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Two-grid methods for finite volume element approximations of nonlinear parabolic equations

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ABSTRACT

Two-grid methods are studied for solving a two dimensional nonlinear parabolic equation using finite volume element method. The methods are based on one coarse-grid space and one fine-grid space. The nonsymmetric and nonlinear iterations are only executed on the coarse grid and the fine-grid solution can be obtained in a single symmetric and linear step. It is proved that the coarse grid can be much coarser than the fine grid. The two-grid methods achieve asymptotically optimal approximation as long as the mesh sizes satisfy $h = O(H^3|\ln H|)$. As a result, solving such a large class of nonlinear parabolic equations will not be much more difficult than solving one single linearized equation.

1. Introduction

We consider the following nonlinear parabolic problem:

$$\begin{cases} u_t - \nabla \cdot (a(x,t)\nabla u) = f(u), & (x,t) \in \Omega \times J, \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times J, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset R^2$ is a convex polygonal domain, $x = (x_1, x_2)$, $J = [0, T]$, $f(u) = f(u, x, t)$ is a given real-valued function on Ω. We assume that

$$0 < a_* \le a(x,t) \le a^*, \quad \forall (x,t) \in \Omega \times J, \qquad |f'(u)| + |f''(u)| \le M, \quad u \in R. \tag{1.2}$$

Under the given assumptions, problem (1.1) has a unique solution in a certain Sobolev space (see, e.g., [1]).

Finite volume element (FVE) method, as a type of important numerical tool for solving differential equations, has a long history. The method has been widely used in several engineering fields, such as fluid mechanics, heat and mass transfer and petroleum engineering. Perhaps the most important property of FVE method is that it can preserve the conservation laws (mass, momentum and heat flux) on each computational cell. This important property, combined with adequate accuracy and ease of implementation, has attracted more people to do research in this field. The theoretical framework and the basic tools for the analysis of FVE method have been developed in the last two decades (see, e.g. [2–10]). The main idea of FVE method is as follows. First, we construct the finite element partition and the relevant dual partition. Second, we choose the solution space of piecewise polynomial functions on the original partition and the test space of piecewise constant functions on the dual partition (control volumes, [2,4,7,9]). Then we use the Petrov–Galerkin technique to construct the variational formulation. Finally the discrete schemes are given.

Two-grid method is a discretization technique for nonlinear equations based on two grids of different sizes. The idea is to use a coarse-grid space to produce a rough approximation of the solution of nonlinear problems, and then use it as

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the initial guess for one Newton-like iteration on the fine grid. This method involves a nonlinear solve on the coarse grid of diameter H and a linear solve on the fine grid of diameter $h \ll H$. Two-grid method was first introduced in [12,13] for linear (nonsymmetric or indefinite) and especially nonlinear elliptic partial equations. Later on, two-grid method was further investigated by many authors (see, e.g., [14–18]). Dawson and Wheeler [14,15] have applied this method combined with mixed finite element method and finite difference method to this model problem; Li and Allen [16] have applied two-grid method combined with mixed finite element method to reaction–diffusion equations; Chen, Guang and Yu [17] have constructed a two-grid method for expanded mixed finite element solution of this model problem. Bi and Ginting [18] have studied two-grid finite volume element method for linear and nonlinear elliptic problems. There are many other efficient methods such as domain decomposition algorithms for parabolic problems, (see, e.g., [19–21]), so that an explicit treatment of the boundary conditions on a coarser mesh will lead to an optimal DD method.

In this paper we will consider FVE method combined with two-grid method to solve (1.1). We choose the two-grid spaces as two conforming finite element spaces V_H and V_h on one coarse grid with mesh size H and one fine grid with mesh size $h \ll H$. We solve a nonsymmetric and nonlinear problem on the coarse-grid space, then we use the known coarse-grid solution and a Taylor expansion to extrapolate the solution on the fine grid. On the fine grid we only need to solve a symmetric and linear system. A remarkable fact about this simple approach is, as shown in [12], that the coarse mesh can be quite coarse and still maintain a good accuracy approximation.

As far as we know there is no two-grid method convergence analysis for parabolic equations in the literature that can be applied in finite volume element method. In this paper we present the algorithms and analysis which partly fill this gap. The rest of this paper is organized as follows. In Section 2, we describe FVE method and two-grid FVE method for the nonlinear parabolic equation (1.1). We give two algorithms for the two-grid FVE method. In Section 3 we derive optimal error estimates in the H^1- and L^2-norm for the FVE method. Section 4 is devoted to the error estimates for the two-grid FVE method.

Throughout this paper, C denotes a generic positive constant which does not depend on the spatial and time discretization parameters and may be different at its different occurrences.

2. FVE method and two-grid FVE method

We will use the standard notation for Sobolev spaces $W^{s,p}(\Omega)$ with $1 \le p \le \infty$ consisting of functions that have generalized derivatives of order s in the space $L^p(\Omega)$. The norm of $W^{s,p}(\Omega)$ is defined by

$$\|u\|_{s,p,\Omega} = \|u\|_{s,p} = \left(\int_\Omega \sum_{|\alpha| \le s} |D^\alpha u|^p \mathrm{d}x \right)^{\frac{1}{p}},$$

with the standard modification for $p = \infty$. In order to simplify the notation, we denote $W^{s,2}(\Omega)$ by $H^s(\Omega)$ and omit the index $p = 2$ and Ω whenever possible; i.e., $\|u\|_{s,2,\Omega} = \|u\|_{s,2} = \|u\|_s$. We denote by $H_0^1(\Omega)$ the subspace of $H^1(\Omega)$ of functions vanishing on the boundary $\partial\Omega$.

