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Strong solutions to the equations of a ferrofluid flow model

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ABSTRACT

We study the differential system introduced by M.I. Shliomis to describe the motion of a ferrofluid driven by an external magnetic field. The system is a combination of the Navier–Stokes equations, the magnetization equation and the magnetostatic equations. No regularizing term is added to the magnetization equation. We prove the local-in-time existence of strong solutions to the system.

1. Introduction

In this paper we investigate the question of solvability of the equations proposed by M.I. Shliomis [30,31] to describe the flow of an incompressible ferrofluid submitted to an external magnetic field. Ferrofluids (also called magnetic fluids) are colloidal suspensions of fine magnetic mono domain nanoparticles in nonconducting liquids. Such fluids have found a wide variety of applications in engineering: magnetic liquid seals, cooling and resonance damping for loudspeaker coils, printing with magnetic inks, rotating shaft seals in vacuum chambers used in the semiconductor industry, see [37] for more details. There are also intensive investigations on the possibility of future biomedical applications of magnetic fluids, such as magnetic separation, drugs or radioisotopes targeted by magnetic guidance, hyperthermia treatments, magnetic resonance imaging contrast enhancement, see for example Pankhurst, Connolly, Jones and Dobson [26].

Consider the flow of an incompressible and viscous, Newtonian ferrofluid, filling a bounded domain D of $\mathbb{R}^3$, under the action of an external magnetic field H_{ext}. This magnetic field induces a demagnetizing field H and a magnetic induction B satisfying the law $B = H + \chi(D)M$ where M is the magnetization inside D and $\chi(D)$ is the characteristic function of D. Let $T > 0$ be a fixed time, $D_T = (0, T) \times D$ and let n denote the outward unit normal to D. The equations proposed by M.I. Shliomis [30,31] for this flow are

$$\operatorname{div} U = 0 \quad \text{in } D_T, \tag{1}$$

$$\rho\big(\partial_t U + (U \cdot \nabla)U\big) - \eta\Delta U + \nabla p = \mu_0(M \cdot \nabla)\mathrm{H} + \frac{\mu_0}{2}\operatorname{curl}(M \wedge H) \quad \text{in } D_T, \tag{2}$$

$$\partial_t M + (U \cdot \nabla)M = \frac{1}{2}\operatorname{curl} U \wedge M - \frac{1}{\tau}(M - \chi_0 \mathrm{H}) - \beta M \wedge (M \wedge H) \quad \text{in } D_T, \tag{3}$$

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where U is the fluid velocity, p is the pressure and the parameters ρ, η, μ_0, χ_0, τ and β are positive and their physical meaning can be found in [28,30,31], for example. The magnetic field H satisfies the magnetostatic equations

$$\operatorname{curl} H = 0, \qquad \operatorname{div}\big(H + \chi(D)M\big) = -\operatorname{div} H_{ext} \quad \text{in } (0,T)\times\mathbb{R}^3. \tag{4}$$

Eqs. (1) and (2) are the Navier–Stokes equations and (3) is the magnetization equation.

In [2] we considered a regularized system where Eq. (3), which is of Bloch-type, is replaced by the following

$$\partial_t M + (U\cdot\nabla)M - \sigma\Delta M = \frac{1}{2}\operatorname{curl} U \wedge M - \frac{1}{\tau}(M - \chi_0 H) - \beta M \wedge (M \wedge H)$$

which is of Bloch–Torrey type, $\sigma > 0$ being a diffusion coefficient that carry spins. The Bloch–Torrey equations were proposed by Torrey [34] as a generalization of the Bloch equations to describe situations when the diffusion of the spin magnetic moment is not negligible; see also G.D. Gaspari [13] for the derivation of the Bloch–Torrey equations. We proved existence of global-in-time weak solutions with finite energy to the system posed in a bounded domain of $\mathbb{R}^3$ and supplemented with initial and boundary conditions.

S. Venkatasubramanian and P. Kaloni [35] considered the differential system introduced by R.E. Rosensweig [28] to describe the flow of an incompressible ferrofluid under the action of a magnetic field. The Rosensweig system consists of the Navier–Stokes equations, the angular momentum equation, the magnetization equation and the magnetostatic equations (see also [27]); in [35] the authors studied the stability and uniqueness of smooth solutions to the system. In a recent paper [3] we studied the local-in-time existence of strong solutions to the Rosensweig system. In [1] we considered a regularized system of the Rosensweig system and proved existence of global-in-time weak solutions with finite energy to the system posed in a bounded domain of $\mathbb{R}^3$ and supplemented with initial and boundary conditions.

The study of magnetic fluids differs from magnetohydrodynamics (MHD) that concerns itself with nonmagnetizable but electrically conducting fluids. The set of equations which describe MHD is a combination of the Navier–Stokes equations of fluid dynamics and Maxwell's equations of electromagnetism; see the papers by G. Duvaut and J.-L. Lions [8], M. Sermange and R. Temam [29], J.F. Gerbeau and C. Le Bris [14,15] and H. Inoue [20] for some results of existence of solutions. In a recent paper B. Ducomet and E. Feireisl [7] proved existence of global-in-time weak solutions to the equations of MHD, specifically, the Navier–Stokes–Fourier system describing the evolution of a compressible, viscous, and heat conducting fluid coupled with the Maxwell equations. Let us also mention some works on equations arising in the theory of micropolar fluids introduced by A.C. Eringen [9] which focuses on the fluids consisting of randomly oriented particles suspended in a viscous medium when the deformation of fluid particles is ignored; we refer to the papers by G.P. Galdi and S. Rionero [12], G. Lukaszewicz [24].

In this paper we consider system (1)–(3) equipped with the boundary and initial conditions

$$U = 0 \quad \text{on } (0,T)\times\partial D, \tag{5}$$

$$U|_{t=0} = U_0, \qquad M|_{t=0} = M_0 \quad \text{in } D, \tag{6}$$

where U_0 and M_0 are given data. For simplicity, we assume that the magnetic field H satisfies, instead of (4), the magnetostatic equations

$$\operatorname{curl} H = 0, \qquad \operatorname{div}(H + M) = F \quad \text{in } D_T, \tag{7}$$

and the boundary condition

$$(H + M)\cdot n = 0 \quad \text{on } (0,T)\times\partial D \tag{8}$$

where F is a given function in D_T such that $\int_D F\,dx = 0$, for all $t \in [0,T]$.

Our aim is to show that problem (1)–(3), (5)–(8), denoted problem $(\mathcal{P})$ in the sequel, admits a local-in-time strong solution (U, M, H), in the sense of Definition 1 (below). We assume that D is an open bounded domain in $\mathbb{R}^3$ of smooth boundary. Let $L^q(D)$ and $W^{s,q}(D)$ $(1 \leqslant q \leqslant \infty,\ s \in \mathbb{R})$ be the usual Lebesgue and Sobolev spaces of scalar-valued functions, respectively. When $q = 2$, $W^{s,q}(D)$ is denoted $H^s(D)$. By $\|\cdot\|$ and $(\cdot;\cdot)$ we denote the L^2-norm and scalar product, respectively. The Hölder spaces $C^{k,\alpha}(\overline{D})$ $(k \in \mathbb{N},\ 0 < \alpha < 1)$ are defined as the subspaces of $C^k(\overline{D})$ consisting of functions whose kth order partial derivatives are Hölder continuous with exponent α. We denote $\mathbb{L}^q(D) = (L^q(D))^3$, $\mathbb{W}^{s,q}(D) = (W^{s,q}(D))^3$, $\mathbb{H}^s(D) = (H^s(D))^3$ and $C^{k,\alpha}(\overline{D},\mathbb{R}^3) = (C^{k,\alpha}(\overline{D}))^3$.

We introduce the following spaces of divergence-free functions, see Galdi [10,11], Ladyzhenskaya [21], J.L. Lions [22], P.L. Lions [23], Tartar [32], Temam [33]:

$$\mathcal{C}^\infty_{0,s}(D) = \big\{ v \in \mathcal{C}^\infty_0\big(D,\mathbb{R}^3\big):\ \operatorname{div} v = 0 \text{ in } D \big\},$$

$$\mathcal{U} = \text{closure of } \mathcal{C}^\infty_{0,s}(D) \quad \text{in } \mathbb{H}^1(D),$$

$$\mathcal{U}_0 = \text{closure of } \mathcal{C}^\infty_{0,s}(D) \quad \text{in } \mathbb{L}^2(D).$$

It is well known that

$$\mathcal{U} = \{v \in \mathbb{H}^1(D):\ \operatorname{div} v = 0 \text{ in } D\},$$
$$\mathcal{U}_0 = \{v \in \mathbb{L}^2(D):\ \operatorname{div} v = 0 \text{ in } D,\ v \cdot n = 0 \text{ on } \partial D\},$$
$$\mathcal{U} \subset \mathcal{U}_0 \subset \mathcal{U}' = \text{dual space of } \mathcal{U} \quad \text{when } \mathcal{U}_0 \text{ is identified with its dual.}$$

We assume that

$$U_0 \in \mathbb{H}^2(D) \cap \mathcal{U} \tag{9}$$

and

$$M_0 \in \mathbb{W}^{1,q}(D), \qquad F \in W^{1,\infty}(0, T; L^q(D)), \quad q > 3, \qquad \int_D F\, dx = 0 \quad \text{in } (0, T). \tag{10}$$

Definition 1. Let $q > 3$ and $r = \min\{q, 6\}$. We say that (U, M, H) is a strong solution in D_T of problem $(\mathcal{P})$ if the conditions (i)–(iv) below are satisfied:

(i) $U \in C([0, T]; \mathcal{U} \cap \mathbb{H}^2(D)) \cap W^{1,\infty}(0, T; \mathbb{L}^2(D)) \cap L^2(0, T; \mathbb{W}^{2,r}(D))$,
$M, H \in L^\infty(0, T; \mathbb{W}^{1,r}(D)) \cap W^{1,\infty}(0, T; \mathbb{L}^r(D))$;

(ii) the function H is such that $H = \nabla\varphi$ where $\varphi \in L^\infty(0, T; W^{2,r}(D))$ and solves the problem

$$-\Delta\varphi = \operatorname{div} M - F \quad \text{in } D_T,$$
$$\frac{\partial\varphi}{\partial n} = -M \cdot n \quad \text{on } (0, T) \times \partial D, \qquad \int_D \varphi\, dx = 0 \quad \text{in } (0, T);$$

(iii) Eqs. (1), (2) hold weakly, that is, for every $v \in \mathcal{U}$, we have

$$\rho \frac{d}{dt} \int_D U \cdot v\, dx + \rho \int_D (U \cdot \nabla)U \cdot v\, dx + \eta \int_D \nabla U \cdot \nabla v\, dx$$
$$= \mu_0 \int_D (M \cdot \nabla)H \cdot v\, dx + \frac{\mu_0}{2} \int_D (\operatorname{curl}(M \wedge H)) \cdot v\, dx \quad \text{in } \mathcal{D}'(]0, T[),$$
$$U|_{t=0} = U_0;$$

(iv) there exists $p \in L^2(0, T; W^{1,r}(D))$ such that Eqs. (1)–(3) hold a.e. in D_T and the initial condition on M holds in the sense of traces.

Our main result is the following one.

Theorem 1. *Under assumptions* (9) *and* (10), *there is a time $T^* > 0$ such that problem $(\mathcal{P})$ admits a unique strong solution (U, M, H) in D_{T^*}, in the sense of Definition* 1.

To prove Theorem 1 we study a linearized problem of problem $(\mathcal{P})$. Assume that $(U_\sharp, M_\sharp, H_\sharp)$ is given, $U_\sharp$ belongs to $L^\infty(0, T; \mathcal{U} \cap \mathbb{H}^2(D)) \cap L^2(0, T; \mathbb{W}^{2,r}(D))$, $\partial_t U_\sharp$ belongs to $L^\infty(0, T; \mathbb{L}^2(D))$ and $M_\sharp$, $H_\sharp$ belong to $L^\infty(0, T; \mathbb{W}^{1,r}(D))$. Observe that, since $r > 3$ and due to the Sobolev embedding

$$W^{1,r}(D) \hookrightarrow C^{0,\alpha}(\overline{D}) \quad \left(\alpha = 1 - \frac{3}{r}\right) \tag{11}$$

where $\hookrightarrow$ denotes the continuous embedding, the function $\nabla U_\sharp$ belongs to $L^2(0, T; C^{0,\alpha}(\overline{D}, \mathbb{R}^9))$, and $M_\sharp \wedge H_\sharp$ belongs to $L^\infty(0, T; \mathbb{W}^{1,r}(D))$.

We define the function M as the solution of the linear hyperbolic system

$$\partial_t M + (U_\sharp \cdot \nabla)M - \frac{1}{2} \operatorname{curl} U_\sharp \wedge M + \frac{1}{\tau} M + \beta M \wedge (M_\sharp \wedge H_\sharp) = \frac{\chi_0}{\tau} H_\sharp \quad \text{in } D_T, \tag{12}$$
$$M|_{t=0} = M_0 \quad \text{in } D. \tag{13}$$

Note that the condition $U = 0$ on $(0, T) \times \partial D$ prevents the necessity of using boundary conditions for the solution to (12), see for instance DiPerna and Lions [6].

Then we define the function H as the solution of

$$\operatorname{curl} H = 0, \qquad \operatorname{div}(H + M) = F \quad \text{in } D_T, \tag{14}$$

$$(H + M) \cdot n = 0 \quad \text{on } (0, T) \times \partial D. \tag{15}$$

The functions M and H being defined by (12), (13) and (14), (15), respectively, we define the function U as the solution of the linear system

$$\rho\big(\partial_t U + (U_\sharp \cdot \nabla)U\big) - \eta \Delta U + \nabla p = \mu_0 (M \cdot \nabla) H + \frac{\mu_0}{2} \operatorname{curl}(M \wedge H) \quad \text{in } D_T, \tag{16}$$

$$\operatorname{div} U = 0 \quad \text{in } D_T, \tag{17}$$

supplemented by the boundary and initial conditions

$$U = 0 \quad \text{on } (0, T) \times \partial D, \tag{18}$$

$$U|_{t=0} = U_0 \quad \text{in } D. \tag{19}$$

We construct a sequence of approximate solutions to problem $(\mathcal{P})$, derive some uniform bounds of the sequence and then prove the convergence of the sequence to a strong solution of problem $(\mathcal{P})$. The method and techniques we use here are inspired from the paper by Y. Cho, H.J. Choe and H. Kim [4] on Navier–Stokes equations for compressible barotropic fluids; see also the paper by Y. Cho and H. Kim [5] on the incompressible Navier–Stokes equations with a density-dependent viscosity. In the final part of the paper (Section 4.4.2) we prove the uniqueness of strong solutions to problem $(\mathcal{P})$.

In the paper, C indicates a generic constant, depending only on some bounds of the physical data, which can take different values in different occurrences.

2. Solvability of problems (12), (13) and (14), (15)

We first show the following results.

Lemma 1. *Problem* (12), (13) *has a unique global-in-time solution* $M \in L^\infty(0, T; \mathbb{W}^{1,r}(D)) \cap W^{1,\infty}(0, T; \mathbb{L}^r(D))$.

