A simplified algorithm for calculating the Padé table derived from Baker and Longman schemes

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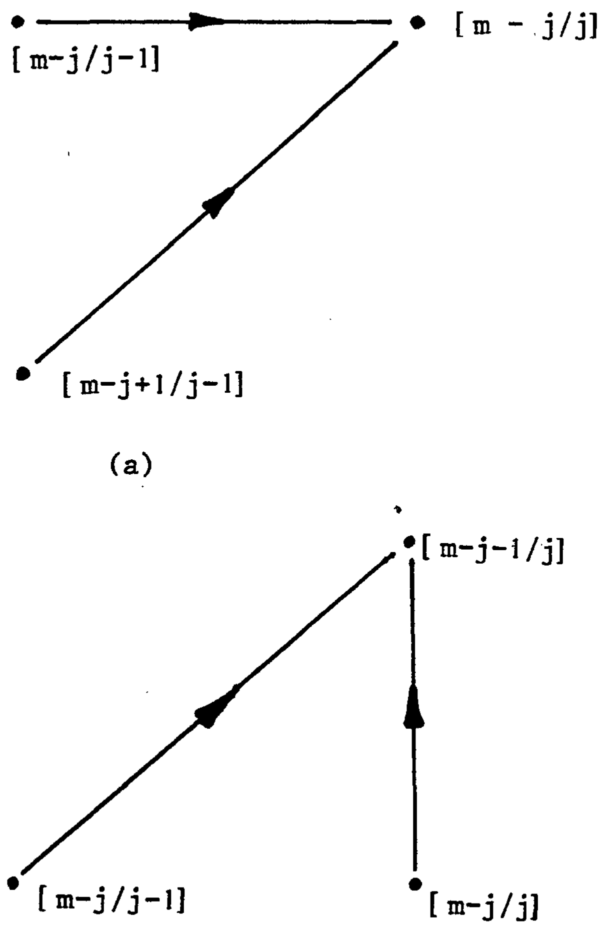
In view of the growing interest in applications of Padé approximants it seems important to have an algorithm as simple and manageable as possible for calculating the Padé table.

An excellent review of many different algorithms has appeared recently [1]. It seems, however, that the author missed some interesting relations that can lead to a remarkable simplification in some algorithms described there.

To demonstrate this let us represent graphically the Baker Algorithm (B.A.).



(a)

(b)

Fig. 1.

Analytically B. A. is expressed as follows :

$$
\left.
\begin{aligned}
a^{(i)}_{m-j/j} &= a^{(i)}_{m-j+1/j-1} - \frac{a^{(m-j+1)}_{m-j+1/j-1}}{a^{(m-j)}_{m-j/j-1}} \cdot a^{(i-1)}_{m-j/j-1} \\
b^{(i)}_{m-j/j} &= b^{(i)}_{m-j+1/j-1} - \frac{a^{(m-j+1)}_{m-j+1/j-1}}{a^{(m-j)}_{m-j/j-1}} \cdot b^{(i-1)}_{m-j/j-1} \\
& j = 1,\ldots,m
\end{aligned}
\right] \quad (1)
$$

This corresponds to fig. 1(a), and

$$
\left.
\begin{aligned}
a^{(i)}_{m-j-1/j} &= \frac{a^{(m-j)}_{m-j/j-1} \cdot a^{(i)}_{m-j/j} - a^{(m-j)}_{m-j/j} \cdot a^{(i)}_{m-j/j-1}}{a^{(m-j)}_{m-j/j-1} - a^{(m-j)}_{m-j/j}} \\
b^{(i)}_{m-j-1/j} &= \frac{a^{(m-j)}_{m-j/j-1} \cdot b^{(i)}_{m-j/j} - a^{(m-j)}_{m-j/j} \cdot b^{(i)}_{m-j/j-1}}{a^{(m-j)}_{m-j/j-1} - a^{(m-j)}_{m-j/j}} \\
& j = 1,\ldots,m-1
\end{aligned}
\right] \quad (2)
$$

This corresponds to fig. 1(b).

$a^{(i)}_{m/n}$ ($b^{(i)}_{m/n}$) is a coefficient of x^i in the numerator (denominator) of [m/n].

However, there is a puzzling asymmetry between a's and b's in (1) and (2) - puzzling, as it is well known [2] that if $P_n(X)/Q_m(x)$ is $[n/m]_f$ - P. A. to f(x) then $Q_m(x)/P_n(x)$ is $[m/n]_{f^{-1}}$ - P. A. to $f^{-1}(x)$ - if we forget for a moment about normalizing condition $Q_n(0) = 1$.