For the polygonal domain Ω, we consider a quasi-uniform regular triangulation T_h consisting of closed triangle elements K such that $\bar{\Omega} = \cup_{K \in T_h} K$. We will use $\mathcal{N}_h$ to denote the set of all nodes or vertices of T_h and $\mathcal{N}_h^0 = \mathcal{N}_h \cap \Omega$.

Then we introduce a dual mesh T_h^* based on T_h. There are various ways to introduce the dual mesh. Almost all approaches can be described by the following general scheme. In each element $K \in T_h$ consisting of vertices x_i, x_j, x_k, select a point Q in the interior of the element K, and select a point x_{ij} on each of the three edges $\overline{x_i x_j}$ of K. Then connect Q to the points x_{ij} by straight lines r_{ij}. Then for a vertex x_i, let V_i be the polygon whose edges are r_{ij} in which x_i is a vertex of the element K. We call V_i a control volume centered at x_i. Obviously we have $\cup_{x_i \in \mathcal{N}_h} V_i = \bar{\Omega}$, and the dual mesh T_h^* is then defined as the set of these control volumes.

We call the control volume mesh T_h^* quasi-uniform regular if there exists a positive constant $C > 0$ such that

$$C^{-1}h^2 \le \text{meas}(V_i) \le Ch^2, \quad \forall V_i \in T_h^*,$$

where h is the maximum diameter of all elements $K \in T_h$.

There are various ways to introduce a regular dual mesh T_h^* depending on the choice of the point Q in each element $K \in T_h$ and the points x_{ij} on its edges. In this paper, we use a popular configuration in which Q is chosen to be the barycenter of the element $K \in T_h$, and the points x_{ij} are chosen to be the midpoints of the edges of K. This type of control volume can be introduced for any triangulation T_h and leads to relatively simple calculations. In addition, if T_h is locally regular, then the corresponding dual mesh T_h^* is also locally regular. Let S_h be the standard piecewise linear finite element space defined on the triangulation T_h,

$$S_h = \{v \in C(\Omega) : v|_K \text{ is linear}, \forall K \in T_h; v|_{\partial\Omega} = 0\},$$

and its dual volume element space S_h^* on T_h^*,

$$S_h^* = \{v \in L^2(\Omega) : v|_{V_i} \text{ is constant for all } V_i \in T_h^*; v|_{V_i} = 0, \text{ if } x_i \in \partial\Omega\}.$$

Then we obtain $S_h = \text{span}\{\phi_i(x) : x_i \in \mathcal{N}_h^0\}$ and $S_h^* = \text{span}\{\phi_i^*(x) : x_i \in \mathcal{N}_h^0\}$, where $\phi_i(x)$ is the standard nodal basis function associated with the node x_i, and $\phi_i^*(x)$ is the characteristic function of V_i.

For any $v \in H_0^1(\Omega) \cap H^2(\Omega)$, we define an interpolation operator $I_h : H_0^1(\Omega) \cap H^2(\Omega) \to S_h$, such that

$$I_h v = \sum_{x_i \in \mathcal{N}_h} v(x_i)\phi_i(x).$$

For any $v_h \in S_h$, we define another interpolation operator $I_h^* : S_h \to S_h^*$, such that

$$I_h^* v_h = \sum_{x_i \in \mathcal{N}_h} v_h(x_i)\phi_i^*(x).$$

Now we formulate FVE method for problem (1.1) as follows. Given a vertex x_i, integrating (1.1) over the associated control volume V_i and using the Green formula, we obtain

$$\int_{V_i} \frac{\partial u}{\partial t} \mathrm{d}x - \int_{\partial V_i} (a\nabla u) \cdot \mathbf{n} \mathrm{d}s = \int_{V_i} f(u) \mathrm{d}x, \tag{2.1}$$

where $\mathbf{n}$ denotes the unit outward normal on ∂V_i. It should be noted that the above formulation is a way of stating that we have an integral conservation form on the dual element (V_i).

Then the semi-discrete FVE method of (1.1) is written as follows: Find $u_h \in S_h$, such that

$$\int_{V_i} \frac{\partial u_h}{\partial t} \mathrm{d}x - \int_{\partial V_i} (a\nabla u_h) \cdot \mathbf{n} \mathrm{d}s = \int_{V_i} f(u_h) \mathrm{d}x. \tag{2.2}$$

Now we rewrite (2.2) to a variational form similar to finite element problems. For any $v_h \in S_h$, we multiply the integral in (2.2) by $v_h(x_i)$, and sum over all $x_i \in \mathcal{N}_h$ to obtain

$$\left(\frac{\partial u_h}{\partial t}, I_h^* v_h\right) + a_h(u_h, I_h^* v_h) = (f(u_h), I_h^* v_h), \tag{2.3}$$

where $a_h(\cdot, I_h^* \cdot)$ is defined by, for any $u_h, v_h \in S_h$,

$$a_h(u_h, I_h^* v_h) = -\sum_{x_i \in \mathcal{N}_h} \int_{\partial V_i} (a\nabla u_h) \cdot \mathbf{n} I_h^* v_h \mathrm{d}s = -\sum_{x_i \in \mathcal{N}_h} v_h(x_i) \int_{\partial V_i} (a\nabla u_h) \cdot \mathbf{n} \mathrm{d}s.$$