Proof. Clearly, (12) is a linear hyperbolic system with regular coefficients. Recall that, due to the Sobolev embedding (11), $\nabla U_\sharp$ belongs to $L^2(0, T; C^{0,\alpha}(\overline{D}, \mathbb{R}^9))$ and $M_\sharp \wedge H_\sharp$ belongs to $L^\infty(0, T; \mathbb{W}^{1,r}(D))$. The existence and uniqueness of solutions to (12) with the initial condition (13) are classical: problem (12), (13) has a unique global-in-time weak solution $M \in L^\infty(0, T; \mathbb{L}^r(D))$, i.e. M is the unique function satisfying the integral identity

$$\int_0^T \int_D \left(M \cdot w_t + (U_\sharp \cdot \nabla) w \cdot M + \left(\frac{1}{2} \operatorname{curl} U_\sharp \wedge M - \frac{1}{\tau} M - \beta M \wedge (M_\sharp \wedge H_\sharp) \right) \cdot w \right) dx\, dt$$

$$= -\frac{\chi_0}{\tau} \int_0^T \int_D H_\sharp \cdot w \, dx\, dt - \int_D M_0(x) \cdot w(0, x)\, dx$$

for any $w \in C^1(\overline{D_T}, \mathbb{R}^3)$ with compact support in $[0, T[\times \overline{D}$. Since the right-hand side of (12) belongs to $L^\infty(0, T; \mathbb{W}^{1,r}(D))$, it implies that the weak solution M belongs to $L^\infty(0, T; \mathbb{W}^{1,r}(D))$ and (12) holds a.e. in D_T. Then we deduce from Eq. (12) that the function M belongs to $W^{1,\infty}(0, T; \mathbb{L}^r(D))$. Lemma 1 is proved. $\square$

Lemma 2. *Let M be the solution of problem* (12), (13). *The following estimates hold*:

(i) $$\|M(t)\|^r_{\mathbb{L}^r(D)} \leqslant b_1(t), \quad t \in (0, T), \tag{20}$$

where

$$b_1(t) = \|M_0\|^r_{\mathbb{L}^r(D)} + C \int_0^t \|H_\sharp(s)\|^r_{\mathbb{L}^r(D)}\, ds; \tag{21}$$

(ii) $$\|\nabla M(t)\|^r_{(\mathbb{L}^r(D))^3} \leqslant b_2(t), \quad t \in (0, T), \tag{22}$$

$$\|M_t(t)\|_{\mathbb{L}^r(D)} \leqslant C b_3(t), \quad t \in (0, T), \tag{23}$$

where

$$b_2(t) = b_2^1(t) + \int_0^t b_2^1(s) b_2^2(s) \exp\left(\int_s^t b_2^2(\sigma)\, d\sigma\right) ds, \quad t \in (0,T), \tag{24}$$

and

$$\begin{aligned} b_2^1(t) = {} & \|\nabla M_0\|^r_{(\mathbb{L}^r(D))^3} + C\int_0^t \left(\|\nabla H_\sharp(s)\|^r_{(\mathbb{L}^r(D))^3} + \|M(s)\|^r_{\mathbb{L}^r(D)} \|U_\sharp(s)\|_{\mathbb{W}^{2,r}(D)}\right) ds \\ & + C\int_0^t \|M(s)\|^r_{\mathbb{L}^r(D)} \|H_\sharp(s)\|_{\mathbb{W}^{1,r}(D)} \|\nabla M_\sharp(s)\|_{(\mathbb{L}^r(D))^3}\, ds \\ & + C\int_0^t \|M(s)\|^r_{\mathbb{L}^r(D)} \|M_\sharp(s)\|_{\mathbb{W}^{1,r}(D)} \|\nabla H_\sharp(s)\|_{(\mathbb{L}^r(D))^3}\, ds \end{aligned} \tag{25}$$

and

$$b_2^2(t) = C\left(\|H_\sharp(t)\|_{\mathbb{W}^{1,r}(D)} \|\nabla M_\sharp(t)\|_{(\mathbb{L}^r(D))^3} + \|M_\sharp(t)\|_{\mathbb{W}^{1,r}(D)} \|\nabla H_\sharp(t)\|_{(\mathbb{L}^r(D))^3}\right) + C\left(1 + \|U_\sharp(t)\|_{\mathbb{W}^{2,r}(D)}\right) \tag{26}$$

and

$$b_3(t) = \|H_\sharp(t)\|_{\mathbb{L}^r(D)} + b_1^{1/r}(t)\left(\|U_\sharp(t)\|_{\mathbb{W}^{1,r}(D)} + \|M_\sharp(t)\|_{\mathbb{W}^{1,r}(D)} \|H_\sharp(t)\|_{\mathbb{W}^{1,r}(D)}\right) + b_2^{1/r}(t) \|U_\sharp(t)\|_{\mathbb{W}^{1,r}(D)}. \tag{27}$$

Proof. (i) Multiplying Eq. (12) by $|M|^{r-2}M$ and integrating over D yields

$$\frac{d}{dt}\left(\frac{\|M\|^r_{\mathbb{L}^r(D)}}{r}\right) + \frac{1}{\tau}\|M\|^r_{\mathbb{L}^r(D)} = \frac{\chi_0}{\tau}\int_D H_\sharp \cdot |M|^{r-2} M\, dx. \tag{28}$$

We estimate the right-hand side of (28) by using the Hölder and Young inequalities and then obtain

$$\frac{d}{dt}\left(\frac{\|M\|^r_{\mathbb{L}^r(D)}}{r}\right) + \frac{1}{2\tau}\|M\|^r_{\mathbb{L}^r(D)} \leqslant C\|H_\sharp\|^r_{\mathbb{L}^r(D)}.$$

Integrating from 0 to t we obtain (20).

(ii) We differentiate (12) with respect to x_i ($1 \leqslant i \leqslant 3$) to obtain

$$\partial_t N + (U_\sharp \cdot \nabla) N - \frac{1}{2}\operatorname{curl} U_\sharp \wedge N + \beta N \wedge (M_\sharp \wedge H_\sharp) + \frac{1}{\tau} N = S \tag{29}$$

where we set $N = \partial_{x_i} M$, $K_\sharp = \partial_{x_i} H_\sharp$, $N_\sharp = \partial_{x_i} M_\sharp$, $V_\sharp = \partial_{x_i} U_\sharp$ and

$$S = \frac{\chi_0}{\tau} K_\sharp - (V_\sharp \cdot \nabla) M + \frac{1}{2}\operatorname{curl} V_\sharp \wedge M - \beta M \wedge (N_\sharp \wedge H_\sharp) - \beta M \wedge (M_\sharp \wedge K_\sharp).$$

Then we multiply (29) by $|N|^{r-2}N$ and integrate over D. Observe that $(\operatorname{curl} U_\sharp \wedge N) \cdot |N|^{r-2}N = 0$ and $\beta N \wedge (M_\sharp \wedge H_\sharp) \cdot |N|^{r-2} N = 0$. We obtain

$$\frac{d}{dt}\left(\frac{\|N\|^r_{\mathbb{L}^r(D)}}{r}\right) + \frac{1}{\tau}\|N\|^r_{\mathbb{L}^r(D)} = \int_D S \cdot |N|^{r-2} N\, dx. \tag{30}$$

The right-hand side is estimated as follows:

$$\begin{aligned} \left|\int_D S \cdot |N|^{r-2} N\, dx\right| &\leqslant \int_D \left(\frac{\chi_0}{\tau}|K_\sharp| + \left|(V_\sharp \cdot \nabla) M\right| + \frac{1}{2}|\operatorname{curl} V_\sharp \wedge M|\right)|N|^{r-1}\, dx \\ &\quad + \int_D \left(\beta\left|M \wedge (N_\sharp \wedge H_\sharp)\right| + \beta\left|M \wedge (M_\sharp \wedge K_\sharp)\right|\right)|N|^{r-1}\, dx \\ &\equiv I_1 + I_2. \end{aligned} \tag{31}$$

Hölder's inequality yields

$$I_1 \leqslant C\left(\|\nabla H_\sharp\|_{(\mathbb{L}^r(D))^3} + \left\|(V_\sharp \cdot \nabla) M\right\|_{\mathbb{L}^r(D)} + \|\operatorname{curl} V_\sharp \wedge M\|_{\mathbb{L}^r(D)}\right)\|\nabla M\|^{r-1}_{(\mathbb{L}^r(D))^3}.$$

Due the Sobolev embedding (11) we have

$$\|V_\sharp\|_{\mathbb{L}^\infty(D)} \leqslant \|\nabla U_\sharp\|_{(\mathbb{L}^\infty(D))^3} \leqslant C\|U_\sharp\|_{\mathbb{W}^{2,r}(D)}$$

and

$$\|M\|_{\mathbb{L}^\infty(D)} \leqslant C\big(\|M\|_{\mathbb{L}^r(D)} + \|\nabla M\|_{(\mathbb{L}^r(D))^3}\big)$$

then

$$\big\|(V_\sharp \cdot \nabla)M\big\|_{\mathbb{L}^r(D)} \leqslant C\|U_\sharp\|_{\mathbb{W}^{2,r}(D)}\|\nabla M\|_{(\mathbb{L}^r(D))^3}$$

and

$$\|\operatorname{curl} V_\sharp \wedge M\|_{\mathbb{L}^r(D)} \leqslant C\|U_\sharp\|_{\mathbb{W}^{2,r}(D)}\big(\|M\|_{\mathbb{L}^r(D)} + \|\nabla M\|_{(\mathbb{L}^r(D))^3}\big).$$

It results that

$$\begin{aligned} I_1 &\leqslant C\|\nabla H_\sharp\|_{(\mathbb{L}^r(D))^3}\|\nabla M\|^{r-1}_{(\mathbb{L}^r(D))^3} + C\|U_\sharp\|_{\mathbb{W}^{2,r}(D)}\|\nabla M\|^r_{(\mathbb{L}^r(D))^3} \\ &\quad + C\|U_\sharp\|_{\mathbb{W}^{2,r}(D)}\big(\|M\|_{\mathbb{L}^r(D)} + \|\nabla M\|_{(\mathbb{L}^r(D))^3}\big)\|\nabla M\|^{r-1}_{(\mathbb{L}^r(D))^3}. \end{aligned}$$

Applying the Young inequality we obtain

$$I_1 \leqslant C\big(\|\nabla H_\sharp\|^r_{(\mathbb{L}^r(D))^3} + \|M\|^r_{\mathbb{L}^r(D)}\|U_\sharp\|_{\mathbb{W}^{2,r}(D)}\big) + C\big(1 + \|U_\sharp\|_{\mathbb{W}^{2,r}(D)}\big)\|\nabla M\|^r_{(\mathbb{L}^r(D))^3}. \tag{32}$$

Using similar arguments we show that

$$\big\|M \wedge (N_\sharp \wedge H_\sharp)\big\|_{\mathbb{L}^r(D)} \leqslant C\big(\|M\|_{\mathbb{L}^r(D)} + \|\nabla M\|_{(\mathbb{L}^r(D))^3}\big)\|\nabla M_\sharp\|_{(\mathbb{L}^r(D))^3}\|H_\sharp\|_{\mathbb{W}^{1,r}(D)}$$

and

$$\big\|M \wedge (M_\sharp \wedge K_\sharp)\big\|_{\mathbb{L}^r(D)} \leqslant C\big(\|M\|_{\mathbb{L}^r(D)} + \|\nabla M\|_{(\mathbb{L}^r(D))^3}\big)\|M_\sharp\|_{\mathbb{W}^{1,r}(D)}\|\nabla H_\sharp\|_{(\mathbb{L}^r(D))^3}$$

then we derive the estimate

$$I_2 \leqslant C\big(\|H_\sharp\|_{\mathbb{W}^{1,r}(D)}\|\nabla M_\sharp\|_{(\mathbb{L}^r(D))^3} + \|M_\sharp\|_{\mathbb{W}^{1,r}(D)}\|\nabla H_\sharp\|_{(\mathbb{L}^r(D))^3}\big)\big(\|M\|^r_{\mathbb{L}^r(D)} + \|\nabla M\|^r_{(\mathbb{L}^r(D))^3}\big). \tag{33}$$

Combining (30)–(33) we deduce that

$$\begin{aligned} &\frac{d}{dt}\left(\frac{\|\nabla M\|^r_{(\mathbb{L}^r(D))^3}}{r}\right) + \frac{1}{\tau}\|\nabla M\|^r_{\mathbb{L}^r(D)} \\ &\quad \leqslant C\big(\|\nabla H_\sharp\|^r_{(\mathbb{L}^r(D))^3} + \|M\|^r_{\mathbb{L}^r(D)}\|U_\sharp\|_{\mathbb{W}^{2,r}(D)}\big) + C\|M\|^r_{\mathbb{L}^r(D)}\big(\|H_\sharp\|_{\mathbb{W}^{1,r}(D)}\|\nabla M_\sharp\|_{(\mathbb{L}^r(D))^3} + \|M_\sharp\|_{\mathbb{W}^{1,r}(D)}\|\nabla H_\sharp\|_{(\mathbb{L}^r(D))^3}\big) \\ &\qquad + C\big(1 + \|U_\sharp\|_{\mathbb{W}^{2,r}(D)}\big)\|\nabla M\|^r_{(\mathbb{L}^r(D))^3} + C\big(\|H_\sharp\|_{\mathbb{W}^{1,r}(D)}\|\nabla M_\sharp\|_{(\mathbb{L}^r(D))^3} + \|M_\sharp\|_{\mathbb{W}^{1,r}(D)}\|\nabla H_\sharp\|_{(\mathbb{L}^r(D))^3}\big)\|\nabla M\|^r_{(\mathbb{L}^r(D))^3} \end{aligned}$$

and Gronwall's inequality yields estimate (22).

Using again the Sobolev embedding (11) we deduce from (12) that

$$\begin{aligned} \|M_t\|_{\mathbb{L}^r(D)} &\leqslant C\|U_\sharp\|_{\mathbb{W}^{1,r}(D)}\|\nabla M\|_{(\mathbb{L}^r(D))^3} + C\|U_\sharp\|_{\mathbb{W}^{1,r}(D)}\big(\|M\|_{\mathbb{L}^r(D)} + \|\nabla M\|_{(\mathbb{L}^r(D))^3}\big) \\ &\quad + C\|M\|_{\mathbb{L}^r(D)}\|M_\sharp\|_{\mathbb{W}^{1,r}(D)}\|H_\sharp\|_{\mathbb{W}^{1,r}(D)} + C\|H_\sharp\|_{\mathbb{L}^r(D)} \end{aligned}$$

and using (20) and (22) we obtain (23). The proof of Lemma 2 is finished. □

Then we establish the following results.

