \sigma_1]_r, [p, \sigma_2]_r)$, *where*

$$\|\mathrm{d}\varphi_q\| = \sup_{\substack{|X|=1 \\ X \in T_q(\mathbb{R}^n)}} |\mathrm{d}\varphi(X)|.$$

LEMMA 3.4′. *Let* $[p, \sigma_1]_0$ *and* $[p, \sigma_2]_0$ *be given, then* $\rho_0(\Phi_0[p, \sigma_1]_0, \Phi_0[p, \sigma_2]_0) \leq \sup_{t\in[0,1]} \|(\mathrm{d}\varphi)_{q_t}\| \cdot \rho_0([p, \sigma_1]_0, [p, \sigma_2]_0)$, *where* $q_t = t \cdot \sigma_1(p) + (1-t) \cdot \sigma_2(p)$.

Proof. Follows immediately from the definitions.

The metric on $\mathbb{R}^n$. We define for $p \in W^{cu}$ the following numbers:

$$\tilde{M}_p = \|(\mathrm{d}\varphi)_p\|$$
$$\tilde{N}_p = \|\mathrm{d}((S\varphi)^{-1})_p\|$$

and

$$\tilde{m}_p = \begin{cases} \text{for } p \notin W^c = \rho(S\varphi(p), W^c) \cdot (\rho(p, W^c))^{-1} \\ \text{for } p \in W^c = \lim_{\substack{p' \in W^{cu} \\ p' \notin W^c \\ p' \to p}} \inf(\tilde{m}_{p'}). \end{cases}$$

(W^{cu}, W^c etc. have here the same meaning as in Section 2). They are, for p = origin, closely related with the invariants $\overline{M}$, $\overline{N}$ and $\overline{m}$, defined in Section 1, which depend only on the eigenvalues of $(\mathrm{d}\varphi)_0$. In fact, for every $\varepsilon > 0$, we can choose our Euclidean metric on $\mathbb{R}^n$ so that $\tilde{M}_0 < \overline{M} + \varepsilon$, $\tilde{N}_0 < \overline{N} + \varepsilon$ and $\tilde{m}_0 > \overline{m} - \varepsilon$. According to the definition of $\alpha((\mathrm{d}\varphi)_0, k)$ and $\beta((\mathrm{d}\varphi)_0, k)$ (Section 1) we know that $\overline{M} \cdot \overline{N}^r \cdot \overline{m}^{r-\alpha((\mathrm{d}\varphi)_0,k)} < 1$ for all $r \leq \beta((\mathrm{d}\varphi)_0, k)$. Hence it follows that we can choose a metric on $\mathbb{R}^n$ such that $\tilde{M}_0 \cdot \tilde{N}_{0r} \cdot \tilde{m}^{r-\alpha((\mathrm{d}\varphi)_0,k)} < \mu$ for some fixed $\mu < 1$ and all $r \leq \beta((\mathrm{d}\varphi)_0, k)$. From now on we assume that our metric on $\mathbb{R}^n$ is fixed and is such that the above inequalities are satisfied; the fiber metrics (Definition 3.3) are also assumed to be derived from this metric on $\mathbb{R}^n$. From now on μ will always stand for the fixed constant in the above inequality.

Modification of φ. Now we modify our diffeomorphism φ (outside a neighborhood of 0). Take $\kappa_1 : \{y_1 = \cdots = y_s = z_1 = \cdots = z_u = 0\} = W^c \to \mathbb{R}$ a non-negative partition function, 1 on a neighborhood of the origin and zero at distance ≥ 1 from the origin. $\kappa_b : W^c \to \mathbb{R}$ is defined, for $b > 0$, by $\kappa_b(x_1, \ldots, x_c) = \kappa_1(x_1/b, \ldots, x_c/b)$. We modify $\varphi = S\varphi + R\varphi$ as follows:

We replace $S\varphi$ by $\bar{\kappa}_b \cdot S\varphi + (1 - \bar{\kappa}_b) \cdot L\varphi$ and we replace $R\varphi$ by $\bar{\kappa}_b \cdot R\varphi$, where

$$\bar{\kappa}_b(x_1, \ldots, x_c, y_1, \ldots, y_s, z_1, \ldots, z_u) = \kappa_b(x_1, \ldots, x_c)$$

and $L\varphi$ is the linear (with respect to $x_1, \ldots, z_u$) map $L_\varphi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ with $\mathrm{d}(L\varphi)_0 = (\mathrm{d}\varphi)_0$.