We consider a time step Δt and approximate the solution at times $t^n = n\Delta t$, $n = 0, 1, \ldots, N$; $\Delta t = T/N$. Then we obtain the Euler backward fully-discrete FVE method for (1.1): Find $u_h \in S_h$, such that

$$\begin{cases} \left(\dfrac{u_h^n - u_h^{n-1}}{\Delta t}, I_h^* v_h\right) + a_h(u_h^n, I_h^* v_h) = (f(u_h^n), I_h^* v_h), \quad \forall v_h \in S_h, \\ u_h^0 = u_0. \end{cases} \tag{2.4}$$

From Lemma 2 in the next section we can know that there exists a unique local solution for (2.4), (see, e.g., [9]). In order to present two-grid FVE method for the nonlinear parabolic problem (1.1), we introduce two quasi-uniform triangulations of Ω, T_H and T_h with two different mesh sizes H and h $(H > h)$. We introduce the corresponding finite element spaces S_H and S_h which satisfy $S_H \subset S_h$. They will be called the coarse-grid and fine-grid spaces, respectively.

To solve problem (1.1), we introduce two-grid algorithms into the FVE method. The idea is to use a coarse-grid space to produce a rough approximation of the solution, and then use it as the initial guess for one Newton-like iteration on the fine grid. This method involves a nonlinear solve on the coarse-grid space and a linear solve on the fine-grid space. We present the two-grid FVE method as two steps:

Algorithm 1. Step 1: On the coarse grid T_H, find $u_H^n \in S_H$ $(n = 1, 2, \ldots)$, such that

$$\begin{cases} \left(\dfrac{u_H^n - u_H^{n-1}}{\Delta t}, I_H^* v_H\right) + a_H(u_H^n, I_H^* v_H) = (f(u_H^n), I_H^* v_H), \quad \forall v_H \in S_H, \\ u_H^0 = u_0. \end{cases} \tag{2.5}$$

Step 2: On the fine grid T_h, find $u_h^n \in S_h$ $(n = 1, 2, \ldots)$, such that

$$\begin{cases} \left(\dfrac{u_h^n - u_h^{n-1}}{\Delta t}, I_h^* v_h\right) + a_h(u_h^n, I_h^* v_h) = (f(u_H^n) + f'(u_H^n)(u_h^n - u_H^n), I_h^* v_h), \quad \forall v_h \in S_h, \\ u_h^0 = u_0. \end{cases} \tag{2.6}$$

We note that the system in the second step of Algorithm 1 is a linear problem but still nonsymmetric. In order to get a symmetric system, we introduce the following bilinear forms

$$a_c(u_h, v_h) = \int_\Omega \bar{a}\nabla u_h \cdot \nabla v_h \mathrm{d}x, \quad \forall u_h, v_h \in S_h, \tag{2.7}$$

$$\begin{aligned} a_{h,c}(u_h, I_h^* v_h) &= -\sum_{x_i \in \mathcal{N}_h} \int_{\partial V_i} (\bar{a}\nabla u_h) \cdot \mathbf{n} I_h^* v_h \mathrm{d}s \\ &= -\sum_{x_i \in \mathcal{N}_h} v_h(x_i) \int_{\partial V_i} (\bar{a}\nabla u_h) \cdot \mathbf{n} \mathrm{d}s, \quad \forall u_h, v_h \in S_h, \end{aligned} \tag{2.8}$$

where $\bar{a} = \bar{a}|_K = a_K$ and

$$a_K = \frac{1}{\mathrm{meas}(K)} \int_K a(x)\mathrm{d}x, \quad \forall K \in T_h.$$

Then from [5,24], we have the following lemma.

Lemma 1. *For any $u_h, v_h \in S_h$, we have*

$$a_{h,c}(u_h, I_h^* v_h) = a_c(u_h, v_h).$$

From this lemma we can see that $a_{h,c}(u_h, I_h^* v_h)$ is symmetric. Then we obtain the second algorithm.

Algorithm 2. Step 1: The same as in Algorithm 1.

Step 2: On the fine grid T_h, find $u_h^n \in S_h$ $(n = 1, 2, \ldots)$, such that

$$\begin{cases} \left(\dfrac{u_h^n - u_h^{n-1}}{\Delta t}, I_h^* v_h\right) + a_{h,c}(u_h^n, I_h^* v_h) = (f(u_H^n) + f'(u_H^n)(u_h^n - u_H^n), I_h^* v_h), \quad \forall v_h \in S_h, \\ u_h^0 = u_0. \end{cases} \tag{2.9}$$

We note that the system in the first step of Algorithm 2 is the same as in Algorithm 1. But on the fine grid in the second step the coefficient matrix of the system is symmetric. So the system is easier to solve (e.g. conjugate-gradient-like methods can be applied effectively). We call these algorithms two-grid FVE methods.

3. Error analysis for FVE method

To describe error estimates for the FVE method, we define $|||u_h|||_0^2 = (u_h, I_h^* u_h), \forall u_h, v_h \in S_h$. Further $(u_h, I_h^* v_h)$ is symmetric and positive definite and the corresponding discrete norm is equivalent to the L^2-norm, i.e., that there exist two positive constants $C_*, C^* > 0$, independent of h such that

$$C_* \|u_h\| \le |||u_h|||_0 \le C^* \|u_h\|, \quad \forall u_h \in S_h. \tag{3.1}$$

The following two lemmas have been proved in [9,24], where Lemma 2 indicates that the bilinear form $a(\cdot, I_h^* \cdot)$ is continuous and coercive on S_h, while Lemma 3 shows that $a(\cdot, I_h^* \cdot)$ is generally not symmetric and how far it is from being symmetric.