Lemma 3. *Assume that M is a given function. Then*:

(i) *If $M \in L^\infty(0,T;\mathbb{L}^r(D))$ then there exists a unique function $\varphi \in L^\infty(0,T;W^{1,r}(D))$ such that $\int_D \varphi\,dx = 0$ and $H = \nabla\varphi$ satisfies*

$$\int_D H \cdot \nabla v\,dx = -\int_D M \cdot \nabla v\,dx - \int_D F v\,dx, \quad \forall v \in \mathcal{C}^\infty(\overline{D}).$$

Moreover, we have the estimate

$$\big\|H(t)\big\|_{\mathbb{L}^r(D)} \leqslant C\big(\big\|M(t)\big\|_{\mathbb{L}^r(D)} + \big\|F(t)\big\|_{L^r(D)}\big), \quad t \in (0,T). \tag{34}$$

(ii) *If* $M \in L^\infty(0,T;\mathbb{W}^{1,r}(D))$ *then* $H \in L^\infty(0,T;\mathbb{W}^{1,r}(D))$ *and we have the estimate*

$$\|\nabla H(t)\|_{(\mathbb{L}^r(D))^3} \leqslant C\big(\|\nabla M(t)\|_{(\mathbb{L}^r(D))^3} + \|F(t)\|_{L^r(D)}\big), \quad t \in (0,T). \tag{35}$$

Note that Lemma 3 is valid for any $1 < r < \infty$.

Proof. Introduce the boundary-value problem, for a.e. $t \in (0,T)$,

$$\Delta\psi = F \quad \text{in } D, \tag{36}$$

$$\frac{\partial\psi}{\partial n} = 0 \quad \text{on } \partial D, \qquad \int_D \psi\,dx = 0. \tag{37}$$

Problem (36), (37) has a unique solution $\psi \in W^{2,r}(D)$ satisfying the estimate

$$\|\psi\|_{W^{2,r}(D)} \leqslant C\|F\|_{L^r(D)},$$

see for instance Grisvard [19]. Denoting $\Psi = \nabla\psi$ and $N = M - \Psi$, we have to find a function φ satisfying

$$\int_D \nabla\varphi\cdot\nabla v\,dx = -\int_D N\cdot\nabla v\,dx, \quad \forall v \in \mathcal{C}^\infty(\overline{D}).$$

Employing Lemma 4.27 in Novotný and Straškraba [25, p. 211] we easily prove Lemma 3. Note that (35) can also be proved as follows. Since $\operatorname{curl}(H+M) = \operatorname{curl} M$, $\operatorname{div}(H+M) = F$ in D and $(H+M)\cdot n = 0$ on ∂D, assuming that the first Betti number of the domain D is zero, by a result of von Wahl [36, Theorem 3.2] we have, a.e. in $(0,T)$,

$$\|\nabla(H+M)\|_{(\mathbb{L}^r(D))^3} \leqslant C\big(\|\operatorname{curl} M\|_{\mathbb{L}^r(D)} + \|F\|_{L^r(D)}\big) \leqslant C\big(\|\nabla M\|_{(\mathbb{L}^r(D))^3} + \|F\|_{L^r(D)}\big)$$

and (35) follows readily. □

Lemma 4. *Let* M *be the solution of problem* (12), (13), *let* H *be the solution of problem* (14), (15) *and let* b_1, b_2, b_3 *be the functions defined in Lemma* 2. *Then*:

(i) $H_t \in L^\infty(0,T;\mathbb{L}^r(D))$ *and we have*

$$\|H_t(t)\|_{\mathbb{L}^r(D)} \leqslant b_4(t), \quad t \in (0,T), \tag{38}$$

where

$$b_4(t) = C\big(1 + b_3(t)\big); \tag{39}$$

(ii) $(M\cdot\nabla)H$, $M\wedge H$ *and* $\operatorname{curl}(M\wedge H)$ *belong to* $L^\infty(0,T;\mathbb{L}^r(D))$ *and we have a.e. in* $(0,T)$

$$\|(M\cdot\nabla)H(t)\|^r_{\mathbb{L}^r(D)} \leqslant C(1+b_2)(b_1+b_2)(t), \tag{40}$$

$$\|M\wedge H(t)\|^r_{\mathbb{L}^r(D)} \leqslant C(1+b_1)(b_1+b_2)(t), \tag{41}$$

$$\|\operatorname{curl}(M\wedge H)(t)\|^r_{\mathbb{L}^r(D)} \leqslant C(1+b_2)(b_1+b_2)(t); \tag{42}$$

(iii) $[(M\cdot\nabla)H]_t$ *and* $\operatorname{curl}[M\wedge H]_t$ *belong to* $L^\infty(0,T;\mathbb{H}^{-1}(D))$ *and* $[M\wedge H]_t$ *belongs to* $L^\infty(0,T;\mathbb{L}^r(D))$ *and we have a.e. in* $(0,T)$

$$\|[(M\cdot\nabla)H]_t(t)\|_{\mathbb{H}^{-1}(D)} \leqslant C(1+b_1+b_2)^{1/r}(b_3+b_4)(t), \tag{43}$$

$$\|[M\wedge H]_t(t)\|^r_{\mathbb{L}^r(D)} \leqslant C(1+b_1+b_2)\big(b_3^r+b_4^r\big)(t), \tag{44}$$

$$\|\operatorname{curl}[M\wedge H]_t(t)\|_{\mathbb{H}^{-1}(D)} \leqslant C(1+b_1+b_2)^{1/r}(b_3+b_4)(t). \tag{45}$$

Proof. (i) Differentiating (14) and (15) with respect to t we have $\operatorname{curl} H_t = 0$, $\operatorname{div}(H_t + M_t) = F_t$ in D, $(H_t+M_t)\cdot n = 0$ on ∂D from which, similarly to (34), we deduce that

$$\|H_t\|_{\mathbb{L}^r(D)} \leqslant C\big(\|M_t\|_{\mathbb{L}^r(D)} + \|F_t\|_{L^r(D)}\big) \leqslant C(1+b_3).$$

(ii) Using the Sobolev embedding (11) and the estimates (20), (22) and (35) we have

$$\|(M\cdot\nabla)H\|^r_{\mathbb{L}^r(D)} \leqslant C\|M\|^r_{\mathbb{W}^{1,r}(D)}\|\nabla H\|^r_{(\mathbb{L}^r(D))^3} \leqslant C(1+b_2)(b_1+b_2).$$

Similarly,

$$\|M\wedge H\|^r_{\mathbb{L}^r(D)} \leqslant C\|M\|^r_{\mathbb{W}^{1,r}(D)}\|H\|^r_{\mathbb{L}^r(D)} \leqslant C(1+b_1)(b_1+b_2)$$

and

$$\begin{aligned}\big\|\operatorname{curl}(M\wedge H)\big\|^r_{\mathbb{L}^r(D)} &\leqslant C\big(\|M\|^r_{\mathbb{W}^{1,r}(D)}\|\nabla H\|^r_{(\mathbb{L}^r(D))^3} + \|H\|^r_{\mathbb{W}^{1,r}(D)}\|\nabla M\|^r_{(\mathbb{L}^r(D))^3}\big)\\ &\leqslant C(1+b_2)(b_1+b_2).\end{aligned}$$

(iii) Let $w \in \mathbb{H}^1_0(D)$. Integrating by parts and using Eq. (14) we have

$$\begin{aligned}\int_D (M\cdot\nabla)H\cdot w\,dx &= \int_D M_i(\partial_i H_j)w_j\,dx\\ &= \int_D \partial_i(M_iH_j)w_j\,dx - \int_D (\partial_i M_i)H_jw_j\,dx\\ &= -\int_D M_i(\partial_i w_j)H_j\,dx - \int_D (F-\operatorname{div}H)H_jw_j\,dx\\ &= -\int_D (M\cdot\nabla w)\cdot H\,dx - \int_D FH\cdot w\,dx + \int_D (\operatorname{div}H)H\cdot w\,dx. \end{aligned}\tag{46}$$

Since $\operatorname{curl} H = 0$ we have $\partial_i H_j = \partial_j H_i$ and using also integrations by parts, the last term of (46) can be written as

$$\begin{aligned}\int_D (\operatorname{div}H)H\cdot w\,dx = \int_D (\partial_i H_i)H_jw_j\,dx &= \int_D \partial_i(H_iH_j)w_j\,dx - \int_D H_i(\partial_j H_i)w_j\,dx\\ &= -\int_D (H\cdot\nabla)w\cdot H\,dx + \int_D \frac{|H|^2}{2}\operatorname{div} w\,dx.\end{aligned}$$

Thus

$$\int_D (M\cdot\nabla)H\cdot w\,dx = -\int_D \big((M+H)\cdot\nabla\big)w\cdot H\,dx - \int_D FH\cdot w\,dx + \int_D \frac{|H|^2}{2}\operatorname{div} w\,dx. \tag{47}$$

By differentiation with respect to t we have

$$\begin{aligned}\int_D \big[(M\cdot\nabla)H\big]_t\cdot w\,dx = &-\int_D \big((M_t+H_t)\cdot\nabla\big)w\cdot H\,dx - \int_D \big((M+H)\cdot\nabla\big)w\cdot H_t\,dx\\ &- \int_D F_tH\cdot w\,dx - \int_D FH_t\cdot w\,dx + \int_D H_t\cdot H\operatorname{div} w\,dx.\end{aligned}\tag{48}$$

Using the Cauchy–Schwarz inequality and the Sobolev embedding (11) we deduce from (48) that

$$\begin{aligned}\left|\int_D \big[(M\cdot\nabla H)\big]_t\cdot w\,dx\right| &\leqslant C\big(\|M_t+H_t\|\|H\|_{\mathbb{W}^{1,r}(D)} + \|M+H\|_{\mathbb{W}^{1,r}(D)}\|H_t\|\big)\|\nabla w\|\\ &\quad + C\big(\|F_t\|\|H\|_{\mathbb{W}^{1,r}(D)} + \|F\|_{W^{1,r}(D)}\|H_t\|\big)\|w\| + C\|H_t\|\|H\|_{\mathbb{W}^{1,r}(D)}\|\nabla w\|.\end{aligned}\tag{49}$$

Using the estimates (20), (22), (23), (34), (35) and (38), we obtain

$$\left|\int_D \big[(M\cdot\nabla H)\big]_t\cdot w\,dx\right| \leqslant C(1+b_1+b_2)^{1/r}(b_3+b_4)\|w\|_{\mathbb{H}^1_0(D)}$$

hence

$$\big\|\big[(M\cdot\nabla)H\big]_t\big\|_{\mathbb{H}^{-1}(D)} \leqslant C(1+b_1+b_2)^{1/r}(b_3+b_4).$$

Arguing as in the previous item we have

$$\|M_t\wedge H\|^r_{\mathbb{L}^r(D)} \leqslant C\|M_t\|^r_{\mathbb{L}^r(D)}\|H\|^r_{\mathbb{W}^{1,r}(D)} \leqslant Cb_3^r(1+b_1+b_2)$$

and

$$\|M\wedge H_t\|^r_{\mathbb{L}^r(D)} \leqslant C\|M\|^r_{\mathbb{W}^{1,r}(D)}\|H_t\|^r_{\mathbb{L}^r(D)} \leqslant Cb_4^r(b_1+b_2)$$

and since $[M \wedge H]_t = M_t \wedge H + M \wedge H_t$ we obtain

$$\big\|[M \wedge H]_t\big\|^r_{\mathbb{L}^r(D)} \leqslant C(1 + b_1 + b_2)\big(b_3^r + b_4^r\big).$$

For $w \in \mathbb{H}^1_0(D)$ we have, by integrating by parts and using the Cauchy–Schwarz inequality,

$$\left|\int_D \big[\operatorname{curl}(M \wedge H)\big]_t \cdot w\,dx\right| \leqslant C\big\|[M \wedge H]_t(t)\big\|_{\mathbb{L}^r(D)}\|w\|_{\mathbb{H}^1_0(D)}$$

and using (44) we obtain (45). The proof of Lemma 2 is finished. □

3. Solvability of problem (16)–(19)

We denote by M the unique solution of problem (12), (13) and by H the unique solution of problem (14), (15) satisfying $M, H \in L^\infty(0, T; \mathbb{W}^{1,r}(D)) \cap W^{1,\infty}(0, T; \mathbb{L}^r(D))$ and the estimates (20), (22), (23), (34), (35), (38) and (40)–(45). Since the uniqueness of strong solutions can be easily proved, we will show the existence of a solution to (16)–(19) and establish some uniform estimates. For this purpose we first construct approximate solutions by using the Galerkin method.

3.1. Approximate solutions

Let P denote the orthogonal projection from $\mathbb{L}^2(D)$ onto $\mathcal{U}_0$ and consider the Stokes operator $-P\Delta : \mathbb{H}^2(D) \cap \mathcal{U} \to \mathcal{U}_0$. The operator $-P\Delta$ is a self-adjoint operator and its inverse is compact. Thus there exist countable sets $(\mu)_{j\geqslant 1}$, $0 < \mu_1 \leqslant \mu_2 \leqslant \cdots$ and $(a_j)_{j\geqslant 1} \subset \mathbb{H}^2(D) \cap \mathcal{U}$ such that $-P\Delta a_j = \mu_j a_j$ $(j \geqslant 1)$ and $(a_j)_{j\geqslant 1}$ form an orthogonal basis in $\mathcal{U}_0$ and an orthogonal basis in $\mathcal{U}$ and $\mathbb{H}^2(D) \cap \mathcal{U}$, with the scalar product $(\nabla u, \nabla v)$ and $(-P\Delta u, -P\Delta v)$, respectively. Moreover, by the Sobolev embedding and a classical regularity result we have $a_j \in C^1(\overline{D}, \mathbb{R}^3)$, for each $j \geqslant 1$. Let us also recall the following result, see Ladyzhenskaya [21, p. 65]: There is a positive number C such that, for any $v \in \mathbb{H}^2(D) \cap \mathcal{U}$,

$$\|\Delta v\| \leqslant C\|P\Delta v\|. \tag{50}$$

For more details on the Stokes operator, see for instance Temam [33, pp. 38, 39] and Ladyzhenskaya [21, pp. 43–45].

We define an approximate solution U_n of problem (16)–(19) by the following scheme. We look for U_n in the form

$$U_n = \sum_{j=1}^n \alpha_j^n(t) a_j.$$

The functions $\alpha_j^n(t)$ will be found from the equation of U_n

$$\begin{aligned}
&\rho \frac{d}{dt}\int_D U_n \cdot a_j\,dx + \rho\int_D (U_\sharp \cdot \nabla)U_n \cdot a_j\,dx + \eta\int_D \nabla U_n \cdot \nabla a_j\,dx \\
&\quad = \mu_0 \int_D (M \cdot \nabla)H \cdot a_j\,dx + \frac{\mu_0}{2}\int_D \big(\operatorname{curl}(M \times H)\big) \cdot a_j\,dx \quad (j = 1, \dots, n)
\end{aligned} \tag{51}$$

with the initial condition

$$U_n|_{t=0} = U_{0n}. \tag{52}$$

Let $\mathcal{X}_n$ denote the space spanned by $a_1, \dots, a_n$. Since $U_0 \in \mathbb{H}^2(D) \cap \mathcal{U}$, we can choose U_{0n} as the orthogonal projection in $\mathbb{H}^2(D) \cap \mathcal{U}$ of U_0 onto $\mathcal{X}_n$.

3.2. Solvability of problem (51), (52) *and uniform estimates*

We have the following result.