It is clear that there is a small neighborhood of 0 in which φ is not changed. Also the new φ satisfies the conditions in the assumptions of Proposition 2. We may, and do, assume that b is so small that for any three points p, q and v in W^c and $r \leq \beta((\mathrm{d}\varphi)_0, k)$ we have $\tilde{M}_p \cdot \tilde{N}_q^r \cdot \tilde{m}_v^{r-\alpha((\mathrm{d}\varphi)_0,k)} < \mu$ where $\mu < 1$ is the constant which occurred in the discussions on the metric on $\mathbb{R}^n$. From now on φ, $S\varphi$ and $R\varphi$ will refer to the functions after the above modification.

We define $\tilde{K}_\delta \subset W^{cu}$ to be the closed δ neighbourhood of W^c in W^{cu}. We can choose δ so small that for all, p, q and $v \in \tilde{K}_\delta$ and $r \leq \beta((\mathrm{d}\varphi)_0, k)$ we have $\tilde{M}_p \cdot \tilde{N}_q \cdot \tilde{m}^{r-\alpha((\mathrm{d}\varphi)_0,k)} < \mu$; μ as above. From now δ as above is fixed.

LEMMA 3.5. *Let $[p, \sigma_1]_r$ and $[p, \sigma_2]_r$ be two jets in the same fiber of π_r with p, $S\varphi(p)$, $\sigma_1(p)$, $\sigma_2(p) \in \tilde{K}_\delta$ but $p \notin W^c$, and $r \leq \beta((\mathrm{d}\varphi)_0, k)$; then*:

$$\rho_r(\Phi_r[p, \sigma_1]_r, \Phi_r[p, \sigma_2]_r) \cdot (\rho(S\varphi(p), W^c))^{r-\alpha((\mathrm{d}\varphi)_0, k)} < \mu \cdot \rho_r([p, \sigma_1], [p, \sigma_2]) \cdot \rho(p, W^c)^{r-\alpha((\mathrm{d}\varphi_0), k)}.$$

Proof. By the definition of $\tilde{m}_p$ we have $\rho(S\varphi(p), W^c) = \tilde{m}_p \cdot \rho(p, W^c)$ and by Lemmas 3.4 and 3.4′ we have

$$\rho_r(\Phi_r[p,\sigma_1]_r, \Phi[p,\sigma_2]_r) \le \sup_{\substack{v\in\tilde K_\delta\\ q\in\tilde K_\delta}} \tilde M_v \cdot \tilde N_q{}^r \cdot \rho_r([p,\sigma_1]_r, [p,\sigma_2]_r).$$

This, together with the above property of $\tilde K_\delta$, proves the lemma.

We are now in a position where we can prove Proposition 2 by giving a convergent sequence of "jets of coordinates along W^{cu}" the limit of which is invariant under Φ. Consider the map $\vartheta_r : W^{cu} \to J^r((\mathbb{R}^n, W^{cu}), (\mathbb{R}^n, W^{cu}))$ defined by $\vartheta_r(p) = [p, \text{identity}]_r$ for all $p \in W^{cu} \cdot \vartheta_r$ is a cross-section of the bundle

$$\Pi_r = \pi_0 \circ \cdots \circ \pi_{r-1} \circ \pi_r : J^r((\mathbb{R}^n, W^{cu}), (\mathbb{R}^n, W^{cu})) \to W^{cu}.$$

To a section of Π_r we can apply the transformation Φ_r as follows: If $\kappa : W^{cu} \to J^r((\mathbb{R}^n, W^{cu}), (\mathbb{R}^n, W^{cu}))$ is a section of Π_r then $\Phi_r\kappa$ is the section which assigns to $p \in W^{cu}$ the jet $(\Phi_r\kappa)p = \Phi_r(\kappa(q))$, where $q = (S\varphi)^{-1}(p)$.

We shall prove the following Proposition 2′ and then derive Proposition 2 from it.

PROPOSITION 2′. *The sequence of sections of* Π_r, *defined by* $\{(\Phi_r)^i\vartheta_r\}_{i=1}^{\infty}$ *converges to a continuous section of* Π_r *for* $r \le \beta((\mathrm{d}\varphi)_0, k)$.

Proof. We shall prove that for some fixed $0 < \delta' < \delta$ the above sections, restricted to $\tilde K_{\delta'}$, converge (under the above hypothesis). This is enough because the "unrestricted limit" equals the iterated "restricted limit" (since if we apply $(\Phi_r)^i$ to a section over $\tilde K_{\delta'}$, we obtain a section over $(S\varphi)^i(\tilde K_{\delta'})$ and $\lim_{i\to\infty}((S\varphi)^i\tilde K_{\delta'}) = W^{cu}$).