Lemma 2. *For h sufficiently small, there exist two positive constants $\alpha, M > 0$ such that, for all $u_h, v_h \in S_h$, the coercive property*

$$a_h(u_h, I_h^* u_h) \ge \alpha \|u_h\|_1^2$$

and the boundedness property

$$|a_h(u_h, I_h^* v_h)| \le M \|u_h\|_1 \|v_h\|_1$$

hold true.

Lemma 3. *For h sufficiently small, there exists a positive constant $C > 0$ such that*

$$|a_h(u_h, I_h^* v_h) - a_h(v_h, I_h^* u_h)| \le Ch \|u_h\|_1 \|v_h\|_1, \quad \forall u_h, v_h \in S_h. \tag{3.2}$$

Let $R_h : H^2(\Omega) \cap H_0^1(\Omega) \to S_h$ be the standard Ritz projection such that

$$a(R_h u, v_h) = a(u, v_h), \quad \forall v_h \in S_h, \tag{3.3}$$

where $a(\cdot, \cdot)$ is the bilinear form related to the finite element scheme, i.e.,

$$a(u, v) = \int_\Omega a\nabla u \cdot \nabla v \mathrm{d}x, \quad \forall u, v \in H_0^1(\Omega). \tag{3.4}$$

It is well-known [22] that

$$\|u - R_h u\|_s \le Ch^{2-s}\|u\|_2, \quad s = 0, 1. \tag{3.5}$$

$$\|u - R_h u\|_{0,\infty} \le Ch|\ln h|\|u\|_{1,\infty}. \tag{3.6}$$

For error analysis we introduce two error functions

$$\varepsilon_h(f, \chi) = (f, \chi) - (f, I_h^* \chi), \quad \forall \chi \in S_h, \tag{3.7}$$

$$\varepsilon_a(\chi, \psi) = a(\chi, \psi) - a_h(\chi, I_h^* \psi), \quad \forall \chi, \psi \in S_h. \tag{3.8}$$

The two error functions are defined in [10,11] and the bounds for (3.7) and (3.8) are shown as the following lemma.

Lemma 4. *Let $\chi \in S_h$, then*

$$|\varepsilon_h(f, \chi)| \le Ch^{i+j}\|f\|_i\|\chi\|_j, \quad f \in H^i(\Omega), i, j = 0, 1,$$

$$|\varepsilon_a(R_h v, \chi)| \le Ch^{i+j}\|v\|_{1+i}\|\chi\|_j, \quad v \in H^{1+i}(\Omega) \cap H_0^1(\Omega), i, j = 0, 1.$$

Now we turn to the error estimate for the FVE method. First, we will give the optimal error estimates in the H^1-norm.

Theorem 1. *Let u and u_h be the solutions of (1.1) and (2.4), respectively. Assume that (1.2) is satisfied and $u \in L^\infty(H^2(\Omega))$, $u_t \in L^2(H^2(\Omega))$, $u_{tt} \in L^2(L^2(\Omega))$. For Δt small enough, if $u_h^0 = R_h u_0$ with R_h defined by (3.3), we have, for $t^n \le T$,*

$$\|u^n - u_h^n\|_1 \le \mathcal{C}(\Delta t + h), \tag{3.9}$$

where $\mathcal{C} = C(\|u\|_{L^\infty(H^2)}, \|u_t\|_{L^2(H^2)}, \|u_{tt}\|_{L^2(L^2)})$ is independent of h and Δt.

Proof. For convenience, let $u^n - u_h^n = (u^n - R_h u^n) + (R_h u^n - u_h^n) =: \eta^n + \xi^n$. Denote $\partial_t \xi^n = \frac{\xi^n - \xi^{n-1}}{\Delta t}$. Then from (2.1) and (2.4), we get the following error equation at $t = t^n$

$$(\partial_t \xi^n, I_h^* v_h) + a_h(\xi^n, I_h^* v_h) = (\partial_t u^n - u_t^n, I_h^* v_h) - (\partial_t \eta^n, I_h^* v_h) - a_h(\eta^n, I_h^* v_h) + (f(u^n) - f(u_h^n), I_h^* v_h),$$
$$\forall v_h \in S_h. \tag{3.10}$$

By (3.3), (3.7) and (3.8), we have

$$\begin{aligned} a_h(\eta^n, I_h^* v_h) &= a_h(u^n, I_h^* v_h) - a_h(R_h u^n, I_h^* v_h) \\ &= (f(u^n) - u_t^n, I_h^* v_h) - a(R_h u^n, v_h) + [a(R_h u^n, v_h) - a_h(R_h u^n, I_h^* v_h)] \\ &= (f(u^n) - u_t^n, I_h^* v_h) - a(u^n, v_h) + [a(R_h u^n, v_h) - a_h(R_h u^n, I_h^* v_h)] \\ &= (f(u^n) - u_t^n, I_h^* v_h - v_h) + [a(R_h u^n, v_h) - a_h(R_h u^n, I_h^* v_h)] \\ &= \varepsilon_h(u_t^n - f(u^n), v_h) + \varepsilon_a(R_h u^n, v_h). \end{aligned} \tag{3.11}$$