Lemma 5. *Problem* (51), (52) *has a unique global-in-time solution* $U_n \in H^2(0, T; \mathcal{X}_n)$ *and the following estimates hold*:

$$\text{(i)} \qquad \left\|\frac{\rho}{2} U_n(t)\right\|^2 + \frac{\eta}{2}\int_0^t \big\|\nabla U_n(s)\big\|^2\,ds \leqslant d_1(t), \quad t \in (0, T), \tag{53}$$

where

$$d_1(t) = \left\|\frac{\rho}{2} U_0\right\|^2_{\mathbb{H}^2(D)} + C\int_0^t \big(\big\|(M \cdot \nabla)H(s)\big\|^2 + \big\|M \wedge H(s)\big\|^2\big)\,ds; \tag{54}$$

(ii)
$$\frac{\rho}{4}\int_0^t \left\|U_{nt}(s)\right\|^2 ds + \frac{\eta}{2}\left\|\nabla U_n(t)\right\|^2 \leqslant d_2(t), \quad t \in (0,T), \tag{55}$$

$$\|U_n\|^2_{L^2(0,T;\mathbb{H}^2(D))} \leqslant C\|d_2\|_{L^2(0,T)}, \tag{56}$$

where

$$d_2(t) = d_2^1(t) + \int_0^t d_2^1(s)d_2^2(s)\exp\left(\int_s^t d_2^2(\sigma)\,d\sigma\right)ds \tag{57}$$

and

$$d_2^1(t) = \frac{\eta}{2}\|U_0\|^2_{\mathbb{H}^2(D)} + C\int_0^t \left(\left\|(M\cdot\nabla)H(s)\right\|^2 + \left\|\operatorname{curl}(M\wedge H)(s)\right\|^2\right)ds \tag{58}$$

and

$$d_2^2(t) = C\left\|U_\sharp(t)\right\|^2_{\mathbb{L}^\infty(D)}; \tag{59}$$

(iii)
$$\frac{\rho}{2}\left\|U_{nt}(t)\right\|^2 + \frac{\eta}{4}\int_0^t \left\|\nabla U_{nt}(s)\right\|^2 ds \leqslant C\left(d_3^1 + d_3^2\right)(t), \quad t \in (0,T), \tag{60}$$

$$\left\|U_n(t)\right\|^2_{\mathbb{H}^2(D)} \leqslant C\left(d_2 + d_3^1 + d_3^2\right)(t), \quad t \in (0,T), \tag{61}$$

where

$$d_3^1 = \|U_0\|_{\mathbb{H}^2(D)} + \left\|U_\sharp(0)\right\|_{\mathbb{H}^2(D)}\|\nabla U_0\| + \left\|(M_0\cdot\nabla)H_0\right\| + \left\|\operatorname{curl}(M_0\wedge H_0)\right\| \tag{62}$$

and

$$d_3^2(t) = \int_0^t \left(\left\|\left[(M\cdot\nabla)H\right]_t(s)\right\|^2_{\mathbb{H}^{-1}(D)} + \left\|[M\wedge H]_t(s)\right\|^2_{\mathbb{H}^{-1}(D)}\right)ds + \|U_{\sharp t}\|^2_{L^\infty(0,t;\mathbb{L}^2(D))}\|d_2\|_{L^2(0,t)}. \tag{63}$$

Here $H_0 = \nabla\varphi_0$ and φ_0 is the unique weak solution in $H^1(D)$ of

$$-\Delta\varphi_0 = \operatorname{div} M_0 - F_0 \quad \text{in } D,$$
$$\frac{\partial\varphi_0}{\partial n} = -M_0\cdot n \quad \text{on } \partial D, \qquad \int_D \varphi_0\,dx = 0,$$

with $F_0 = F(0)$.

Proof. We can rewrite Eq. (51) as a linear system of ordinary differential equations with regular coefficients. The existence of a unique solution U_n then follows from the theory of linear ordinary differential equations. Note that $U_n \in H^2(0,T;\mathcal{X}_n)$. Let us now establish the estimates stated in the lemma.

Proof of (i). We multiply (51) by $\alpha_j^n(t)$ and add the resulting equations for $j = 1,\dots,n$. Using the relations $\int_D (U_n\cdot\nabla)U_n\cdot U_n\,dx = 0$ and $\int_D(\operatorname{curl}(M\wedge H))\cdot U_n\,dx = \int_D (M\wedge H)\cdot\operatorname{curl} U_n\,dx$ we obtain

$$\frac{d}{dt}\left(\frac{\rho}{2}\|U_n\|^2\right) + \eta\|\nabla U_n\|^2 = \mu_0\int_D (M\cdot\nabla)H\cdot U_n\,dx + \frac{\mu_0}{2}\int_D (M\wedge H)\cdot\operatorname{curl} U_n\,dx. \tag{64}$$

We estimate the right-hand side of (64) by using the Poincaré and Young inequalities and then obtain

$$\frac{d}{dt}\left(\frac{\rho}{2}\|U_n\|^2\right) + \frac{\eta}{2}\|\nabla U_n\|^2 \leqslant C\left(\left\|(M\cdot\nabla)H\right\|^2 + \|M\wedge H\|^2\right).$$

Integrating from 0 to t and using the estimate $\|U_{0n}\| \leqslant \|U_{0n}\|_{\mathbb{H}^2(D)} \leqslant \|U_0\|_{\mathbb{H}^2(D)}$ we obtain (53).

Proof of (ii). We multiply (51) by $\frac{d}{dt}\alpha_j^n(t)$ and add the resulting equations for $j = 1,\dots,n$; this gives

$$\rho\int_D |U_{nt}|^2\,dx + \rho\int_D (U_\sharp\cdot\nabla)U_n\cdot U_{nt}\,dx + \frac{\eta}{2}\frac{d}{dt}\int_D |\nabla U_n|^2\,dx$$
$$= \int_D \mu_0(M\cdot\nabla)H\cdot U_{nt}\,dx + \frac{\mu_0}{2}\int_D \left(\operatorname{curl}(M\wedge H)\right)\cdot U_{nt}\,dx. \tag{65}$$

Applying the Young inequality we have

$$\left|\rho\int_D (U_\sharp\cdot\nabla)U_n\cdot U_{nt}\,dx\right| \leqslant \frac{\rho}{4}\int_D |U_{nt}|^2\,dx + C\int_D |U_\sharp|^2|\nabla U_n|^2\,dx$$
$$\leqslant \frac{\rho}{4}\int_D |U_{nt}|^2\,dx + C\|U_\sharp\|^2_{\mathbb{L}^\infty(D)}\int_D |\nabla U_n|^2\,dx. \tag{66}$$

We also have

$$\left|\int_D \mu_0(M\cdot\nabla)H\cdot U_{nt}\,dx\right| \leqslant \frac{\rho}{4}\int_D |U_{nt}|^2\,dx + C\int_D \left|(M\cdot\nabla)H\right|^2\,dx \tag{67}$$

and

$$\left|\int_D \frac{\mu_0}{2}\operatorname{curl}(M\wedge H)\cdot U_{nt}\,dx\right| \leqslant \frac{\rho}{4}\int_D |U_{nt}|^2\,dx + C\int_D \left|\operatorname{curl}(M\wedge H)\right|^2\,dx. \tag{68}$$

Combining (65)–(68), we find

$$\frac{\rho}{4}\int_D |U_{nt}|^2\,dx + \frac{\eta}{2}\frac{d}{dt}\int_D |\nabla U_n|^2\,dx \leqslant C\left(\left\|(M\cdot\nabla)H\right\|^2 + \left\|\operatorname{curl}(M\wedge H)\right\|^2\right) + C\|U_\sharp\|^2_{\mathbb{L}^\infty(D)}\|\nabla U_n\|^2. \tag{69}$$

Applying Gronwall's inequality we obtain (55).

To show (56), we multiply (51) by $-\mu_j\alpha_j^n(t)$ and add the resulting equations for $j=1,\dots,n$. We obtain

$$\eta\int_D |P\Delta U_n|^2\,dx = \rho\int_D U_{nt}\cdot P\Delta U_n\,dx + \rho\int_D (U_\sharp\cdot\nabla)U_n\cdot P\Delta U_n\,dx$$
$$-\int_D \mu_0(M\cdot\nabla)H\cdot P\Delta U_n\,dx - \frac{\mu_0}{2}\int_D \operatorname{curl}(M\wedge H)\cdot P\Delta U_n\,dx.$$

Using the Young inequality to estimate the right-hand side of this equality we obtain

$$\|P\Delta U_n\|^2 \leqslant C\left(\|U_{nt}\|^2 + \|U_\sharp\|^2_{\mathbb{L}^\infty(D)}\|\nabla U_n\|^2 + \left\|(M\cdot\nabla)H\right\|^2 + \left\|\operatorname{curl}(M\wedge H)\right\|^2\right).$$

Integrating from 0 to t and using (50) and (55) together with the estimate $\|U_n\|^2_{\mathbb{H}^2(D)} \leqslant C\|\Delta U_n\|^2$, we deduce (56).

Proof of (iii). Differentiating Eq. (51) with respect to t yields

$$\rho\frac{d}{dt}\int_D U_{nt}\cdot a_j\,dx + \rho\int_D (U_\sharp\cdot\nabla)U_{nt}\cdot a_j\,dx + \eta\int_D \nabla U_{nt}\cdot\nabla a_j\,dx$$
$$= \int_D \mu_0\big[(M\cdot\nabla)H\big]_t\cdot a_j\,dx + \frac{\mu_0}{2}\int_D \big(\operatorname{curl}(M\wedge H)_t\big)\cdot a_j\,dx - \rho\int_D (U_{\sharp t}\cdot\nabla)U_n\cdot a_j\,dx$$

for $j=1,\dots,n$. We multiply this equality by $\frac{d}{dt}\alpha_j^n(t)$ and add the resulting equations for $j=1,\dots,n$. Using the relation $\int_D (U_\sharp\cdot\nabla)U_{nt}\cdot U_{nt}\,dx = 0$, we obtain

$$\frac{d}{dt}\left(\frac{\rho}{2}\|U_{nt}\|^2\right) + \eta\|\nabla U_{nt}\|^2$$
$$= \int_D \mu_0\big[(M\cdot\nabla)H\big]_t\cdot U_{nt}\,dx + \frac{\mu_0}{2}\int_D \big(\operatorname{curl}(M\wedge H)_t\big)\cdot U_{nt}\,dx - \rho\int_D (U_{\sharp t}\cdot\nabla)U_n\cdot U_{nt}\,dx. \tag{70}$$

Using Cauchy–Schwarz, Poincaré and Young inequalities we have

$$\left|\int_D \big[\mu_0(M\cdot\nabla)H\big]_t\cdot U_{nt}\,dx\right| \leqslant \frac{\eta}{4}\|\nabla U_{nt}\|^2 + C\left\|\big[(M\cdot\nabla)H\big]_t\right\|^2_{\mathbb{H}^{-1}(D)},$$
$$\left|\frac{\mu_0}{2}\int_D \big(\operatorname{curl}(M\wedge H)_t\big)\cdot U_{nt}\,dx\right| \leqslant \frac{\eta}{4}\|\nabla U_{nt}\|^2 + C\left\|[M\wedge H]_t\right\|^2_{\mathbb{H}^{-1}(D)}.$$

An integration by parts gives

$$\int_D (U_{\sharp t}\cdot\nabla)U_n\cdot U_{nt}\,dx = -\int_D (U_{\sharp t}\cdot\nabla)U_{nt}\cdot U_n\,dx$$

then, using Hölder and Young inequalities we have

$$\begin{aligned}\left|\rho\int_D (U_{\sharp t}\cdot\nabla)U_n\cdot U_{nt}\,dx\right| &\leqslant \rho\|U_{\sharp t}\|\|\nabla U_{nt}\|\|U_n\|_{\mathbb{L}^\infty(D)}\\ &\leqslant \frac{\eta}{4}\|\nabla U_{nt}\|^2 + C\|U_{\sharp t}\|^2\|U_n\|^2_{\mathbb{H}^2(D)}.\end{aligned}$$

Substitution of these estimates into (70) yields

$$\frac{d}{dt}\left(\frac{\rho}{2}\|U_{nt}\|^2\right) + \frac{\eta}{4}\|\nabla U_{nt}\|^2 \leqslant C\left(\left\|\left[(M\cdot\nabla)H\right]_t\right\|^2_{\mathbb{H}^{-1}(D)} + \left\|[M\wedge H]_t\right\|^2_{\mathbb{H}^{-1}(D)} + \|U_{\sharp t}\|^2\|U_n\|^2_{\mathbb{H}^2(D)}\right).$$

Integrating with respect to t we deduce that

$$\begin{aligned}&\frac{\rho}{2}\left\|U_{nt}(t)\right\|^2 + \frac{\eta}{4}\int_0^t \left\|\nabla U_{nt}(s)\right\|^2 ds\\ &\quad\leqslant \frac{\rho}{2}\left\|U_{nt}(0)\right\|^2 + C\int_0^t \left(\left\|\left[(M\cdot\nabla)H\right]_t(s)\right\|^2_{\mathbb{H}^{-1}(D)} + \left\|[M\wedge H]_t(s)\right\|^2_{\mathbb{H}^{-1}(D)}\right) ds + C\|U_{\sharp t}\|^2_{L^\infty(0,t;\mathbb{L}^2(D))}\|d_2\|_{L^2(0,t)}.\end{aligned} \tag{71}$$

By virtue of (65), at time $t=0$ we have

$$\begin{aligned}\rho\left\|U_{nt}(0)\right\|^2 &= -\rho\int_D \big(U_\sharp(0)\cdot\nabla\big)U_{0n}\cdot U_{nt}(0)\,dx + \eta\int_D \Delta U_{0n}\cdot U_{nt}(0)\,dx\\ &\quad+\int_D \mu_0(M_0\cdot\nabla)H_0\cdot U_{nt}(0)\,dx + \frac{\mu_0}{2}\int_D \big(\operatorname{curl}(M_0\wedge H_0)\big)\cdot U_{nt}(0)\,dx.\end{aligned} \tag{72}$$

Recall that the function $U_\sharp$ belongs to $C([0,T];\mathbb{H}^2(D))$. It is clear that

$$\|\Delta U_{0n}\| \leqslant C\|U_{0n}\|_{\mathbb{H}^2(D)} \leqslant C\|U_0\|_{\mathbb{H}^2(D)} \tag{73}$$

and by the Hölder inequality and the Sobolev embedding $H^2(D)\hookrightarrow L^\infty(D)$ we have

$$\begin{aligned}\left|\int_D \big(U_\sharp(0)\cdot\nabla\big)U_{0n}\cdot U_{nt}(0)\,dx\right| &\leqslant C\left\|U_\sharp(0)\right\|_{\mathbb{L}^\infty(D)}\|\nabla U_{0n}\|\left\|U_{nt}(0)\right\|\\ &\leqslant C\left\|U_\sharp(0)\right\|_{\mathbb{H}^2(D)}\|\nabla U_0\|\left\|U_{nt}(0)\right\|,\end{aligned} \tag{74}$$

and

$$\left|\int_D \mu_0(M_0\cdot\nabla)H_0\cdot U_{nt}(0)\,dx\right| \leqslant C\left\|(M_0\cdot\nabla)H_0\right\|\left\|U_{nt}(0)\right\| \tag{75}$$

and

$$\left|\int_D \big(\operatorname{curl}(M_0\wedge H_0)\big)\cdot U_{nt}(0)\,dx\right| \leqslant C\left\|\operatorname{curl}(M_0\wedge H_0)\right\|\left\|U_{nt}(0)\right\|. \tag{76}$$

Combining (72)–(76) we obtain

$$\left\|U_{nt}(0)\right\| \leqslant C\left(\|U_0\|_{\mathbb{H}^2(D)} + \left\|U_\sharp(0)\right\|_{\mathbb{H}^2(D)}\|\nabla U_0\| + \left\|(M_0\cdot\nabla)H_0\right\| + \left\|\operatorname{curl}(M_0\wedge H_0)\right\|\right). \tag{77}$$

Then, integrating (71) from 0 to t and using (77) we obtain (60).