Let

$$C = \sup_{\substack{p\in\tilde K_\delta\\ p\notin W^c}} \frac{\rho_0((\Phi_0\,\vartheta_0)(p), \vartheta_0(p))}{\rho(p, W^c)^{\alpha((\mathrm{d}\varphi)_0,k)}}.$$

C is finite because of the definition of Φ_0 and the fact that $R\varphi$ is zero up to order $\alpha((\mathrm{d}\varphi)_0, k)$ along W^c and has compact support. Take $D > C/1-\mu$ (this is the same μ which occurred in Lemma 3.5) and take $0 < \delta' < \delta$ such that $\delta' + D\cdot(\delta')^{\alpha((\mathrm{d}\varphi)_0,k)} < \delta$.

Definition 3.6 (THE SPACE $\mathscr{F}_0$). $\mathscr{F}_0$ is the space of continuous sections κ *of* Π_0 defined on $\tilde K_{\delta'}$ for which $\rho_0(\kappa(p), \vartheta_0(p) \le D\cdot(\rho(p, W^c))^{\alpha((\mathrm{d}\varphi)_0,k)}$, for all $p \in \tilde K_{\delta'}$. The topology on $\mathscr{F}_0$ is given by the following metric: For $\kappa_1, \kappa_2 \in \mathscr{F}_0$, $\tilde\rho_0(\kappa_1,\kappa_2)$ is the smallest number such that $\rho_0(\kappa_1(p), \kappa_2(p)) \le \tilde\rho_0(\kappa_1,\kappa_2)\cdot(\rho(p, W^c)^{\alpha((\mathrm{d}\varphi)_0,k)}$ for all $p \in \tilde K_{\delta'}$.

LEMMA 3.7. Φ_0 *induces a map from* $\mathscr{F}_0$ *into itself which is a contraction with respect to the metric* $\tilde\rho_0$.

Proof. The map from $\mathscr{F}_0$ to itself, induced by Φ_0 is the following: Let $\kappa \in \mathscr{F}_0$, then $\Phi_0\kappa$ is a section of Π_0 defined over $(S\varphi)(\tilde K_{\delta'}) \supset \tilde K_{\delta'}$; in order to get again an element of $\mathscr{F}_0$ we restrict $\Phi_0\kappa$ to $\tilde K^{\delta'}$. We call this induced map also Φ_0. Now we show that Φ_0 maps $\mathscr{F}_0$ into itself.

Let $\kappa \in \mathscr{F}_0$ and $p \in \tilde K_{\delta'}$ such that $S\varphi(p) \in \tilde K_{\delta'}$. Then $\rho_0(\kappa(p), \vartheta_0(p)) = \rho(\kappa(p), p) \le D\cdot(\delta')^{\alpha((\mathrm{d}\varphi)_0,k)}$ so $\rho(\kappa(p), W^c) < \delta$ (we used here $\kappa(p)$ also for the image point of p under the jet $\kappa(p)$). Therefore we can apply Lemma 3.5 and obtain:

$$\rho_0((\Phi_0\kappa)(q), (\Phi_0\vartheta_0(q)) \cdot (\rho(q, W^c))^{-\alpha((d\varphi)_0, k)} < \mu \cdot \rho_0(\kappa(p), \vartheta_0(p)) \cdot (\rho(p, W^c))^{-\alpha((d\varphi)_0, k)} \leq \mu \cdot D,$$

where $q = S\varphi(p)$; we also have

$$\rho_0((\Phi_0\,\vartheta_0)(q), \vartheta_0(q)) \cdot (\rho(q, W^c))^{-\alpha((d\varphi)_0, k)} < C$$

(this is the same C we defined in the beginning of the proof of Proposition 2′) so

$$\rho_0((\Phi_0\kappa)(q), \vartheta_0(q)) \cdot (\rho(q, W^c))^{-\alpha((d\varphi)_0, k)} < C + \mu \cdot D < D.$$

This shows that Φ_0 maps $\mathscr{F}_0$ into itself.

The fact that Φ_0 is contracting follows from Lemma 3.5.

Remark 3.8. Because Φ_0 is a contraction on $\mathscr{F}_0$, there is a unique $\kappa_0 \in \mathscr{F}_0$ such that for any $\kappa \in \mathscr{F}_0$, $\lim_{i\to\infty} (\Phi_0)^i\kappa = \kappa_0$; in particular $\vartheta_0 \,|\, \tilde{K}_{\delta'}$, and hence ϑ_0, converges to a continuous section of Π_0.

Definition 3.9 (*the spaces* $\mathscr{F}_i$). We first define sections $S_r : J^{r-1}(\mathbb{R}^n, \mathbb{R}^n) \to J^r(\mathbb{R}^n, \mathbb{R}^n)$ for $r \geq 1$. Let $[p, \sigma]_{r-1}$ represent an element of $J^{r-1}(\mathbb{R}^n, \mathbb{R}^n)$. If we require that, for $r \geq 2$, σ be a polynomial map of degree $\leq r - 1$, and, for $r = 1$, σ be an affine translation, then the map σ is, for the given jet, unique. We then define $S_r([p, \sigma]_{r-1})$ to be the r-jet, at p, represented by this unique map σ.