Choosing $v_h = \partial_t \xi^n$, we obtain

$$\begin{aligned} (\partial_t \xi^n, I_h^* \partial_t \xi^n) + a_h(\xi^n, I_h^* \partial_t \xi^n) &= (\partial_t u^n - u_t^n, I_h^* \partial_t \xi^n) - (\partial_t \eta^n, I_h^* \partial_t \xi^n) - \varepsilon_h(u_t^n - f(u^n), \partial_t \xi^n) \\ &\quad - \varepsilon_a(R_h u^n, \partial_t \xi^n) + (f(u^n) - f(u_h^n), I_h^* \partial_t \xi^n). \end{aligned} \tag{3.12}$$

Now we estimate (3.12). First

$$\begin{aligned} a_h(\xi^n, I_h^* \partial_t \xi^n) &= \frac{1}{2}[a_h(\xi^n + \xi^{n-1}, I_h^*(\xi^n - \xi^{n-1})) + a_h(\xi^n - \xi^{n-1}, I_h^*(\xi^n - \xi^{n-1}))] \\ &\ge \frac{1}{2\Delta t} a_h(\xi^n + \xi^{n-1}, I_h^*(\xi^n - \xi^{n-1})) \\ &= \frac{1}{2\Delta t}[a_h(\xi^n, I_h^* \xi^n) - a_h(\xi^{n-1}, I_h^* \xi^{n-1})] - \frac{1}{2}[a_h(\partial_t \xi^n, I_h^* \xi^n) - a_h(\xi^n, I_h^* \partial_t \xi^n)]. \end{aligned} \tag{3.13}$$

By (3.12) and (3.13), we have

$$\begin{aligned} &(\partial_t \xi^n, I_h^* \partial_t \xi^n) + \frac{1}{2\Delta t}[a_h(\xi^n, I_h^* \xi^n) - a_h(\xi^{n-1}, I_h^* \xi^{n-1})] \\ &\le (\partial_t u^n - u_t^n, I_h^* \partial_t \xi^n) - (\partial_t \eta^n, I_h^* \partial_t \xi^n) - \varepsilon_h(u_t^n - f(u^n), \partial_t \xi^n) - \varepsilon_a(R_h u^n, \partial_t \xi^n) \\ &\quad + \frac{1}{2}[a_h(\partial_t \xi^n, I_h^* \xi^n) - a_h(\xi^n, I_h^* \partial_t \xi^n)] + (f(u^n) - f(u_h^n), I_h^* \partial_t \xi^n) \\ &= (\partial_t u^n - u_t^n, \partial_t \xi^n) - (\partial_t \eta^n, I_h^* \partial_t \xi^n) - \varepsilon_h(\partial_t u^n - f(u^n), \partial_t \xi^n) - \varepsilon_a(R_h u^n, \partial_t \xi^n) \\ &\quad + \frac{1}{2}[a_h(\partial_t \xi^n, I_h^* \xi^n) - a_h(\xi^n, I_h^* \partial_t \xi^n)] + (f(u^n) - f(u_h^n), I_h^* \partial_t \xi^n). \end{aligned} \tag{3.14}$$

Multiplying by Δt and summing over n from 1 to l $(1 \le l \le N)$ at both sides of (3.14), by (3.1) and Lemma 3, since $\xi^0 = 0$ we have

$$\frac{\alpha}{2}\|\xi^l\|_1^2 + C\sum_{n=1}^{l}\|\partial_t\xi^n\|^2\Delta t \le \sum_{n=1}^{l}(\partial_t u^n - u_t^n, \partial_t\xi^n)\Delta t - \sum_{n=1}^{l}(\partial_t\eta^n, I_h^*\partial_t\xi^n)\Delta t - \sum_{n=1}^{l}\varepsilon_h(\partial_t u^n - f(u^n), \partial_t\xi^n)\Delta t - \sum_{n=1}^{l}\varepsilon_a(R_h u^n, \partial_t\xi^n)\Delta t + \sum_{n=1}^{l}\frac{1}{2}[a_h(\partial_t\xi^n, I_h^*\xi^n) - a_h(\xi^n, I_h^*\partial_t\xi^n)]\Delta t + \sum_{n=1}^{l}(f(u^n) - f(u_h^n), I_h^*\partial_t\xi^n)\Delta t \equiv \sum_{i=1}^{6}T_i. \tag{3.15}$$

We now estimate the right-hand terms of (3.15), from the results given in [23], we have

$$|T_1| \le C(\epsilon)\sum_{n=1}^{l}\left(\int_{t^{n-1}}^{t^n}\|u_{tt}\|\mathrm{d}t\right)^2\Delta t + \epsilon\sum_{n=1}^{l}\|\partial_t\xi^n\|^2\Delta t \le C(\epsilon)\left(\int_0^{t^l}\|u_{tt}\|^2\mathrm{d}t\right)(\Delta t)^2 + \epsilon\sum_{n=1}^{l}\|\partial_t\xi^n\|^2\Delta t, \tag{3.16}$$

where ϵ denotes a small positive constant.