We prove (61) by arguing as for (56) and using the new estimate (60). The proof of Lemma 5 is complete. □

3.3. Passing to the limit as $n \to \infty$

Lemma 6. *Problem* (16)–(19) *admits a unique global-in-time strong solution* U *satisfying*:

(i) $U \in C([0,T]; \mathcal{U} \cap \mathbb{H}^2(D)) \cap W^{1,\infty}(0,T; \mathbb{L}^2(D)) \cap L^2(0,T; \mathbb{W}^{2,r}(D))$;

(ii)
$$\left\| \frac{\rho}{2} U(t) \right\|^2 + \frac{\eta}{2} \int_0^t \|\nabla U(s)\|^2 \, ds \leqslant d_1(t), \quad t \in (0,T), \tag{78}$$

where d_1 *is given by* (54);

(iii)
$$\frac{\rho}{4} \int_0^t \|U_t(s)\|^2 \, ds + \frac{\eta}{2} \|\nabla U(t)\|^2 \leqslant d_2(t), \quad t \in (0,T), \tag{79}$$

$$\|U\|^2_{L^2(0,T;\mathbb{H}^2(D))} \leqslant C \|d_2\|_{L^2(0,T)}, \tag{80}$$

where d_2 *is given by* (57)–(59);

(iv)
$$\frac{\rho}{2} \|U_t(t)\|^2 + \frac{\eta}{4} \int_0^t \|\nabla U_t(s)\|^2 \, ds \leqslant C\big(d_3^1 + d_3^2\big)(t), \quad t \in (0,T), \tag{81}$$

$$\|U(t)\|^2_{\mathbb{H}^2(D)} \leqslant C\big(d_2 + d_3^1 + d_3^2\big)(t), \quad t \in (0,T), \tag{82}$$

where d_3^1 *and* d_3^2 *are given by* (62) *and* (63);

(v)
$$\|U\|^2_{L^2(0,T;\mathbb{W}^{2,r}(D))} \leqslant C d_4 \tag{83}$$

where

$$d_4 = d_3^1 + d_3^2(T) + \|d_2\|_{L^2(0,T)} \|U_\sharp\|^2_{\mathbb{L}^\infty(D_T)} + \|(M \cdot \nabla)H\|^2_{L^2(0,T;\mathbb{L}^r(D))} + \|\mathrm{curl}(M \wedge H)\|^2_{L^2(0,T;\mathbb{L}^r(D))}. \tag{84}$$

Proof. According to the estimates (53), (55), (56), (60) and (61), there is a subsequence of (U_n) converging to a limit U in a weak sense. The function U belongs to $L^\infty(0,T;\mathcal{U} \cap \mathbb{H}^2(D)) \cap W^{1,\infty}(0,T;\mathbb{L}^2(D)) \cap L^2(0,T;\mathbb{H}^2(D))$ and by a classical embedding result $U \in C([0,T];\mathcal{U} \cap \mathbb{H}^2(D))$. It is a simple matter to show that U is a weak solution to problem (16)–(19); we have, for every $v \in \mathcal{U}$,

$$\rho \frac{d}{dt} \int_D U \cdot v \, dx + \rho \int_D (U_\sharp \cdot \nabla) U \cdot v \, dx + \eta \int_D \nabla U \cdot \nabla v \, dx$$
$$= \mu_0 \int_D (M \cdot \nabla) H \cdot v \, dx + \frac{\mu_0}{2} \int_D \big(\mathrm{curl}(M \wedge H)\big) \cdot v \, dx \quad \text{in } \mathcal{D}'(]0,T[), \tag{85}$$

$$U|_{t=0} = U_0. \tag{86}$$

Moreover, according to the lower semi-continuity of various norms, we have the regularity estimates (78)–(82).

From (85) and the classical results for Stokes equations, see for instance Temam [33, Lemma 2.1, p. 22] it follows that, for a.e. $t \in (0,T)$, there exists $p \in L^2(D)$ such that (U,p) is a weak solution of the Stokes system

$$-\eta \Delta U + \nabla p = G, \qquad \mathrm{div}\, U = 0 \quad \text{in } D, \qquad U = 0 \quad \text{on } \partial D,$$

with $G = -\rho(\partial_t U + (U_\sharp \cdot \nabla)U) + \mu_0 (M \cdot \nabla) H + \frac{\mu_0}{2} \mathrm{curl}(M \wedge H)$. According to the regularity of $U_\sharp$, U, $(M \cdot \nabla)H$ and $\mathrm{curl}(M \wedge H)$, in particular $\partial_t U \in L^2(0,T;\mathbb{L}^6(D))$ according to (i) and the Sobolev embedding $H^1(D) \hookrightarrow L^6(D)$, we have $G \in L^2(0,T;\mathbb{L}^r(D))$. From the regularity results for the Stokes system, see Giaquinta and Modica [16], Giga and Sohr [17], M. Giga, Y. Giga and H. Sohr [18], we have $U \in L^2(0,T;\mathbb{W}^{2,r}(D))$, $p \in L^2(0,T;\mathbb{W}^{1,r}(D))$ and

$$\|U\|_{L^2(0,T;\mathbb{W}^{2,r}(D))} + \|p\|_{L^2(0,T;\mathbb{W}^{1,r}(D))} \leqslant C \|G\|_{L^2(0,T;\mathbb{L}^r(D))}.$$

Using the Hölder and Young inequalities we obtain

$$\begin{aligned}
\|U\|^2_{L^2(0,T;\mathbb{W}^{2,r}(D))} &\leqslant C\big(\|U_t\|^2_{L^2(0,T;\mathbb{L}^6(D))} + \|\nabla U\|^2_{L^2(0,T;(\mathbb{L}^6(D))^3)} \|U_\sharp\|^2_{\mathbb{L}^\infty(D_T)}\big) \\
&\quad + C\big(\|(M \cdot \nabla)H\|^2_{L^2(0,T;\mathbb{L}^r(D))} + \|\mathrm{curl}(M \wedge H)\|^2_{L^2(0,T;\mathbb{L}^r(D))}\big) \\
&\leqslant C\big(\|U_t\|^2_{L^2(0,T;\mathbb{H}^1(D))} + \|U\|^2_{L^2(0,T;\mathbb{H}^2(D))} \|U_\sharp\|^2_{\mathbb{L}^\infty(D_T)}\big) \\
&\quad + C\big(\|(M \cdot \nabla)H\|^2_{L^2(0,T;\mathbb{L}^r(D))} + \|\mathrm{curl}(M \wedge H)\|^2_{L^2(0,T;\mathbb{L}^r(D))}\big)
\end{aligned} \tag{87}$$

from which follows (83), according to (80) and (81). Lemma 6 is proved. □

4. Proof of Theorem 1

To prove the existence we define a sequence of approximate solutions to problem $(\mathcal{P})$, derive some uniform bounds and then prove the convergence of the approximate solutions to a strong solution of problem $(\mathcal{P})$.

4.1. Approximate solutions

Set $(U^0, M^0, H^0) = (0,0,0)$, assuming that (U^n, M^n, H^n) is defined, let $(U^{n+1}, M^{n+1}, H^{n+1})$ be the unique strong solution to the linearized problem (12)–(19) with $(U_\sharp, M_\sharp, H_\sharp)$ replaced by (U^n, M^n, H^n). Thus M^{n+1} satisfies the linear system

$$\partial_t M^{n+1} + \big(U^n \cdot \nabla\big) M^{n+1} - \frac{1}{2}\operatorname{curl} U^n \wedge M^{n+1} + \frac{1}{\tau} M^{n+1} + \beta M^{n+1} \wedge \big(M^n \wedge H^n\big) = \frac{\chi_0}{\tau} H^n \quad \text{in } D_T, \tag{88}$$

supplemented by the initial condition

$$M^{n+1}(0) = M_0 \quad \text{in } D; \tag{89}$$

the function H^{n+1} satisfies the equations and boundary conditions

$$\operatorname{curl} H^{n+1} = 0, \qquad \operatorname{div}\big(H^{n+1} + M^{n+1}\big) = F \quad \text{in } D_T, \tag{90}$$

$$\big(H^{n+1} + M^{n+1}\big)\cdot n = 0 \quad \text{on } (0,T)\times\partial D; \tag{91}$$

the function U^{n+1} satisfies the linear system

$$\rho\big(\partial_t U^{n+1} + \big(U^n\cdot\nabla\big)U^{n+1}\big) - \eta\Delta U^{n+1} + \nabla p^{n+1} = \mu_0\big(M^{n+1}\cdot\nabla\big)H^{n+1} + \frac{\mu_0}{2}\operatorname{curl}\big(M^{n+1}\wedge H^{n+1}\big) \quad \text{in } D_T, \tag{92}$$

$$\operatorname{div} U^{n+1} = 0 \quad \text{in } D_T, \tag{93}$$

supplemented by the boundary and initial conditions

$$U^{n+1} = 0 \quad \text{on } (0,T)\times\partial D, \qquad U^{n+1}(0) = U_0 \quad \text{in } D. \tag{94}$$

We will show that the sequence $(U^n, M^n, H^n)_{n\geqslant 0}$ satisfies some uniform bounds and converges to a local-in-time strong solution to problem $(\mathcal{P})$.

4.2. Uniform bounds

Introduce the function Φ_N defined on $(0,T)$ by

$$\Phi_N(t) = \max_{0\leqslant n\leqslant N}\Big(\sup_{0\leqslant s\leqslant t}\big(1 + \big\|\nabla U^{n+1}(s)\big\| + \big\|M^{n+1}(s)\big\|_{\mathbb{W}^{1,r}(D)}\big)\Big)$$

where N is a large fixed integer.

Lemma 7. *We have*

$$\big\|\nabla U^{n+1}(t)\big\|^2 + \int_0^t \big\|U_t^{n+1}(s)\big\|^2\,ds \leqslant C + C\int_0^t \Phi_N^6(s)\,ds \tag{95}$$

for any $0\leqslant n\leqslant N$ *and* $t\in(0,T)$.

Proof. Multiplying Eq. (92) by U_t^{n+1} we derive the analogue of (65):

$$\rho\int_D \big|U_t^{n+1}\big|^2\,dx + \rho\int_D \big(U^n\cdot\nabla\big)U^{n+1}\cdot U_t^{n+1}\,dx + \frac{\eta}{2}\frac{d}{dt}\int_D \big|\nabla U^{n+1}\big|^2\,dx$$
$$= \int_D \mu_0\big(M^{n+1}\cdot\nabla\big)H^{n+1}\cdot U_t^{n+1}\,dx + \frac{\mu_0}{2}\int_D \big(\operatorname{curl}\big(M^{n+1}\wedge H^{n+1}\big)\big)\cdot U_t^{n+1}\,dx.$$

Using the Hölder and Young inequalities, we estimate the second, fourth and fifth term and then obtain the inequality

$$\frac{\rho}{2}\int_D \big|U_t^{n+1}\big|^2\,dx + \frac{\eta}{2}\frac{d}{dt}\int_D \big|\nabla U^{n+1}\big|^2\,dx$$
$$\leqslant C\big\|U^n\big\|^2_{\mathbb{L}^6(D)}\big\|\nabla U^{n+1}\big\|^2_{(\mathbb{L}^3(D))^3} + C\big\|\big(M^{n+1}\cdot\nabla\big)H^{n+1}\big\|^2 + C\big\|\operatorname{curl}\big(M^{n+1}\wedge H^{n+1}\big)\big\|^2$$
$$\equiv C(I_1 + I_2 + I_3). \tag{96}$$

Arguing as in the proof of Lemma 4 we have

$$\big\|\big(M^{n+1}\cdot\nabla\big)H^{n+1}\big\| \leqslant C\big\|M^{n+1}\big\|_{\mathbb{W}^{1,r}(D)}\big(1+\big\|M^{n+1}\big\|_{\mathbb{W}^{1,r}(D)}\big) \tag{97}$$

and

$$\big\|\operatorname{curl}\big(M^{n+1}\wedge H^{n+1}\big)\big\| \leqslant C\big\|M^{n+1}\big\|_{\mathbb{W}^{1,r}(D)}\big(1+\big\|M^{n+1}\big\|_{\mathbb{W}^{1,r}(D)}\big). \tag{98}$$

Using the interpolation inequality

$$\|\nabla v\|^2_{\mathbb{L}^3(D)} \leqslant C\|\nabla v\|\|v\|_{H^2(D)}, \quad \forall v\in H^2(D)\cap H^1_0(D), \tag{99}$$

the Sobolev embedding $H^1(D)\hookrightarrow L^6(D)$ and the Poincaré inequality, we have

$$I_1 \leqslant C\big\|\nabla U^n\big\|^2\big\|\nabla U^{n+1}\big\|\big\|U^{n+1}\big\|_{\mathbb{H}^2(D)}. \tag{100}$$

On the other hand, there exists $p^{n+1}\in L^2(0,T;H^1(D))$ such that (U^{n+1},p^{n+1}) is a strong solution (for a.e. $t\in(0,T)$) of the Stokes system

$$-\eta\Delta U^{n+1}+\nabla p^{n+1}=G^{n+1}, \qquad \operatorname{div}U^{n+1}=0 \quad \text{in } D, \qquad U^{n+1}=0 \quad \text{on } \partial D, \tag{101}$$

where

$$G^{n+1}=-\rho\big(U^{n+1}_t+\big(U^n\cdot\nabla\big)U^{n+1}\big)+\mu_0\big(M^{n+1}\cdot\nabla\big)H^{n+1}+\frac{\mu_0}{2}\operatorname{curl}\big(M^{n+1}\wedge H^{n+1}\big). \tag{102}$$

From the regularity results for the Stokes system we have

$$\begin{aligned}\big\|U^{n+1}\big\|_{\mathbb{H}^2(D)}+\big\|p^{n+1}\big\|_{H^1(D)} &\leqslant C\big\|G^{n+1}\big\|\\ &\leqslant C\big(\big\|U^{n+1}_t\big\|+\big\|\big(U^n\cdot\nabla\big)U^{n+1}\big\|+\big\|\big(M^{n+1}\cdot\nabla\big)H^{n+1}\big\|+\big\|\operatorname{curl}\big(M^{n+1}\wedge H^{n+1}\big)\big\|\big).\end{aligned}$$