We now define the space $\mathscr{F}_i$ of continuous sections $\tilde{K}_{\delta'} \to J^i((\mathbb{R}^n, W^{cu}), (\mathbb{R}^n, W^{cu}))$ by induction ($\mathscr{F}_0$ is already defined (Definition 3.6)): $\mathscr{F}_i$ is the set of those continuous sections κ such that the corresponding $(i-1)$-jet section $\pi_i \circ \kappa$ is in $\mathscr{F}_{i-1}$ and such that there is a constant $A(\kappa)$ such that for all $p \in \tilde{K}_{\delta'}$

$$\rho_i(\kappa(p), (S_i \circ \pi_i \circ \kappa)(p)) \leq A(\kappa) \cdot (\rho(p, W^c))^{\alpha((d\varphi)_0, k)-i}.$$

The natural projection $\mathscr{F}_i \to \mathscr{F}_{i-1}$ is denoted by Π_i. In each fibre of Π_i we define the following metric $\tilde{\rho}_i$: If κ_1 and κ_2 are in the same fibre of Π_i then $\tilde{\rho}_i(\kappa_1, \kappa_2)$ is the smallest number, such that for any $p \in \tilde{K}_{\delta'}$

$$\rho_i(\kappa_1(p), \kappa_2(p)) \leq \tilde{\rho}_i(\kappa_1, \kappa_2) \cdot \rho(p, W^c)^{\alpha((d\varphi)_0, k)-i}.$$

LEMMA 3.10. *For each $i \leq \beta((d\varphi)_0, k)$, Φ_i induces a map from $\mathscr{F}_i$ into itself; for any $\kappa \in \mathscr{F}_{i-1}$, the map induced by Φ_i from $\tilde{\pi}_i^{-1}(\kappa)$ to $\tilde{\pi}_i^{-1}(\Phi_{i-1}(\kappa))$ is a contraction with respect to the fiber metric $\tilde{\rho}_i$.*

Proof. We first show that Φ_i induces a map from $\mathscr{F}_i$ into itself. We know that this is true for $i = 0$ (Lemma 3.7) so we can apply induction. Suppose that Φ_{i-1} induces a map from $\mathscr{F}_{i-1}$ into itself. Take $\kappa \in \mathscr{F}_i$. Then:

(i) $\pi_i \circ \kappa \in \mathscr{F}_{i-1}$

(ii) $\rho_i(\kappa(p), (S_i \circ \pi_i \circ \kappa)(p)) \leq A(\kappa) \cdot (\rho(p, W^c))^{\alpha((d\varphi)_0, k)-i}$ for some $A(\kappa)$ and all $p \in \tilde{K}_{\delta'}$.

We have to show that $\Phi_i\kappa$, restricted to $\tilde{K}_{\delta'}$, also satisfies the above two conditions. From the induction hypothesis it follows that $\pi_i \circ (\Phi_i\kappa) = \Phi_{i-1}(\pi_i \circ \kappa) \in \mathscr{F}_{i-1}$, so $\Phi_i\kappa$ satisfies (i).

To show that $\Phi_i\kappa$ satisfies condition (ii) we first observe that for $p \in \tilde{K}_{\delta'}$, sufficiently far away from the origin we have $(\Phi_i(S_i \circ \pi_i \circ \kappa))(p) = (S_i \circ \pi_i \circ (\Phi_i \kappa))(p)$ because then φ and $S\varphi$ are linear (see "modification of φ"). This means that, for p far enough from the origin,

$$\rho_i(\Phi_i \kappa(p), (S_i \circ \pi_i \circ (\Phi_i \kappa))(p) \leq \mu \cdot A(\kappa) \cdot (\rho(p, W^c))^{\alpha((\mathrm{d}\varphi)_0, k) - i}$$

(see Lemma 3.5).

Next we show that the images of ϑ_i and $\Phi_i \kappa$, as submanifolds of $J^r((\mathbb{R}^n, W^{cu}), (\mathbb{R}^n, W^{cu}))$ have, along $\vartheta_i(W^c) = \kappa(W^c)$ contact of order $\alpha((\mathrm{d}\varphi)_0, k) - i$. Because $\kappa \in \mathscr{F}_i$, $\mathrm{Im}(\kappa)$ and $\mathrm{Im}(\vartheta_i)$ have, along $\vartheta_i(W^c)$, contact of order $\alpha((\mathrm{d}\varphi)_0, k) - i$ (see (ii) above). Hence $\mathrm{Im}(\Phi_i \vartheta_i)$ and $\mathrm{Im}(\Phi_i \kappa)$ have contact of order $\alpha((\mathrm{d}\varphi)_0, k) - i$. Because $R\varphi$ is zero up to order $\alpha((\mathrm{d}\varphi)_0, k)$ along W^c, $\mathrm{Im}(\vartheta_i)$ and $\mathrm{Im}(\Phi_i \vartheta_i)$ have contact of order $((\mathrm{d}\varphi)_0, k) - i$ along $\vartheta_i(W^c)$. Consequently $\mathrm{Im}(\vartheta_i)$ and $\mathrm{Im}(\Phi_i \kappa)$ have contact of order $((\mathrm{d}\varphi)_0, k)$ along $\vartheta_i(W^c)$.