For T_2, from (3.5), we get

$$|T_2| \le C(\epsilon)\sum_{n=1}^{l}\int_{t^{n-1}}^{t^n}\|\eta_t\|^2\mathrm{d}t + \epsilon\sum_{n=1}^{l}\|\partial_t\xi^n\|^2\Delta t \le C(\epsilon)h^4\left(\int_0^{t^l}\|u_t\|_2^2\mathrm{d}t\right) + \epsilon\sum_{n=1}^{l}\|\partial_t\xi^n\|^2\Delta t. \tag{3.17}$$

For T_3 and T_4, by (3.7) and (3.8) we have

$$|T_3| \le C\sum_{n=1}^{l}h\|\partial_t u^n - f(u^n)\|_1\|\partial_t\xi^n\|\Delta t \le C(\epsilon)h^2\left(\int_0^{t^l}\|u_t\|_1^2\mathrm{d}t\right) + C(\epsilon)h^2\left(\sum_{n=1}^{l}\|f^n\|_1^2\Delta t\right) + \epsilon\sum_{n=1}^{l}\|\partial_t\xi^n\|^2\Delta t. \tag{3.18}$$

$$|T_4| \le C\sum_{n=1}^{l}h\|u^n\|_2\|\partial_t\xi^n\|\Delta t \le C(\epsilon)h^2\left(\sum_{n=1}^{l}\|u^n\|_2^2\Delta t\right) + \epsilon\sum_{n=1}^{l}\|\partial_t\xi^n\|^2\Delta t. \tag{3.19}$$

By Lemma 3 and the inverse estimate, we have

$$|T_5| \le C\sum_{n=1}^{l}h\|\xi^n\|_1\|\partial_t\xi^n\|_1\Delta t \le C\sum_{n=1}^{l}\|\xi^n\|_1\|\partial_t\xi^n\|\Delta t \le C(\epsilon)\sum_{n=1}^{l}\|\xi^n\|_1^2\Delta t + \epsilon\sum_{n=1}^{l}\|\partial_t\xi^n\|^2\Delta t. \tag{3.20}$$

For T_6, at any point $x \in \Omega$, by the Taylor expansion, we have

$$f(u^n) - f(u_h^n) = f'(\tilde{u}^n)(u^n - u_h^n) = f'(\tilde{u}^n)(\eta^n + \xi^n),$$

for some value $\tilde{u}^n$ between u^n and u_h^n. From (3.5), we have

$$|T_6| \le C(\epsilon)\sum_{n=1}^{l}(\|\xi^n\|^2 + \|\eta^n\|^2)\Delta t + \epsilon\sum_{n=1}^{l}\|\partial_t\xi^n\|^2\Delta t \le C(\epsilon)\sum_{n=1}^{l}(\|\xi^n\|^2 + \|u^n\|_2^2h^4)\Delta t + \epsilon\sum_{n=1}^{l}\|\partial_t\xi^n\|^2\Delta t. \tag{3.21}$$

Combining (3.15)-(3.21), we have

$$\|\xi^l\|_1^2+\sum_{n=1}^{l}\|\partial_t\xi^n\|^2\Delta t \le C(\epsilon)h^2\left[\int_0^{t^l}\|u_t\|_2^2\mathrm{d}t+\sum_{n=1}^{l}\|u^n\|_2^2\Delta t+\sum_{n=1}^{l}\|f^n\|_1^2\Delta t\right]$$
$$+C(\epsilon)(\Delta t)^2\int_0^{t^l}\|u_{tt}\|^2\mathrm{d}t+C(\epsilon)\sum_{n=1}^{l}\|\xi^n\|_1^2\Delta t+C\epsilon\sum_{n=1}^{l}\|\partial_t\xi^n\|^2\Delta t. \tag{3.22}$$

Choosing proper ϵ and kicking the last term into the left side of (3.22), and applying discrete Gronwall lemma [15], for small Δt, we have

$$\|\xi^l\|_1^2+\sum_{n=1}^{l}\|\partial_t\xi^n\|^2\Delta t \le Ch^2\left[\int_0^{t^l}\|u_t\|_2^2\mathrm{d}t+\sum_{n=1}^{l}\|u^n\|_2^2\Delta t+\sum_{n=1}^{l}\|f^n\|_1^2\Delta t\right]+C(\Delta t)^2\int_0^{t^l}\|u_{tt}\|^2\mathrm{d}t. \tag{3.23}$$

Together with (3.5) we get (3.9). □

Then we state and show the optimal error estimate in the L^2-norm for the FVE method.

Theorem 2. *Let u and u_h be the solutions of (1.1) and (2.4), respectively. Assume that (1.2) is satisfied and $u\in L^\infty(H^2(\Omega))$, $u_t\in L^2(H^2(\Omega))$, $u_{tt}\in L^2(L^2(\Omega))$. For Δt small enough, if $u_h^0=R_hu_0$ with R_h defined by (3.3), we have, for $t^n\le T$,*

$$\|u^n-u_h^n\|\le \mathcal{C}(\Delta t+h^2), \tag{3.24}$$

where $\mathcal{C}=C(\|u\|_{L^\infty(H^2)},\|u_t\|_{L^2(H^2)},\|u_{tt}\|_{L^2(L^2)})$ is independent of h and Δt.

Proof. Choosing $v_h=\xi^n$ instead of $v_h=\partial_t\xi^n$ in (3.12), we obtain

$$(\partial_t\xi^n,I_h^*\xi^n)+a_h(\xi^n,I_h^*\xi^n)=(\partial_tu^n-u_t^n,I_h^*\xi^n)-(\partial_t\eta^n,I_h^*\xi^n)$$
$$-\varepsilon_h(u_t^n-f(u^n),\xi^n)-\varepsilon_a(R_hu^n,\xi^n)+(f(u^n)-f(u_h^n),I_h^*\xi^n). \tag{3.25}$$