This puzzle can easily be solved, just by writing relations expressing "descending" along two parallel antidiagonals - in a way similar as (1) and (2) express "ascending" - the former relations being called hereafter Anti-Baker Algorithm (A-B. A.) :

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$$b^{(i)}_{j/m-j} = b^{(i)}_{j-1/m-j+1} - \frac{b^{(m-j+1)}_{j-1/m-j+1}}{b^{(m-j)}_{j-1/m-j}} \cdot b^{(i-1)}_{j-1/m-j}$$

$$a^{(i)}_{j/m-j} = a^{(i)}_{j-1/m-j+1} - \frac{b^{(m-j+1)}_{j-1/m-j+1}}{b^{(m-j)}_{j-1/m-j}} \cdot a^{(i-1)}_{j-1/m-j} \qquad j = 1,\ldots,m \qquad (1a)$$

$$b^{(i)}_{j/m-j-1} = \frac{b^{(m-j)}_{j-1/m-j} \cdot b^{(i)}_{j/m-j} - b^{m-j}_{j/m-j} \cdot b^{(i)}_{j-1/m-j}}{b^{(m-j)}_{j-1/m-j} - b^{(m-j)}_{j/m-j}} \qquad (2a)$$

$$a^{(i)}_{j/m-j-1} = \frac{b^{(m-j)}_{j-1/m-j} \cdot a^{(i)}_{j/m-j} - b^{(m-j)}_{j/m-j} \cdot a^{(i)}_{j-1/m-j}}{b^{(m-j)}_{j-1/m-j} - b^{(m-j)}_{j/m-j}} \qquad j = 1,\ldots,m-1$$

Notations as in (1) and (2).
Now it can be seen that A-B.A. relations also fulfill normalizing conditions - they just "propagate" it.
If we put $b^{(o)}_{o/m} = 1$ ($b^{(-1)}_{\ldots} \equiv 0$), then by induction all $b^{(o)}_{m-j/j} = 1$.

Graphically A-B. A. is just a mirror image of B. A. with respect to a line parallel to the diagonal of the Padé table.

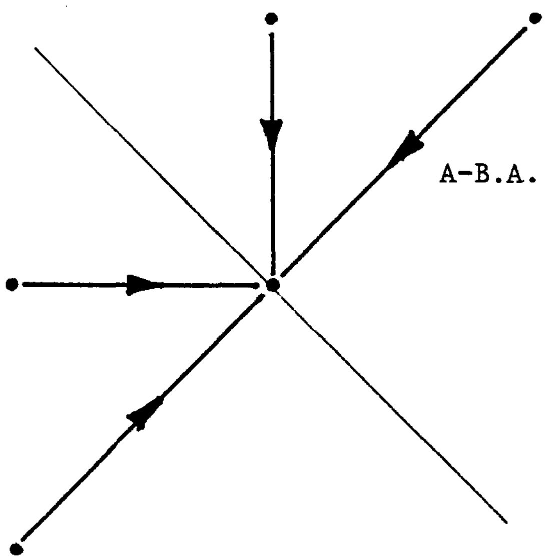

Fig. 2 (a)   B.A.

Once we observe that there exists A-B. A. it is then an obvious task to modify the Longman Algorithm (L. A.) to diminish an amount of calculations to get a desired approximant.
L. A. is an application of fig. 2a - B. A. for numerators

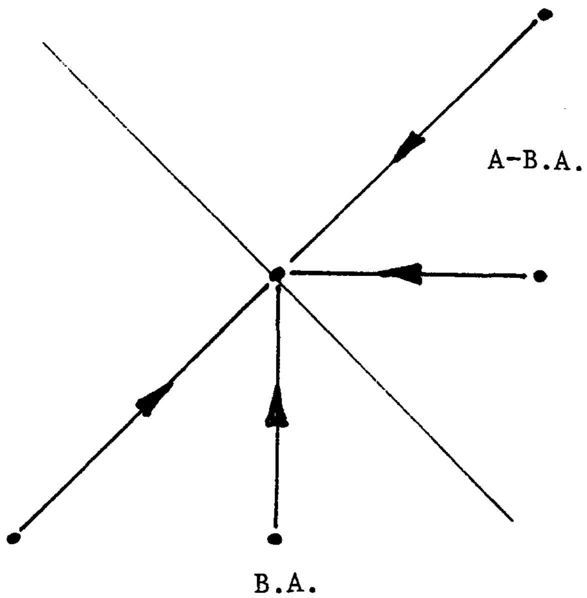

Fig. 2 (b)

and A-B. A. for denominators - in this way to calculate [m-j/j] P. A. one proceeds according to the scheme :