Using the Hölder inequality, the Sobolev embedding $H^1(D)\hookrightarrow L^6(D)$, the Poincaré inequality and (97)–(99), we have

$$\begin{aligned}\big\|U^{n+1}\big\|_{\mathbb{H}^2(D)}+\big\|p^{n+1}\big\|_{H^1(D)} &\leqslant C\big(\big\|U^{n+1}_t\big\|+\big\|U^n\big\|_{\mathbb{L}^6(D)}\big\|\nabla U^{n+1}\big\|_{(\mathbb{L}^3(D)^3)}+\big\|M^{n+1}\big\|_{\mathbb{W}^{1,r}(D)}\big(1+\big\|M^{n+1}\big\|_{\mathbb{W}^{1,r}(D)}\big)\big)\\ &\leqslant C\big(\big\|U^{n+1}_t\big\|+\big\|\nabla U^n\big\|\big\|\nabla U^{n+1}\big\|^{1/2}\big\|U^{n+1}\big\|^{1/2}_{\mathbb{H}^2(D)}+\big\|M^{n+1}\big\|_{\mathbb{W}^{1,r}(D)}\big(1+\big\|M^{n+1}\big\|_{\mathbb{W}^{1,r}(D)}\big)\big)\end{aligned}$$

then Young's inequality yields

$$\begin{aligned}\big\|U^{n+1}\big\|_{\mathbb{H}^2(D)}+\big\|p^{n+1}\big\|_{H^1(D)} &\leqslant \frac{1}{2}\big\|U^{n+1}\big\|_{\mathbb{H}^2(D)}+C\big\|U^{n+1}_t\big\|+C\big\|\nabla U^n\big\|^2\big\|\nabla U^{n+1}\big\|\\ &\quad +C\big\|M^{n+1}\big\|_{\mathbb{W}^{1,r}(D)}\big(1+\big\|M^{n+1}\big\|_{\mathbb{W}^{1,r}(D)}\big)\end{aligned}$$

from which follows

$$\big\|U^{n+1}\big\|_{\mathbb{H}^2(D)} \leqslant C\big(\big\|U^{n+1}_t\big\|+\big\|\nabla U^n\big\|^2\big\|\nabla U^{n+1}\big\|+\big\|M^{n+1}\big\|_{\mathbb{W}^{1,r}(D)}\big(1+\big\|M^{n+1}\big\|_{\mathbb{W}^{1,r}(D)}\big)\big). \tag{103}$$

Combining (100) and (103), together with the Young inequality, we obtain

$$\begin{aligned}I_1 &\leqslant C\big\|\nabla U^n\big\|^2\big\|\nabla U^{n+1}\big\|\big(\big\|U^{n+1}_t\big\|+\big\|\nabla U^n\big\|^2\big\|\nabla U^{n+1}\big\|+\big\|M^{n+1}\big\|_{\mathbb{W}^{1,r}(D)}\big(1+\big\|M^{n+1}\big\|_{\mathbb{W}^{1,r}(D)}\big)\big)\\ &\leqslant \frac{\rho}{4}\big\|U^{n+1}_t\big\|^2+C\big\|\nabla U^n\big\|^4\big\|\nabla U^{n+1}\big\|^2+C\big\|\nabla U^n\big\|^2\big\|\nabla U^{n+1}\big\|\big\|M^{n+1}\big\|_{\mathbb{W}^{1,r}(D)}\big(1+\big\|M^{n+1}\big\|_{\mathbb{W}^{1,r}(D)}\big)\end{aligned}$$

and thus

$$I_1 \leqslant \frac{\rho}{4}\big\|U^{n+1}_t\big\|^2+C\Phi^6_N. \tag{104}$$

According to (97) and (98) we have

$$I_2+I_3 \leqslant C\big\|M^{n+1}\big\|^2_{W^{1,r}(D)}\big(1+\big\|M^{n+1}\big\|^2_{\mathbb{W}^{1,r}(D)}\big) \leqslant C\Phi^4_N \tag{105}$$

then we deduce from (104) and (105) that

$$I_1+I_2+I_3 \leqslant \frac{\rho}{4}\big\|U^{n+1}_t\big\|^2+C\Phi^6_N.$$

Reporting this in (96) and integrating from 0 to t we obtain (95). The proof of Lemma 7 is complete. □

Lemma 8. *We have*

$$\|U_t^{n+1}(t)\|^2 + \int_0^t \|\nabla U_t^{n+1}(s)\|^2\,ds \leqslant C\exp\left(C\int_0^t \Phi_N^8(s)\,ds\right) \tag{106}$$

for any $0 \leqslant n \leqslant N$ *and* $t \in (0, T)$.

Proof. Differentiating Eq. (92) with respect to t and multiplying the result by U_t^{n+1}, we derive the analogue of (70):

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{\rho}{2}\|U_t^{n+1}\|^2\right) + \eta\|\nabla U_t^{n+1}\|^2 \\
&\quad = \int_D \mu_0\big[(M^{n+1}\cdot\nabla)H^{n+1}\big]_t\cdot U_t^{n+1}\,dx + \frac{\mu_0}{2}\int_D \big[\operatorname{curl}(M^{n+1}\wedge H^{n+1})\big]_t\cdot U_t^{n+1}\,dx - \rho\int_D (U_t^n\cdot\nabla)U^{n+1}\cdot U_t^{n+1}\,dx \\
&\quad \equiv J_1 + J_2 + J_3.
\end{aligned} \tag{107}$$

Denote

$$A_N(t) = \sup_{0\leqslant n\leqslant N}\big(1 + \|U_t^{n+1}(t)\|^2\big).$$

We estimate each term J_j $(1 \leqslant j \leqslant 3)$ by using the Sobolev embedding, the Poincaré and Hölder and Young inequalities. For J_1, using (49) we can write

$$\begin{aligned}
|J_1| &\leqslant C\big(\|M_t^{n+1}\| + \|H_t^{n+1}\|\big)\|H^{n+1}\|_{\mathbb{W}^{1,r}(D)}\|\nabla U_t^{n+1}\| + C\big(\|M^{n+1}\|_{\mathbb{W}^{1,r}(D)} + \|H^{n+1}\|_{\mathbb{W}^{1,r}(D)}\big)\|H_t^{n+1}\|\|\nabla U_t^{n+1}\| \\
&\quad + C\big(\|F_t\|\|H^{n+1}\|_{\mathbb{W}^{1,r}(D)} + \|F\|_{\mathbb{W}^{1,r}(D)}\|H_t^{n+1}\|\big)\|U_t^{n+1}\| + C\|H_t^{n+1}\|\|H^{n+1}\|_{\mathbb{W}^{1,r}(D)}\|\nabla U_t^{n+1}\|
\end{aligned}$$

and using the following inequalities (see Lemmas 3 and 4)

$$\|H^{n+1}\|_{\mathbb{W}^{1,r}(D)} \leqslant C\big(\|M^{n+1}\|_{\mathbb{W}^{1,r}(D)} + \|F\|_{L^r(D)}\big), \qquad \|H_t^{n+1}\| \leqslant C\big(\|M_t^{n+1}\| + \|F_t\|\big), \tag{108}$$

we obtain

$$|J_1| \leqslant C\big(1 + \|M_t^{n+1}\|\big)\big(1 + \|M^{n+1}\|_{\mathbb{W}^{1,r}(D)}\big)\|\nabla U_t^{n+1}\| + C\big(1 + \|M^{n+1}\|_{\mathbb{W}^{1,r}(D)}\big)\|U_t^{n+1}\| + C\big(1 + \|M_t^{n+1}\|\big)\|U_t^{n+1}\|.$$

On the other hand, it follows from (88) that

$$\begin{aligned}
\|M_t^{n+1}\| &\leqslant C\big(\|U^n\|_{\mathbb{L}^6(D)}\|\nabla M^{n+1}\|_{(\mathbb{L}^3(D))^3} + \|\nabla U^n\|\|M^{n+1}\|_{\mathbb{W}^{1,r}(D)} + \|M^{n+1}\|\big) \\
&\quad + C\big(\|M^{n+1}\|\|M^n\|_{\mathbb{W}^{1,r}(D)}\|H^n\|_{\mathbb{W}^{1,r}(D)} + \|H^n\|\big) \\
&\leqslant C\big(\|\nabla U^n\|\|M^{n+1}\|_{\mathbb{W}^{1,r}(D)} + \|\nabla U^n\|\|M^{n+1}\|_{\mathbb{W}^{1,r}(D)} + \|M^{n+1}\|\big) \\
&\quad + C\big(\|M^{n+1}\|\|M^n\|_{\mathbb{W}^{1,r}(D)}\big(1 + \|M^n\|_{\mathbb{W}^{1,r}(D)}\big) + \big(1 + \|M^n\|_{\mathbb{W}^{1,r}(D)}\big)\big) \\
&\leqslant C\Phi_N^3.
\end{aligned} \tag{109}$$

Then

$$\begin{aligned}
|J_1| &\leqslant C\big(\Phi_N^4\|\nabla U_t^{n+1}\| + \Phi_N^3\|U_t^{n+1}\|\big) \\
&\leqslant \frac{\eta}{8}\|\nabla U_t^{n+1}\|^2 + C\big(\Phi_N^8 + \Phi_N^3 A_N^{1/2}\big).
\end{aligned} \tag{110}$$

An integration by parts gives

$$\int_D \big[\operatorname{curl}(M^{n+1}\wedge H^{n+1})\big]_t\cdot U_t^{n+1}\,dx = \int_D \big[M^{n+1}\wedge H^{n+1}\big]_t\cdot\operatorname{curl}U_t^{n+1}\,dx$$

then applying the Young inequality we obtain

$$|J_2| \leqslant \frac{\eta}{8}\|\nabla U_t^{n+1}\|^2 + C\big\|\big[M^{n+1}\wedge H^{n+1}\big]_t\big\|^2. \tag{111}$$

Arguing as in Lemma 4(iii) we have

$$\big\|\big[M^{n+1}\wedge H^{n+1}\big]_t\big\| \leqslant C\|M_t^{n+1}\|\big(1 + \|M^{n+1}\|_{\mathbb{W}^{1,r}(D)}\big) + C\|M^{n+1}\|_{\mathbb{W}^{1,r}(D)}\big(1 + \|M_t^{n+1}\|\big).$$

From this estimate, (109) and (111), we deduce that

$$|J_2| \leqslant \frac{\eta}{8}\left\|\nabla U_t^{n+1}\right\|^2 + C\Phi_N^8. \tag{112}$$

Using the Sobolev embedding, the Hölder and Young inequalities and (99) we have

$$\begin{aligned}
|J_3| &\leqslant C\left\|U_t^n\right\|_{\mathbb{L}^6(D)}\left\|U_t^{n+1}\right\|_{\mathbb{L}^3(D)}\left\|\nabla U^{n+1}\right\| \\
&\leqslant C\left\|\nabla U_t^n\right\|\left\|U_t^{n+1}\right\|^{1/2}\left\|\nabla U_t^{n+1}\right\|^{1/2}\left\|\nabla U^{n+1}\right\| \\
&\leqslant \frac{\eta}{4}\left\|\nabla U_t^n\right\|^2 + C\left\|U_t^{n+1}\right\|\left\|\nabla U_t^{n+1}\right\|\left\|\nabla U^{n+1}\right\|^2 \\
&\leqslant \frac{\eta}{4}\left\|\nabla U_t^n\right\|^2 + \frac{\eta}{4}\left\|\nabla U_t^{n+1}\right\|^2 + C\Phi_N^4 A_N.
\end{aligned} \tag{113}$$

Combining (107), (110), (112) and (113), we deduce that

$$\frac{d}{dt}\left(\frac{\rho}{2}\left\|U_t^{n+1}\right\|^2\right) + \frac{\eta}{2}\left\|\nabla U_t^{n+1}\right\|^2 \leqslant \frac{\eta}{4}\left\|\nabla U_t^n\right\|^2 + C\Phi_N^8 A_N,$$

integrating with respect to t we obtain

$$\frac{\rho}{2}\left\|U_t^{n+1}(t)\right\|^2 + \frac{\eta}{2}\int_0^t\left\|\nabla U_t^{n+1}(s)\right\|^2 ds \leqslant \frac{\rho}{2}\left\|U_t^{n+1}(0)\right\|^2 + C\int_0^t \Phi_N^8(s)A_N(s)\,ds + \frac{\eta}{4}\int_0^t\left\|\nabla U_t^n(s)\right\|^2 ds$$

and since, arguing as in the proof of Lemma 5(iii), $\|U_t^{n+1}(0)\|^2 \leqslant C$, we have

$$\frac{\rho}{2}\left\|U_t^{n+1}(t)\right\|^2 + \frac{\eta}{2}\int_0^t\left\|\nabla U_t^{n+1}(s)\right\|^2 ds \leqslant C\left(1 + \int_0^t \Phi_N^8(s)A_N(s)\,ds\right) + \frac{\eta}{4}\int_0^t\left\|\nabla U_t^n(s)\right\|^2 ds. \tag{114}$$

Using induction, this inequality implies

$$\begin{aligned}
\int_0^t\left\|\nabla U_t^{n+1}(s)\right\|^2 ds &\leqslant C\left(1 + \int_0^t \Phi_N^8(s)A_N(s)\,ds\right) + \frac{1}{2}\int_0^t\left\|\nabla U_t^n(s)\right\|^2 ds \\
&\leqslant C\left(1 + \frac{1}{2} + \frac{1}{4}\cdots\right)\left(1 + \int_0^t \Phi_N^8(s)A_N(s)\,ds\right) \\
&\leqslant C\left(1 + \int_0^t \Phi_N^8(s)A_N(s)\,ds\right)
\end{aligned}$$

for any $0 \leqslant n \leqslant N$. Then we deduce from (114) that

$$\left\|U_t^{n+1}(t)\right\|^2 + \int_0^t\left(\left\|\nabla U_t^{n+1}(s)\right\|^2\right) ds \leqslant C\left(1 + \int_0^t \Phi_N^8(s)A_N(s)\,ds\right) \tag{115}$$

for any $0 \leqslant n \leqslant N$. We then have

$$A_N(t) \leqslant C\left(1 + \int_0^t \Phi_N^8(s)A_N(s)\,ds\right)$$

and Gronwall's inequality yields

$$A_N(t) \leqslant C\exp\left(C\int_0^t \Phi_N^8\,ds\right) \tag{116}$$

and the lemma follows from (115) and (116). □

Lemma 9. *There is a positive number K_1, depending only on r, such that*

$$\left\|M^{n+1}(t)\right\|_{\mathbb{W}^{1,r}(D)} \leqslant C\exp\left(C\exp\left(C\int_0^t \Phi_N^{K_1}(s)\,ds\right)\right) \tag{117}$$

for any $0 \leqslant n \leqslant N$ *and* $t \in (0,T)$.