For the first term of the left-hand side of (3.25), we have

$$\begin{aligned}(\partial_t\xi^n,I_h^*\xi^n)&=\frac{1}{2\Delta t}[(\xi^n-\xi^{n-1},I_h^*(\xi^n+\xi^{n-1}))+(\xi^n-\xi^{n-1},I_h^*(\xi^n-\xi^{n-1}))]\\&\ge\frac{1}{2\Delta t}(\xi^n-\xi^{n-1},I_h^*(\xi^n+\xi^{n-1}))\\&=\frac{1}{2\Delta t}[(\xi^n,I_h^*\xi^n)-(\xi^{n-1},I_h^*\xi^{n-1})]\\&=\frac{1}{2\Delta t}(|||\xi^n|||_0^2-|||\xi^{n-1}|||_0^2).\end{aligned} \tag{3.26}$$

Then we have

$$\frac{1}{2\Delta t}(|||\xi^n|||_0^2-|||\xi^{n-1}|||_0^2)+a_h(\xi^n,I_h^*\xi^n)\le(\partial_tu^n-u_t^n,\xi^n)-(\partial_t\eta^n,I_h^*\xi^n)-\varepsilon_h(\partial_tu^n-f(u^n),\xi^n)$$
$$-\varepsilon_a(R_hu^n,\xi^n)+(f(u^n)-f(u_h^n),I_h^*\xi^n). \tag{3.27}$$

By Lemma 2, (3.1), (3.26), multiplying Δt and summing over n from 1 to l $(1\le l\le N)$ at both sides of (3.27), since $\xi^0=0$ we have

$$\|\xi^l\|^2+\sum_{n=1}^{l}\|\xi^n\|_1^2\Delta t\le C\sum_{n=1}^{l}(\partial_tu^n-u_t^n,\xi^n)\Delta t-C\sum_{n=1}^{l}(\partial_t\eta^n,I_h^*\xi^n)\Delta t-C\sum_{n=1}^{l}\varepsilon_h(\partial_tu^n-f(u^n),\xi^n)\Delta t$$
$$-C\sum_{n=1}^{l}\varepsilon_a(R_hu^n,\xi^n)\Delta t+C\sum_{n=1}^{l}(f(u^n)-f(u_h^n),I_h^*\xi^n)\Delta t\equiv\sum_{i=1}^{5}Q_i. \tag{3.28}$$

Now we estimate the right-hand terms of (3.28), from the results given in [23], we have

$$\begin{aligned}|Q_1|&\le C\sum_{n=1}^{l}\left(\int_{t^{n-1}}^{t^n}\|u_{tt}\|\mathrm{d}t\right)^2\Delta t+C\sum_{n=1}^{l}\|\xi^n\|^2\Delta t\\&\le C\left(\int_0^{t^l}\|u_{tt}\|^2\mathrm{d}t\right)(\Delta t)^2+C\sum_{n=1}^{l}\|\xi^n\|^2\Delta t.\end{aligned} \tag{3.29}$$

For Q_2, from (3.5), we get

$$|Q_2| \le C\sum_{n=1}^{l}\int_{t^{n-1}}^{t^n}\|\eta_t\|^2\mathrm{d}t + C\sum_{n=1}^{l}\|\xi^n\|^2\Delta t$$
$$\le Ch^4\left(\int_0^{t^l}\|u_t\|_2^2\mathrm{d}t\right) + C\sum_{n=1}^{l}\|\xi^n\|^2\Delta t. \tag{3.30}$$

For Q_3 and Q_4, by (3.7) and (3.8) we have

$$|Q_3| \le C\sum_{n=1}^{l}h^2\|\partial_t u^n - f(u^n)\|_1\|\xi^n\|_1\Delta t$$
$$\le C(\epsilon)h^4\left(\int_0^{t^l}\|u_t\|_1^2\mathrm{d}t\right) + C(\epsilon)h^4\left(\sum_{n=1}^{l}\|f^n\|_1^2\Delta t\right) + \epsilon\sum_{n=1}^{l}\|\xi^n\|_1^2\Delta t. \tag{3.31}$$

$$|Q_4| \le C\sum_{n=1}^{l}h^2\|u^n\|_2\|\xi^n\|_1\Delta t$$
$$\le C(\epsilon)h^4\left(\sum_{n=1}^{l}\|u^n\|_2^2\Delta t\right) + \epsilon\sum_{n=1}^{l}\|\xi^n\|_1^2\Delta t. \tag{3.32}$$

For Q_5, by (3.5), we have

$$|Q_5| \le C\sum_{n=1}^{l}(\|\xi^n\|^2 + \|\eta^n\|^2)\Delta t + C\sum_{n=1}^{l}\|\xi^n\|^2\Delta t$$
$$\le Ch^4\left(\sum_{n=1}^{l}\|u^n\|_2^2\Delta t\right) + C\sum_{n=1}^{l}\|\xi^n\|^2\Delta t. \tag{3.33}$$

Combining (3.28)–(3.33), we have

$$\|\xi^l\|^2 + \sum_{n=1}^{l}\|\xi^n\|_1^2\Delta t \le C(\epsilon)h^4\left[\int_0^{t^l}\|u_t\|_2^2\mathrm{d}t + \sum_{n=1}^{l}\|u^n\|_2^2\Delta t + \sum_{n=1}^{l}\|f^n\|_1^2\Delta t\right]$$
$$+ C(\Delta t)^2\int_0^{t^l}\|u_{tt}\|^2\mathrm{d}t + C\sum_{n=1}^{l}\|\xi^n\|^2\Delta t + C\epsilon\sum_{n=1}^{l}\|\xi^n\|_1^2\Delta t. \tag{3.34}$$