It follows that for any compact $L \subset \tilde{K}_{\delta'}$ there is a constant $A(\Phi_i \kappa, L)$ such that the inequality in condition (ii) is satisfied for all $p \in L$, with $\Phi_i \kappa$ instead of κ and $A(\Phi_i \kappa, L)$ instead of $A(\kappa)$. Combining this with the observation about "far away" points we see that $\Phi_i \kappa$ satisfies condition (ii). This proves that Φ_i induces a map from $\mathscr{F}_i$ into itself.

The fact that the map is contracting on fibers follows from Lemma (3.5).

LEMMA 3.11. *For each $i \leq \beta((\mathrm{d}\varphi)_0, k)$ there is a $\kappa_i \in \mathscr{F}_i$ such that for any*

$$\kappa \in \mathscr{F}_i, \quad \lim_{j \to \infty} (\Phi_i)^j \kappa = \kappa_i .$$

Proof. For $i = 0$ the lemma coincides with Remark 3.8. Suppose the lemma is true for $i - 1 < \beta((\mathrm{d}\varphi)_0, k)$. The lemma then follows for i from the "fiber contraction theorem" [1] applied to the map $\Phi_i : \mathscr{F}_i \to \mathscr{F}_i$ which preserves the fibers of Π_i. The assumptions in the fiber contraction theorem are satisfied because of Lemma 3.10 (and the trivial fact that all the fibers of $\tilde{\pi}_i$ are isometric).

Conclusion of the proof of Proposition 2′. The convergence of $\{(\Phi_i)^j \vartheta_i\}_{j=1}^{\infty}$ for $i \leq \beta((\mathrm{d}\varphi)_0, k)$ follows from Lemma 3.11 and the fact that $\vartheta_i \mid \tilde{K}_{\delta'} \in \mathscr{F}_i$.

Proof of Proposition 2. From now on we shall write α, β instead of $\alpha((\mathrm{d}\varphi)_0, k)$ and $\beta((\mathrm{d}\varphi)_0, k)$. We first define a sequence $\{F_i\}_{i=0}^{\infty}$ of C^β-maps $(\mathbb{R}^n, W^{cu}) \to (\mathbb{R}^n, W^{cu})$ such that:

(i) $F_i(x_1, \ldots, z_u) = \sum f^i_{j_1, \ldots, j_s}(x_1, \ldots, x_c, z_1, \ldots, z_u) \cdot y_1^{j_1} \cdot \cdots \cdot y_s^{j_s}$ where the sum is taken over all $(j_1, \ldots, j_s)$ with $j_v \geq 0$ and $\sum j_v \leq \beta$; $f^i_{j_1, \ldots, j_s}$ takes values in $\mathbb{R}^n$, $f^i_{0,0,\ldots,0}$ takes values in W^{cu};

(ii) the jet of $(\Phi_\beta)^i \vartheta_\beta$ in $p \in W^{cu}$ can be represented by $[p, F_i]_\beta$.

Such a sequence is constructed as follows: $F_0 =$ identity; if F_i is given then F_{i+1} is obtained from $\varphi \circ F_i \circ (S\varphi)^{-1}$ by throwing away the terms of order $> \beta$ in $y_1, \ldots, y_s$. By Proposition 2′, $\{F_i\}_{i=0}^{\infty}$ has a limit which is of the form

$$F(x_1, \ldots, z_u) = \sum f_{j_1, \ldots, j_s}(x_1, \ldots, x_c, z_1, \ldots, z_u) \cdot y_1^{j_1} \cdot \cdots \cdot y_s^{j_s},$$

where the summation is taken over the same indices $(j_1, \ldots, j_s)$ as above, but now each $f_{j_1, \ldots, j_s}$ is only a $C^{\beta - \Sigma j_v}$-function. According to Whitney's extension theorem (see, for example [4]) there is a C^β-function $\tilde{F} : (\mathbb{R}^n, W^{cu}) \to (\mathbb{R}^n, W^{cu})$ such that, for each $p \in W^{cu}$,

$[p, \tilde{F}]_\beta$ represents the jet of $(\lim_{i\to\infty} \Phi_\beta{}^i \vartheta_\beta)$ in p. $\tilde{F}$ induces a diffeomorphism from W^{cu} to itself; we may, and do, assume that $\tilde{F}$ is a diffeomorphism of $\mathbb{R}^n$ to itself. Clearly $S\varphi$ and $\tilde{F}^{-1} \circ \varphi \circ \tilde{F}$ have the same β-jet along W^{cu}, so $\tilde{F}$ defines the desired coordinate system. This proves Proposition 2.