Proof. By (20) and (21) it holds that

$$\left\|M^{n+1}(t)\right\|^r_{\mathbb{L}^r(D)} \leqslant C + C\int_0^t \left\|H^n(s)\right\|^r_{\mathbb{L}^r(D)}\,ds$$

and using (34) we obtain

$$\left\|M^{n+1}(t)\right\|^r_{\mathbb{L}^r(D)} \leqslant C + C\int_0^t \Phi_N^r(s)\,ds. \tag{118}$$

By (22) we have

$$\left\|\nabla M^{n+1}(t)\right\|^r_{(\mathbb{L}^r(D))^3} \leqslant b_2(t) \tag{119}$$

with $b_2(t) = b_2^1(t) + \int_0^t b_2^1(s)b_2^2(s)\exp(\int_s^t b_2^2(\sigma)\,d\sigma)\,ds$. Here b_2^1 and b_2^2 defined by (25) and (26) where we replace $M_\sharp$, $H_\sharp$ and $U_\sharp$ by M^n, H^n and U^n, respectively. Clearly,

$$b_2(t) \leqslant b_2^1(t) + \left(\int_0^t b_2^1(s)b_2^2(s)\,ds\right)\exp\left(\int_0^t b_2^2(\sigma)\,d\sigma\right).$$

Using (34) and (35) we deduce that

$$b_2^1(t) \leqslant C + C\int_0^t \Phi_N^{r+2}(s)\,ds + C\int_0^t \Phi_N^r(s)\left\|U^n(s)\right\|_{\mathbb{W}^{2,r}(D)}\,ds,$$

applying Young's inequality to the last term and since $r > 3$ we obtain

$$b_2^1(t) \leqslant C + C\int_0^t \Phi_N^{2r}(s)\,ds + C\int_0^t \left\|U^n(s)\right\|^2_{\mathbb{W}^{2,r}(D)}\,ds.$$

Since Φ_N is a nondecreasing function we also have, for $s \leqslant t$,

$$b_2^1(s) \leqslant C\Phi_N^{2r}(s) + C\int_0^t \left\|U^n(\sigma)\right\|^2_{\mathbb{W}^{2,r}(D)}\,d\sigma. \tag{120}$$

On the other hand we have

$$b_2^2(s) \leqslant C\left(\Phi_N^2(s) + C\left\|U^n(s)\right\|_{\mathbb{W}^{2,r}(D)}\right) \tag{121}$$

then

$$\exp\left(\int_0^t b_2^2(\sigma)\,d\sigma\right)ds \leqslant \exp\left(C\int_0^t \Phi_N^2(s)\,ds\right)\exp\left(C\int_0^t \left\|U^n(s)\right\|_{\mathbb{W}^{2,r}(D)}\,ds\right).$$

Using Young's inequality, we deduce from (120) and (121) that

$$\int_0^t b_2^1(s)b_2^2(s)\,ds \leqslant C\int_0^t \Phi_N^{4r}(s)\,ds + C\left(\int_0^t \left\|U^n(s)\right\|^2_{\mathbb{W}^{2,r}(D)}\,ds\right)^2 + C\int_0^t \left\|U^n(s)\right\|^2_{\mathbb{W}^{2,r}(D)}\,ds$$

therefore

$$b_2(t) \leqslant C + C\int_0^t \Phi_N^{2r}(s)\,ds + C\int_0^t \|U^n(s)\|_{\mathbb{W}^{2,r}(D)}^2\,ds + C\left[\int_0^t \Phi_N^{4r}(s)\,ds + \left(\int_0^t \|U^n(s)\|_{\mathbb{W}^{2,r}(D)}^2\,ds\right)^2 + \int_0^t \|U^n(s)\|_{\mathbb{W}^{2,r}(D)}^2\,ds\right]$$
$$\times \exp\left(C\int_0^t \Phi_N^2(s)\,ds\right)\exp\left(C\int_0^t \|U^n(s)\|_{\mathbb{W}^{2,r}(D)}\,ds\right). \tag{122}$$

Let us now give an estimate of $\|U^n\|_{\mathbb{W}^{2,r}(D)}$ (for any $0 \leqslant n \leqslant N$). For this we apply the elliptic regularity results for the Stokes system (101), (102). Remind that $3 < r \leqslant 6$. Assume first that $r < 6$. We have

$$\begin{aligned}\|U^{n+1}\|_{\mathbb{W}^{2,r}(D)} &\leqslant C\|G^{n+1}\|_{\mathbb{L}^r(D)}\\ &\leqslant C\left(\|U_t^{n+1}\|_{\mathbb{L}^r(D)} + \|(U^n\cdot\nabla)U^{n+1}\|_{\mathbb{L}^r(D)}\right) + C\left(\|(M^{n+1}\cdot\nabla)H^{n+1}\|_{\mathbb{L}^r(D)} + \|\operatorname{curl}(M^{n+1}\wedge H^{n+1})\|_{\mathbb{L}^r(D)}\right).\end{aligned}$$

We use the Hölder inequality, the Sobolev embedding, (35), (97), (98) and the interpolation inequality to obtain

$$\begin{aligned}\|U^{n+1}\|_{\mathbb{W}^{2,r}(D)} &\leqslant C\|\nabla U_t^{n+1}\| + C\|U^n\|_{\mathbb{L}^6(D)}\|\nabla U^{n+1}\|_{(\mathbb{L}^{6r/(6-r)}(D))^3} + C\|M^{n+1}\|_{\mathbb{W}^{1,r}(D)}\left(1+\|M^{n+1}\|_{\mathbb{W}^{1,r}(D)}\right)\\ &\leqslant C\|\nabla U_t^{n+1}\| + C\|\nabla U^n\|\|\nabla U^{n+1}\|^{1-\theta}\|U^{n+1}\|_{\mathbb{W}^{2,r}(D)}^{\theta} + C\|M^{n+1}\|_{\mathbb{W}^{1,r}(D)}\left(1+\|M^{n+1}\|_{\mathbb{W}^{1,r}(D)}\right)\end{aligned}$$

with $\theta = (4r-6)/3r \in (0,1)$. Applying Young's inequality we obtain

$$\|U^{n+1}\|_{\mathbb{W}^{2,r}(D)} \leqslant C\left(\|\nabla U_t^{n+1}\| + \Phi_N^{\delta} + \Phi_N^2\right) + \frac{1}{2}\|U^{n+1}\|_{\mathbb{W}^{2,r}(D)}$$

with $\delta = (2-\theta)/(1-\theta) > 1$ and then

$$\|U^{n+1}\|_{\mathbb{W}^{2,r}(D)} \leqslant C\left(\|\nabla U_t^{n+1}\| + \Phi_N^{\delta} + \Phi_N^2\right),$$

hence

$$\|U^{n+1}\|_{\mathbb{W}^{2,r}(D)}^2 \leqslant C\left(\|\nabla U_t^{n+1}\|^2 + \Phi_N^{2\delta} + \Phi_N^4\right). \tag{123}$$

We estimate similarly $\|U^{n+1}\|_{\mathbb{W}^{2,6}(D)}$ (for any $0 \leqslant n \leqslant N$). Using the Sobolev embedding $W^{1,r'}(D) \hookrightarrow C^{0,\alpha}(\overline{D})$, for $3 < r' < 6$ and $\alpha = 1 - \frac{3}{r'}$, we can write

$$\begin{aligned}\|U^{n+1}\|_{\mathbb{W}^{2,6}(D)} &\leqslant C\|\nabla U_t^{n+1}\| + C\|U^n\|_{\mathbb{L}^6(D)}\|\nabla U^{n+1}\|_{(\mathbb{L}^\infty(D))^3} + C\|M^{n+1}\|_{\mathbb{W}^{1,r}(D)}\left(1+\|M^{n+1}\|_{\mathbb{W}^{1,r}(D)}\right)\\ &\leqslant C\|\nabla U_t^{n+1}\| + C\|\nabla U^n\|\|\nabla U^{n+1}\|_{(\mathbb{W}^{1,r'}(D))^3} + C\|M^{n+1}\|_{\mathbb{W}^{1,r}(D)}\left(1+\|M^{n+1}\|_{\mathbb{W}^{1,r}(D)}\right)\\ &\leqslant C\|\nabla U_t^{n+1}\| + C\|\nabla U^n\|\|U^{n+1}\|_{\mathbb{H}^2(D)}^{1-\theta'}\|U^{n+1}\|_{\mathbb{W}^{2,6}(D)}^{\theta'} + C\|M^{n+1}\|_{\mathbb{W}^{1,r}(D)}\left(1+\|M^{n+1}\|_{\mathbb{W}^{1,r}(D)}\right)\end{aligned}$$

with $\theta' = (3r'-6)/2r' \in (0,1)$. Applying the Young inequality and using the estimate (as a consequence of (103))

$$\|U^{n+1}\|_{\mathbb{H}^2(D)}^2 \leqslant C\left(\Phi_N^6 + A_N\right) \tag{124}$$

we obtain

$$\|U^{n+1}\|_{\mathbb{W}^{2,6}(D)} \leqslant C\left(\|\nabla U_t^{n+1}\| + \Phi_N^{\delta'} + \Phi_N^{\frac{2}{1-\theta'}} + A_N\right) + \frac{1}{2}\|U^{n+1}\|_{\mathbb{W}^{2,6}(D)},$$

with $\delta' = (4-3\theta')/(1-\theta') > 1$ and then

$$\|U^{n+1}\|_{\mathbb{W}^{2,6}(D)}^2 \leqslant C\left(\|\nabla U_t^{n+1}\|^2 + \Phi_N^{2\delta'} + \Phi_N^{\frac{4}{1-\theta'}} + A_N^2\right). \tag{125}$$

Then (117) follows from (106), (118), (119), (122), (123) and (125). Lemma 9 is proved. □

Lemma 10. *There is a time $T^* > 0$ such that*

$$\int_0^{T^*}\left(\|U_t^{n+1}(s)\|_{\mathbb{H}^1(D)}^2 + \|U^{n+1}(s)\|_{\mathbb{W}^{2,r}(D)}^2\right)ds + \sup_{0\leqslant t\leqslant T^*}\left(\|U^{n+1}(t)\|_{\mathbb{H}^2(D)} + \|M^{n+1}(t)\|_{\mathbb{W}^{1,r}(D)}\right) \leqslant C, \tag{126}$$

for any $n \geqslant 0$.

Proof. It follows from Lemmas 7 and 9 that there is a positive number K, depending only on r, such that Φ_N satisfies the integral inequality

$$\Phi_N(t) \leqslant C \exp\left(C \exp\left(C \int_0^t \Phi_N^K(s)\, ds\right)\right).$$

We deduce as in [4] that there is a time $T^* > 0$ such that $\Phi_N(t) \leqslant C$, for all $t \in (0, T^*)$. According to (116) and (124) we have $\|U^{n+1}(t)\|_{\mathbb{H}^2(D)} \leqslant C$, for any $t \in (0, T^*)$, then, using Lemmas 7–9 we easily derive (126). The prof of Lemma 10 is complete. □

Remark 1. We deduce from (108) and (126) that

$$\sup_{0 \leqslant t \leqslant T^*} \left\| H^{n+1}(t) \right\|_{\mathbb{W}^{1,r}(D)} \leqslant C.$$

4.3. Convergence

Denote

$$\widetilde{U}^{n+1} = U^{n+1} - U^n, \qquad \widetilde{M}^{n+1} = M^{n+1} - M^n, \qquad \widetilde{H}^{n+1} = H^{n+1} - H^n,$$

and

$$\widetilde{p}^{n+1} = p^{n+1} - p^n, \qquad \widetilde{Q}_i^{n+1} = Q_i^{n+1} - Q_i^n \quad (i = 1, 2)$$

where $Q_1^m = (M^m \cdot \nabla)H^m$, $Q_2^m = M^m \wedge H^m$ $(m = n, n+1)$. We easily verify that the functions $\widetilde{U}^{n+1}$, $\widetilde{M}^{n+1}$, $\widetilde{H}^{n+1}$ and $\widetilde{p}^{n+1}$ satisfy the system of linear equations

$$\rho\big(\partial_t \widetilde{U}^{n+1} + \big(U^n \cdot \nabla\big)\widetilde{U}^{n+1}\big) - \eta \Delta \widetilde{U}^{n+1} + \nabla \widetilde{p}^{n+1} = -\rho\big(\widetilde{U}^n \cdot \nabla\big)U^n + \mu_0 \widetilde{Q}_1^{n+1} + \frac{\mu_0}{2} \operatorname{curl} \widetilde{Q}_2^{n+1} \quad \text{in } D_T, \tag{127}$$

$$\operatorname{div} \widetilde{U}^{n+1} = 0 \quad \text{in } D_T, \tag{128}$$

$$\begin{aligned} &\partial_t \widetilde{M}^{n+1} + \big(U^n \cdot \nabla\big)\widetilde{M}^{n+1} - \frac{1}{2} \operatorname{curl} U^n \wedge \widetilde{M}^{n+1} + \frac{1}{\tau}\widetilde{M}^{n+1} + \beta \widetilde{M}^{n+1} \wedge Q_2^n \\ &\quad = -\big(\widetilde{U}^n \cdot \nabla\big)M^n + \frac{1}{2} \operatorname{curl} \widetilde{U}^n \wedge M^n - \beta M^n \wedge \widetilde{Q}_2^n + \frac{\chi_0}{\tau}\widetilde{H}^n \quad \text{in } D_T, \end{aligned} \tag{129}$$

$$\operatorname{curl} \widetilde{H}^{n+1} = 0, \qquad \operatorname{div}\big(\widetilde{H}^{n+1} + \widetilde{M}^{n+1}\big) = 0 \quad \text{in } D_T, \tag{130}$$

and the boundary and initial conditions

$$\widetilde{U}^{n+1} = 0, \qquad \big(\widetilde{H}^{n+1} + \widetilde{M}^{n+1}\big) \cdot n = 0 \quad \text{on } (0, T) \times \partial D, \tag{131}$$

$$\widetilde{U}^{n+1}(0) = 0, \qquad \widetilde{M}^{n+1}(0) = 0 \quad \text{in } D. \tag{132}$$

Multiplying Eq. (127) by $\widetilde{U}^{n+1}$ and integrating over D yields

$$\begin{aligned} &\frac{d}{dt}\left(\frac{\rho}{2}\big\|\widetilde{U}^{n+1}\big\|^2\right) + \eta \big\|\nabla \widetilde{U}^{n+1}\big\|^2 \\ &\quad = -\rho \int_D \big(\widetilde{U}^n \cdot \nabla\big)U^n \cdot \widetilde{U}^{n+1}\, dx + \mu_0 \int_D \widetilde{Q}_1^{n+1} \cdot \widetilde{U}^{n+1}\, dx + \frac{\mu_0}{2} \int_D \operatorname{curl} \widetilde{Q}_2^{n+1} \cdot \widetilde{U}^{n+1}\, dx \\ &\quad \equiv R_1 + R_2 + R_3. \end{aligned} \tag{133}$$

Using Hölder, Sobolev and Young inequalities, together with the uniform estimate (126), we have

$$\begin{aligned} |R_1| &\leqslant C \big\|\widetilde{U}^n\big\| \big\|\nabla U^n\big\|_{(\mathbb{L}^3(D))^3} \big\|\widetilde{U}^{n+1}\big\|_{\mathbb{L}^6(D)} \\ &\leqslant C \big\|\widetilde{U}^n\big\| \big\|\nabla U^n\big\|_{(\mathbb{L}^3(D))^3} \big\|\nabla \widetilde{U}^{n+1}\big\| \\ &\leqslant \frac{\eta}{8}\big\|\nabla \widetilde{U}^{n+1}\big\|^2 + C \big\|\widetilde{U}^n\big\|^2 \big\|\nabla U^n\big\|^2_{(\mathbb{L}^3(D))^3} \\ &\leqslant \frac{\eta}{8}\big\|\nabla \widetilde{U}^{n+1}\big\|^2 + C \big\|\widetilde{U}^n\big\|^2. \end{aligned} \tag{134}$$