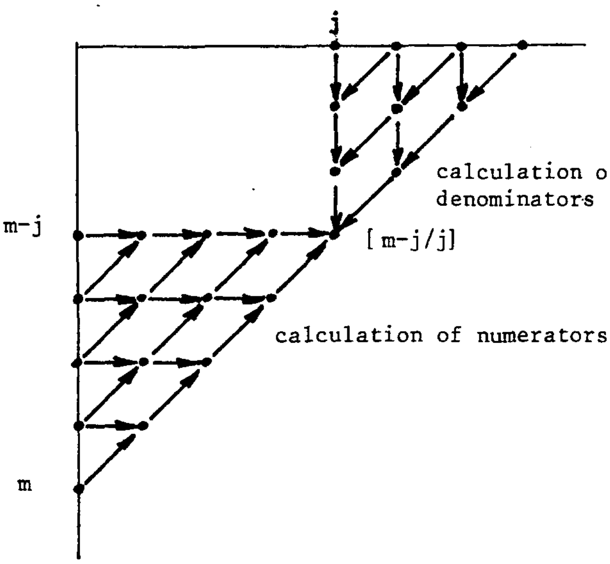

Fig. 3

One can see that L. A. relation for denominators given in [1] is just the first line of (1a).
To avoid calculating too many unnecessary coefficients one can use not only (1) and (1a) but also (2) and (2a) - i.e. use B. A. for numerators and A-B. A. for denominators. In this way only elements lying on the two antidiagonals - one passing through [m-j/j] and the second one just above it must be calculated - like in B. A., but numerators in an "ascending", and denominators in a "descending" order - like in L. A.
It is visualized in fig. 4.

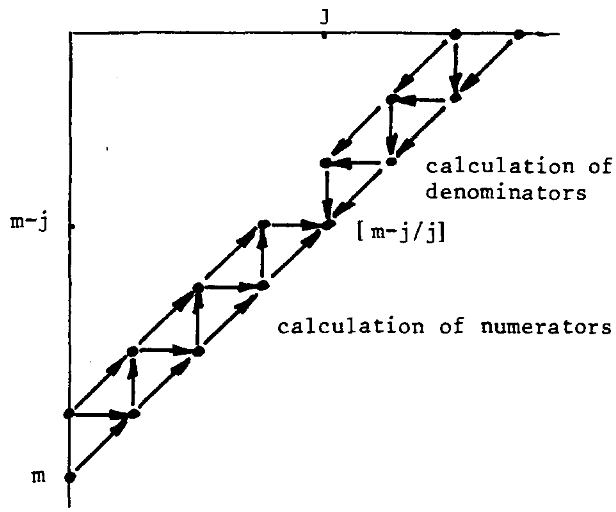

Fig. 4.

Finally we shall pick up formulae representing the proposed algorithm :

Numerators :

$$a^{(i)}_{m-j/j} = a^{(i)}_{m-j+1/j-1} - \frac{a^{(m-j+1)}_{m-j+1/j-1}}{a^{(m-j)}_{m-j/j-1}} \cdot a^{(i-1)}_{m-j/j-1} \qquad j = 1,\ldots,m$$

$$a^{(i)}_{m-j-1/j} = \frac{a^{(m-j)}_{m-j/j-1} \cdot a^{(i)}_{m-j/j} - a^{(m-j)}_{m-j/j} \cdot a^{(i)}_{m-j/j-1}}{a^{(m-j)}_{m-j/j-1} - a^{(m-j)}_{m-j/j}} \qquad j = 1,\ldots,m-1 \tag{3}$$

Denominators :

$$b^{(i)}_{j/m-j} = b^{(i)}_{j-1/m-j+1} - \frac{b^{(m-j+1)}_{j-1/m-j+1}}{b^{(m-j)}_{j-1/m-j}} \cdot b^{(i-1)}_{j-1/m-j} \qquad j = 1,\ldots,m$$

$$b^{(i)}_{j/m-j-1} = \frac{b^{(m-j)}_{j-1/m-j} \cdot b^{(i)}_{j/m-j} - b^{(m-j)}_{j/m-j} \cdot b^{(i)}_{j-1/m-j}}{b^{(m-j)}_{j-1/m-j} - b^{(m-j)}_{j/m-j}} \qquad j = 1,\ldots,m-1 \tag{4}$$

We would like to point out that in many physical applications - e.g. in particle physics, one is interested in looking for poles of P. A. - it is sufficient to use (4) i. e. to calculate a denominator of a required P. A. only from denominators of P. A's lying on two antidiagonals from the first row through the element in question of the Padé table.

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REFERENCES

1. G. CLAESSENS : "A new look at the Padé table and the different methods for computing its elements", J. of Comp. and Appl. Math., vol. I, p. 141 (1975).
2. e.g. G. A. BAKER Jr. : "Advances in theoretical physics", vol. 1, p. 1 (1965).