Proof of Proposition 3. This proof is completely analogous to the proof of Proposition 2 and hence will be omitted; here we only want to make this analogy precise. In the proof of proposition 2 we started with the "good jet" along W^c and ended with the "good jet" along W^{cu}, making essential use of the fact that, in W^{cu}, φ was "expanding away" from W^c. In order to apply the same method in obtaining the "good jet" over all of $\mathbb{R}^n$, we replace φ by φ^{-1}, which is expanding away from W^{cu}. This replacing φ by φ^{-1} implies that $\overline{M}, \overline{m}, \bar{n}, \overline{N}$ are replaced by $\overline{N}, \bar{n}, \overline{m}, \overline{M}$, which is reflected in the definitions of $\alpha((\mathrm{d}\varphi)_0, k)$ and $\beta((\mathrm{d}\varphi)_0, k)$ (see Section 1). Because in the proof of Proposition 2 we did not use the fact that $\mathrm{d}\varphi \mid T_0(W^c)$ has only eigenvalues of absolute value one, the analogy is complete.

§4. PARTIALLY HYPERBOLIC ZERO-POINTS OF VECTOR FIELDS

We consider a C^∞-vectorfield X on $\mathbb{R}^n$ which is zero at the origin. We say that $(X; x_1, \ldots, x_c, y_1, \ldots, y_s, z_1, \ldots, z_u)$ is in *standard form* ($x_1, \ldots, z_u$ are coordinates on $\mathbb{R}^n$) if

$$X = \sum_{i=1}^{c} X_i(x_1, \ldots, x_c)\frac{\partial}{\partial x_i} + \sum_{i,j=1}^{s} A_{ij}(x_1, \ldots, x_c)y_j\frac{\partial}{\partial y_i} + \sum_{i,j=1}^{u} B_{ij}(x_1, \ldots, x_c)z_j\frac{\partial}{\partial z_i},$$

where:

(1) All eigenvalues of $(\partial X_i/\partial x_j)$ in $(x_1 = \cdots = x_c = 0)$ have real part zero;
(2) all eigenvalues of $A_{i,j}(0, \ldots, 0)$ have real part <0;
(3) all eigenvalues of $B_{i,j}(0, \ldots, 0)$ have real part >0.

$(X; x_1, \ldots, z_u)$ is *locally instandard form* if X has, in some neighbourhood of the origin, the above form.

The integral of X will be denoted by $\mathscr{D}_X : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ (i.e. $t \to \mathscr{D}_X(p, t)$ is the integral curve though p, $\mathscr{D}_X(p, 0) = p$; the domain of definition of $\mathscr{D}_X$ may be smaller than $\mathbb{R}^n \times \mathbb{R}$, but certainly contains a neighbourhood of (origin $\times \mathbb{R}$)). $\mathscr{D}_{X,t}$ is defined by $\mathscr{D}_{X,t}(p) = \mathscr{D}_X(p, t)$.

Notice that $(X; x_1, \ldots, z_u)$ is (locally) in standard form if and only if $(\mathscr{D}_{X,t}; x_1, \ldots, z_u)$ is (locally) in standard form for all $t > 0$. The eigenvalues of $\mathrm{d}(\mathscr{D}_{X,t})_0$ are of the form $e^{t\lambda_1}, \ldots, e^{t\lambda_n}$ where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $\mathrm{d}(X)_0$. Hence if $\mathscr{D}_{X,t}$ satisfies the Sternberg k-condition for some $t \neq 0$, then it satisfies the Sternberg k-condition for all $t \neq 0$. We say that X satisfies the *Sternberg k-condition* if $\mathscr{D}_{X,t}$ satisfies it for some $t \neq 0$. The numbers $\alpha(\mathrm{d}(\mathscr{D}_{X,t})_0, k)$ and $\beta(\mathrm{d}(\mathscr{D}_{X,t})_0, k)$, for $t > 0$, do not depend on t; we define $\alpha(\mathrm{d}(X)_0, k)$ and $\beta(\mathrm{d}(X)_0, k)$ to be equal to these numbers. Our main theorem for vectorfields can now be formulated as follows:

THEOREM. *Let X be a C^∞ vectorfield on $\mathbb{R}^n$ which is zero at the origin. If X satisfies the*

Sternberg $\alpha(\mathrm{d}(X)_0, k)$-condition, then there are C^k-coordinates $x_1, \ldots, x_c, y_1, \ldots, y_s$, $z_1, \ldots, z_u$ on $\mathbb{R}^n$ such that $(X; x_1, \ldots, z_u)$ is locally in standard form.