On the other hand, since (see (47)) for any $w \in \mathbb{H}_0^1(D)$ and $m = n, n+1$,

$$\int_D \big(\big(M^m \cdot \nabla\big)H^m\big) \cdot w\, dx = -\int_D \big(\big(M^m + H^m\big) \cdot \nabla\big)w \cdot H^m\, dx - \int_D F H^m \cdot w\, dx + \int_D \frac{|H^m|^2}{2} \operatorname{div} w\, dx,$$

we can write

$$\int_D \widetilde{Q}_1^{n+1} \cdot w\,dx = -\int_D \big(\big(\widetilde{M}^{n+1} + \widetilde{H}^{n+1}\big)\cdot\nabla\big)w \cdot H^{n+1}\,dx - \int_D \big(\big(M^n + H^n\big)\cdot\nabla\big)w \cdot \widetilde{H}^{n+1}\,dx$$
$$- \int_D F\widetilde{H}^{n+1} \cdot w\,dx + \int_D \frac{1}{2}\widetilde{H}^{n+1} \cdot \big(H^{n+1} + H^n\big)\operatorname{div} w\,dx.$$

Then, using the inequality

$$\big\|\widetilde{H}^{n+1}\big\| \leqslant C\big\|\widetilde{M}^{n+1}\big\| \tag{135}$$

and (126) we have

$$\begin{aligned}|R_2| &\leqslant C\big\|\widetilde{M}^{n+1} + \widetilde{H}^{n+1}\big\|\big\|H^{n+1}\big\|_{\mathbb{L}^\infty(D)}\big\|\nabla\widetilde{U}^{n+1}\big\| + C\big\|M^n + H^n\big\|_{\mathbb{L}^\infty(D)}\big\|\widetilde{H}^{n+1}\big\|\big\|\nabla\widetilde{U}^{n+1}\big\| \\ &\quad + C\big\|\widetilde{H}^{n+1}\big\|\big\|\nabla\widetilde{U}^{n+1}\big\| + C\big\|\widetilde{H}^{n+1}\big\|\big\|H^{n+1} + H^n\big\|\big\|\nabla\widetilde{U}^{n+1}\big\| \\ &\leqslant C\big\|\widetilde{M}^{n+1}\big\|\big\|\nabla\widetilde{U}^{n+1}\big\|\end{aligned}$$

and Young's inequality yields

$$|R_2| \leqslant \frac{\eta}{8}\big\|\nabla\widetilde{U}^{n+1}\big\|^2 + C\big\|\widetilde{M}^{n+1}\big\|^2. \tag{136}$$

Writing $\widetilde{Q}_2^{n+1} = \widetilde{M}^{n+1} \wedge H^{n+1} + M^n \wedge \widetilde{H}^{n+1}$ and integrating by parts we obtain

$$R_3 = \frac{\mu_0}{2}\int_D \big(\widetilde{M}^{n+1} \wedge H^{n+1} + M^n \wedge \widetilde{H}^{n+1}\big)\cdot\operatorname{curl}\widetilde{U}^{n+1}\,dx,$$

then

$$\begin{aligned}|R_3| &\leqslant C\big(\big\|\widetilde{M}^{n+1}\big\|\big\|H^{n+1}\big\|_{\mathbb{L}^\infty(D)} + \big\|M^n\big\|_{\mathbb{L}^\infty(D)}\big\|\widetilde{H}^{n+1}\big\|\big)\big\|\nabla\widetilde{U}^{n+1}\big\| \\ &\leqslant \frac{\eta}{4}\big\|\nabla\widetilde{U}^{n+1}\big\|^2 + C\big\|\widetilde{M}^{n+1}\big\|^2\big\|H^{n+1}\big\|^2_{\mathbb{L}^\infty(D)} + C\big\|M^n\big\|^2_{\mathbb{L}^\infty(D)}\big\|\widetilde{H}^{n+1}\big\|^2 \\ &\leqslant \frac{\eta}{4}\big\|\nabla\widetilde{U}^{n+1}\big\|^2 + C\big\|\widetilde{M}^{n+1}\big\|^2.\end{aligned} \tag{137}$$

We deduce from (133), (134), (136) and (137) that

$$\frac{d}{dt}\left(\frac{\rho}{2}\big\|\widetilde{U}^{n+1}\big\|^2\right) + \frac{\eta}{2}\big\|\nabla\widetilde{U}^{n+1}\big\|^2 \leqslant C\big\|\widetilde{U}^n\big\|^2 + C\big\|\widetilde{M}^{n+1}\big\|^2$$

from which follows by integration from 0 to t

$$\frac{\rho}{2}\big\|\widetilde{U}^{n+1}(t)\big\|^2 + \frac{\eta}{2}\int_0^t\big\|\nabla\widetilde{U}^{n+1}(s)\big\|^2\,ds \leqslant C\int_0^t\big\|\widetilde{U}^n(s)\big\|^2\,ds + C\int_0^t\big\|\widetilde{M}^{n+1}(s)\big\|^2\,ds. \tag{138}$$

Multiplying (129) by $\widetilde{M}^{n+1}$ and integrating over D yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\big\|\widetilde{M}^{n+1}\big\|^2 + \frac{1}{\tau}\big\|\widetilde{M}^{n+1}\big\|^2 \\ &\quad = \int_D \left(-\big(\widetilde{U}^n\cdot\nabla\big)M^n + \frac{1}{2}\operatorname{curl}\widetilde{U}^n \wedge M^n - \beta M^n \wedge \widetilde{Q}_2^n + \frac{\chi_0}{\tau}\widetilde{H}^n\right)\cdot\widetilde{M}^{n+1}\,dx \\ &\quad \equiv R_4.\end{aligned} \tag{139}$$

We estimate R_4 by using Hölder and Sobolev and Young inequalities, together with the uniform estimate (126). We have

$$\begin{aligned}|R_4| &\leqslant C\big\|\widetilde{U}^n\big\|_{\mathbb{L}^6(D)}\big\|\nabla M^n\big\|_{(\mathbb{L}^3(D))^3}\big\|\widetilde{M}^{n+1}\big\| + C\big\|\widetilde{H}^n\big\|\big\|\widetilde{M}^{n+1}\big\| + C\big\|M^n\big\|_{\mathbb{L}^\infty(D)}\big\|\widetilde{M}^{n+1}\big\|\big(\big\|\widetilde{U}^n\big\|_{\mathbb{H}^1(D)} + \big\|\widetilde{Q}_2^n\big\|\big) \\ &\leqslant C\big\|\nabla\widetilde{U}^n\big\|\big\|\widetilde{M}^{n+1}\big\| + C\big\|\widetilde{H}^n\big\|\big\|\widetilde{M}^{n+1}\big\| + C\big\|\widetilde{M}^{n+1}\big\|\big\|\widetilde{Q}_2^n\big\|.\end{aligned}$$

Since $\widetilde{Q}_2^n = \widetilde{M}^n \wedge H^n + M^{n-1} \wedge \widetilde{H}^n$ and $\|\widetilde{H}^n\| \leqslant C\|\widetilde{M}^n\|$, we have $\|\widetilde{Q}_2^n\| \leqslant C\|\widetilde{M}^n\|$ and then

$$\begin{aligned}|R_4| &\leqslant C\big\|\nabla\widetilde{U}^n\big\|\big\|\widetilde{M}^{n+1}\big\| + C\big\|\widetilde{M}^{n+1}\big\|\big\|\widetilde{M}^n\big\| \\ &\leqslant \frac{1}{2\tau}\big\|\widetilde{M}^{n+1}\big\|^2 + C\big(\big\|\nabla\widetilde{U}^n\big\|^2 + \big\|\widetilde{M}^n\big\|^2\big).\end{aligned}$$

Then we deduce from (139) that

$$\frac{1}{2}\frac{d}{dt}\|\widetilde{M}^{n+1}\|^2 + \frac{1}{2\tau}\|\widetilde{M}^{n+1}\|^2 \leqslant C\big(\|\widetilde{M}^n\|^2 + \|\nabla\widetilde{U}^n\|^2\big)$$

from which follows

$$\|\widetilde{M}^{n+1}(t)\|^2 \leqslant C\int_0^t \big(\|\widetilde{M}^n(s)\|^2 + \|\nabla\widetilde{U}^n(s)\|^2\big)\,ds. \tag{140}$$

Combining (138) and (140) we obtain

$$\|\widetilde{U}^{n+1}(t)\|^2 + \|\widetilde{M}^{n+1}(t)\|^2 + \int_0^t \|\nabla\widetilde{U}^{n+1}(s)\|^2\,ds \leqslant C\int_0^t \left(\|\widetilde{U}^n(s)\|^2 + \|\widetilde{M}^n(s)\|^2 + \int_0^s \|\nabla\widetilde{U}^n(s_1)\|^2\,ds_1\right)ds.$$

Thus, setting

$$y^{n+1}(t) = \|\widetilde{U}^{n+1}(t)\|^2 + \|\widetilde{M}^{n+1}(t)\|^2 + \int_0^t \|\nabla\widetilde{U}^{n+1}(s)\|^2\,ds,$$

we have $y^{n+1}(t) \leqslant C\int_0^t y^n(s)\,ds$ and then we show by induction that

$$y^{n+1}(t) \leqslant \frac{(Ct)^n}{n!}\sup_{0\leqslant s\leqslant t} y^1(s), \quad t \in (0, T^*).$$

We conclude from this inequality and (135) that the sequence $(U^n, M^n, H^n)_{n\geqslant 0}$ converges to a limit (U, M, H) in $L^\infty(0, T^*; \mathbb{L}^2(D))$. Moreover, $(U^n)_{n\geqslant 0}$ converges in $L^2(0, T^*; \mathbb{H}^1(D))$.

4.4. End of the proof of Theorem 1

4.4.1. Existence

We deduce from the uniform bound (126) that (U, M, H) satisfies the regularity of the item (i) of Definition 1. We easily verify that (U, M, H) satisfies the items (ii)–(iv). From (ii) we deduce that there is a pressure $p \in L^2(0, T^*; W^{1,r}(D))$ such that Eqs. (1) and (2) hold a.e. in D_{T^*}.

4.4.2. Uniqueness

Let (U^1, M^1, H^1) and (U^2, M^2, H^2) be two strong solutions in D_T of problem $(\mathcal{P})$. Set $U = U^2 - U^1$, $M = M^2 - M^1$, $H = H^2 - H^1$, $p = p^2 - p^1$, $V^i = (M^i\cdot\nabla)H^i$, $W^i = M^i \wedge H^i$ $(i = 1, 2)$, $V = V^2 - V^1$ and $W = W^2 - W^1$. We easily verify that the functions U, M, H and p satisfy the equations

$$\rho\big(\partial_t U + \big(U^1\cdot\nabla\big)U\big) - \eta\Delta U + \nabla p = -\rho(U\cdot\nabla)U^2 + \mu_0 V + \frac{\mu_0}{2}\operatorname{curl} W \quad \text{in } D_T, \tag{141}$$

$$\operatorname{div} U = 0 \quad \text{in } D_T, \tag{142}$$

$$\begin{aligned}&\partial_t M + \big(U^1\cdot\nabla\big)M - \frac{1}{2}\operatorname{curl} U^1 \wedge M + \frac{1}{\tau}M + \beta M \wedge W^2 \\ &\quad = -(U\cdot\nabla)M^2 + \frac{1}{2}\operatorname{curl} U \wedge M^2 - \beta M^1 \wedge W + \frac{\chi_0}{\tau}H \quad \text{in } D_T,\end{aligned} \tag{143}$$

$$\operatorname{curl} H = 0, \qquad \operatorname{div}(H + M) = 0 \quad \text{in } D_T, \tag{144}$$

and the boundary and initial conditions

$$U = 0, \qquad (H + M)\cdot n = 0 \quad \text{on } (0, T)\times\partial D, \tag{145}$$

$$U(0) = 0, \qquad M(0) = 0 \quad \text{in } D. \tag{146}$$

Multiplying Eq. (141) by U and integrating over D yields

$$\begin{aligned}\frac{d}{dt}\left(\frac{\rho}{2}\|U\|^2\right) + \eta\|\nabla U\|^2 &= -\rho\int_D (U\cdot\nabla)U^2\cdot U\,dx + \mu_0\int_D V\cdot U\,dx + \frac{\mu_0}{2}\int_D \operatorname{curl} W\cdot U\,dx \\ &\equiv S_1 + S_2 + S_3.\end{aligned} \tag{147}$$

Obviously we have $|S_1| \leqslant C\|U\|^2$. We estimate S_2 and S_3 by arguing as in Section 4.3 for R_2 and R_3; we obtain

$$|S_2| \leqslant \frac{\eta}{4}\|\nabla U\|^2 + C\|M\|^2, \qquad |S_3| \leqslant \frac{\eta}{4}\|\nabla U\|^2 + C\|M\|^2.$$

Then we deduce from (147) that

$$\frac{d}{dt}\left(\frac{\rho}{2}\|U\|^2\right)+\frac{\eta}{2}\|\nabla U\|^2\leqslant C\|U\|^2+C\|M\|^2$$

and by integration from 0 to t

$$\frac{\rho}{2}\|U(t)\|^2+\frac{\eta}{2}\int_0^t\|\nabla U(s)\|^2\,ds\leqslant C\int_0^t\|U(s)\|^2\,ds+C\int_0^t\|M(s)\|^2\,ds. \tag{148}$$

Multiplying (143) by M and integrating over D yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|M\|^2+\frac{1}{\tau}\|M\|^2&=\int_D\left(-(U\cdot\nabla)M^2+\frac{1}{2}\operatorname{curl}U\wedge M^2-\beta M^1\wedge W+\frac{\chi_0}{\tau}H\right)\cdot M\,dx\\&\equiv S_4.\end{aligned} \tag{149}$$

We estimate S_4 by arguing as in Section 4.3 for R_4. We obtain

$$|S_4|\leqslant\varepsilon\|\nabla U\|^2+C\|M\|^2$$

for any $\varepsilon>0$, then we deduce from (149) that

$$\frac{1}{2}\frac{d}{dt}\|M\|^2+\frac{1}{\tau}\|M\|^2\leqslant\varepsilon\|\nabla U\|^2+C\|M\|^2$$

and Gronwall's inequality yields

$$\|M(t)\|^2\leqslant C\varepsilon\int_0^t\|\nabla U(s)\|^2\,ds.$$

Reporting the latter inequality in (148) and choosing ε small enough we can write

$$\frac{\rho}{2}\|U(t)\|^2+\frac{\eta}{4}\int_0^t\|\nabla U(s)\|^2\,ds\leqslant C\int_0^t\|U(s)\|^2\,ds.$$

Applying Gronwall's inequality we obtain $U=0$, then by (140) $M=0$, and (144), (145) imply $H=0$. The uniqueness of strong solutions to problem $(\mathcal{P})$ is proved. The proof of Theorem 1 is complete.

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