Sketch of the proof. First we construct coordinates $x_1', \ldots, z_u'$ and vectorfields SX and RX with $X \mid U = (SX + RX) \mid U$, for some neighbourhood U of the origin, such that:

(1) $(SX; x_1', \ldots, z_u')$ is in standard form;
(2) RX is zero up to order $\alpha(\mathrm{d}(X)_0, k)$ along $\{y_1' = \cdots = y_s' = z_1' = \cdots = z_u' = 0\}$;
(3) RX is tangent to $\{y_1' = \cdots = y_s' = 0\}$.

This is the analogue of Proposition 1. The proof of Proposition 1 is essentially based on the centermanifold theorem; this theorem also exists for vectorfields, so we can indeed find the above coordinates $x_1', \ldots, z_u'$ and vectorfields SX and RX. From now on $SX + RX$ will be denoted by X.

In the proof of Proposition 2 we modified φ (outside a neighbourhood of the origin) and chose an Euclidean metric ρ on $\mathbb{R}^n$ such that certain inequalities were satisfied. It is not difficult to see that in the case when we have a vectorfield X, we can modify X (outside a neighbourhood of the origin) and choose an Euclidean metric ρ on $\mathbb{R}^n$ such that, for every $t \in (0, 1]$, $\mathscr{D}_{X,t}$ has with respect to ρ the same properties as the modified φ. The proof of Proposition 2 then shows that, for every $t \in (0, 1]$, there is a unique $\beta(\mathrm{d}(X)_0, k)$-jet of a coordinate system F_t along $\{y_1' = \cdots = y_s' = 0\}$ which "linearizes $\mathscr{D}_{X,t}$ along

$$\{y_1' = \cdots = y_s' = 0\}$$

in the $z_1', \ldots, z_u'$ directions". Because for every positive integer m $(\mathscr{D}_{X,t/m})^m = \mathscr{D}_{X,t}$ and because F_t is unique, $F_{t/m} = F_t$. Hence, for every rational number $q \in (0, 1]$, $F_q = F_1$ By continuity and unicity one then has $F_t = F_1$ for all $t \in (0, 1]$. But this means that F_1 "linearizes X along $\{y_1' = \cdots = y_s' = 0\}$ in the $z_1', \ldots, z_u'$ directions". Hence we can find $C^{\beta(\mathrm{d}(X)_0, k)}$-coordinates $(x_1'', \ldots, x_n'')$ such that (the modified) X can be written as $X = S'X + R'X$ with:

(1) $(S'X; x_1'', \ldots, z_u'')$ in standard form, and
(2) $R'X$ zero up to order $\beta(\mathrm{d}(X)_0, k)$ along $\{y_1'' = \cdots = y_s'' = 0\}$.

This is the analogue of Proposition 2 for vectorfields. The theorem now follows from the fact that "linearizing in the $y_1'', \ldots, y_s''$ directions" can be done by a procedure completely analogous to the linearization in the $z_1', \ldots, z_u'$ directions in $\{y_1' = \cdots = y_s' = 0\}$.

§5. C^k-LINEARIZING HYPERBOLIC CLOSED ORBITS.

Let X be a C^∞-vectorfield on a manifold M and let γ be a closed orbit of X with period $t_\gamma > 0$, i.e. γ is a subset of M such that for every $m \in \gamma$, $X_m \neq 0$, $\mathscr{D}_X(m, (-\infty, +\infty)) = \gamma$ and $\mathscr{D}_X(m, t) = m$ if and only if t is a integral multiple of t_γ. We call γ a hyperbolic closed orbit if some (and hence each) $m \in \gamma$ is a partially hyperbolic fixed point of $\mathscr{D}_{X,t_\gamma}$ with $\dim(T^c) = 1$ (see Section 1 for the definition of T^c); note that because $\mathrm{d}(\mathscr{D}_{X,t})(X_m) = X_m$, always $\dim(T^c) \geq 1$.

Definition 5.1 (*the normal bundle of a hyperbolic closed orbit* γ). For each $m \in \gamma$ we define $N(m) \subset T_m(M)$ to be the direct sum of all eigenspaces of $\mathrm{d}(\mathscr{D}_{X,t_\gamma}) \mid T_m(M)$ corresponding to eigenvalues with absolute value different from 1. $N(\gamma) = \bigcup_{m \in \gamma} N(m)$ is a smooth co-dimension 1 subbundle of $T_\gamma(M)$ which we call the normal bundle of γ.

X induces a vectorfield $N(X)$ on $N(\gamma)$; $N(X)$ can be defined by $\mathscr{D}_{N(X),t} = \mathrm{d}(\mathscr{D}_{X,t}) \mid N(\gamma)$ for all t.

Definition 5.2. A C^k-linearization of a hyperbolic closed orbit γ is a C^k-embedding $\varphi : U \to M$, where U is a neighbourhood of the zero section in $N(\gamma)$, such that:

(1) for every $m \in \gamma$, $\varphi \circ s_0(m) = m$, where s_0 is the zero section in $N(\gamma)$;
(2) $\mathrm{d}\varphi(N(X) \mid U) = X \mid \varphi(U)$.

THEOREM. *Let γ be a hyperbolic closed orbit of the C^∞-vectorfield X on M with period t_γ. If for some (and hence for all) $m \in \gamma$, $\mathrm{d}(\mathscr{D}_{X,t_\gamma})_m$ satisfies the Sternberg $\alpha(\mathrm{d}(\mathscr{D}_{X,t_\gamma})_m, k)$-condition, then there is a C^k-linearization of γ.*

Remark. C^0-linearizations were obtained by Irwin [3].

Proof. By our main theorem there is a neighborhood W of m in M and C^k-coordinates $x_1, y_1, \ldots, y_s, z_1, \ldots, z_u$ on W such that:
1° $m = (0, 0, \ldots, 0)$;
2° near m, $(\mathscr{D}_{X,t_\gamma}; x_1, y_1, \ldots, y_s, z_1, \ldots, z_u)$ is in standard form, i.e.

$$\mathscr{D}_{X,t_\gamma}(x_1, \ldots, z_u) = \Big(X_1(x_1), \sum_{i=1}^{s} A_{1,i}(x_1) \cdot y_i, \ldots, \sum_{i=1}^{s} A_{s,i}(x_1) \cdot y_i, \sum_{i=1}^{u} B_{1,i}(x_1) \cdot z_i, \ldots, \sum_{i=1}^{u} B_{u,i}(x_1) \cdot z_i\Big).$$

Near m, $\gamma = \{y_1 = \cdots = y_s = z_1 = \cdots = z_u = 0\}$; γ consists of fixed points of $\mathscr{D}_{X,t_\gamma}$, so $X_1(x_1) \equiv 1$. One can also choose the coordinates so that $A_{i,j}(x_1)$ and $B_{i,j}(x_1)$ become independent of x_1 (we shall however not use this).

We first define the map φ on a neighbourhood of the origin in the fiber $N(m)$: The elements of $N(m)$ are vectors in $T_m(M)$ of the form

$$\sum_{i=1}^{s} \alpha_1 \frac{\partial}{\partial y_i} + \sum_{j=1}^{u} \beta_j \frac{\partial}{\partial z_j};$$

such a vector will be denoted by $(\alpha_1, \ldots, \alpha_s, \beta_1, \ldots, \beta_u)$. We now take

$$\varphi(\alpha_1, \ldots, \alpha_s, \beta_1, \ldots, \beta_u) = (x_1 = 0, y_1 = \alpha_1, \ldots, y_s = \alpha_s, z_1 = \beta_1, \ldots, z_u = \beta_u)$$

if this is defined (which is the case for $(\alpha_1, \ldots, \beta_u)$ close to the origin).

Note that with this definition of $\varphi \mid N(m)$ we have, near the origin of $N(m)$,

$$\mathscr{D}_{X,t_\gamma} \circ (\varphi \mid N(m)) = \varphi \circ \mathscr{D}_{N(X),t_\gamma} \mid N(m).$$

The requirement $\mathrm{d}\varphi(N(X) \mid U) = X \mid \varphi(U)$ now fixes the C^r-linearization φ, in a unique way, on a neighbourhood of the zero section.

REFERENCES

1. M. HIRSCH and C. PUGH: Stable manifolds and hyperbolic sets, in: *Proceedings of the A. M. S. Summer Institute on Global Analysis*, University Press, Berkeley (1967).
2. M. HIRSCH, C. PUGH and M. SHUB: *Invariant Manifolds* (to appear).
3. M. C. IRWIN: A classification of elementary cycles, *Topology* 9 (1970), 35–48.
4. B. MALGRANGE: *Ideals of Differentiable Functions*, Oxford University Press, Oxford (1966).
5. S. STERNBERG: On the structure of local homeomorphisms of euclidean n-space—II. *Am. J. Math.* **80**, (1958) 623–631.

Institut des Hautes Etudes Scientifiques
à Bures-sur-Yvette (France), and
Mathematisch Instituut, Universiteit van
Amsterdam, Holland