Choosing proper ϵ and kicking the last term into the left side of (3.34), and applying discrete Gronwall lemma, for small Δt we have

$$\|\xi^l\|^2 + \sum_{n=1}^{l}\|\xi^n\|_1^2\Delta t \le Ch^4\left[\int_0^{t^l}\|u_t\|_2^2\mathrm{d}t + \sum_{n=1}^{l}\|u^n\|_2^2\Delta t + \sum_{n=1}^{l}\|f^n\|_1^2\Delta t\right] + C(\Delta t)^2\int_0^{t^l}\|u_{tt}\|^2\mathrm{d}t. \tag{3.35}$$

Together with (3.5) this yields (3.24). □

4. Error analysis for two-grid FVE method

In this section we consider the error estimates in the H^1-norm for the two-grid FVE method Algorithms 1 and 2. For the two-grid FVE method Algorithm 1, we have:

Theorem 3. *Let u and u_h be the solutions of (1.1) and the two-grid FVE method Algorithm 1, respectively. Assume that (1.2) is satisfied, $u \in L^\infty(H^2(\Omega)\cap W^{1,\infty}(\Omega))$, $u_t \in L^2(H^2(\Omega))$, $u_{tt} \in L^2(L^2(\Omega))$, and the coarse-grid partition H and the time step Δt satisfy $H^{-1}\Delta t < C$. For Δt small enough, if $u_h^0 = R_h u_0$ with R_h defined by (3.3), then we have, for $t^n \le T$,*

$$\|u^n - u_h^n\|_1 \le \mathcal{C}(\Delta t + h + H^3|\ln H|), \tag{4.1}$$

where $\mathcal{C} = C(\|u\|_{L^\infty(H^2)}, \|u\|_{L^\infty(W^{1,\infty})}, \|u_t\|_{L^2(H^2)}, \|u_{tt}\|_{L^2(L^2)})$ is independent of h and Δt.

Proof. Once again, we set $u^n - u_h^n = (u^n - R_h u^n) + (R_h u^n - u_h^n) =: \eta^n + \xi^n$ and choose $v_h = \partial_t \xi^n$. Then for Algorithm 1, we get the error equation

$$\begin{aligned}(\partial_t \xi^n, I_h^* \partial_t \xi^n) + a_h(\xi^n, I_h^* \partial_t \xi^n) &= (\partial_t u^n - u_t^n, I_h^* \partial_t \xi^n) - (\partial_t \eta^n, I_h^* \partial_t \xi^n) - a_h(\eta^n, I_h^* \partial_t \xi^n) \\ &\quad + (f(u^n) - f(u_H^n) - f'(u_H^n)(u_h^n - u_H^n), I_h^* \partial_t \xi^n). \end{aligned} \tag{4.2}$$

Similarly as in Theorem 1, we have

$$\begin{aligned}&(\partial_t \xi^n, I_h^* \partial_t \xi^n) + \frac{1}{2\Delta t}[a_h(\xi^n, I_h^* \xi^n) - a_h(\xi^{n-1}, I_h^* \xi^{n-1})] \\ &\quad \le (\partial_t u^n - u_t^n, \partial_t \xi^n) - (\partial_t \eta^n, I_h^* \partial_t \xi^n) - \varepsilon_h(\partial_t u^n - f(u^n), \partial_t \xi^n) - \varepsilon_a(R_h u^n, \partial_t \xi^n) \\ &\qquad + \frac{1}{2}[a_h(\partial_t \xi^n, I_h^* \xi^n) - a_h(\xi^n, I_h^* \partial_t \xi^n)] + (f(u^n) - f(u_H^n) - f'(u_H^n)(u_h^n - u_H^n), I_h^* \partial_t \xi^n). \end{aligned} \tag{4.3}$$

For the last term of the right-hand side of (4.3), a Taylor expansion about u_H^n yields

$$f(u^n) = f(u_H^n) + f'(u_H^n)(u^n - u_H^n) + \frac{1}{2} f''(\tilde{u})(u^n - u_H^n)^2,$$

for some function $\tilde{u}$. Then

$$\begin{aligned} f(u^n) - f(u_H^n) - f'(u_H^n)(u_h^n - u_H^n) &= f'(u_H^n)(u^n - u_h^n) + \frac{1}{2} f''(\tilde{u})(u^n - u_H^n)^2 \\ &= f'(u_H^n)(\eta^n + \xi^n) + \frac{1}{2} f''(\tilde{u})(u^n - u_H^n)^2. \end{aligned}$$

By (1.2), we have

$$|(f(u^n) - f(u_H^n) - f'(u_H^n)(u_h^n - u_H^n), I_h^* \partial_t \xi^n)| \le C(\epsilon)(\|\xi^n\|^2 + \|\eta^n\|^2) + C(\epsilon)\|(u^n - u_H^n)^2\|^2 + \epsilon \|\partial_t \xi^n\|^2. \tag{4.4}$$

For the first five terms of the right-hand side of (4.3), we can estimate them similarly as in Theorem 1. Multiplying Δt and summing over n from 1 to l $(1 \le l \le N)$, since $\xi^0 = 0$, for small Δt, we have

$$\begin{aligned} \|\xi^l\|_1^2 + \sum_{n=1}^{l} \|\partial_t \xi^n\|^2 \Delta t &\le C(\epsilon) h^2 \left[\int_0^{t^l} \|u_t\|_2^2 dt + \sum_{n=1}^{l} \|u^n\|_2^2 \Delta t + \sum_{n=1}^{l} \|f^n\|_1^2 \Delta t \right] + C(\epsilon)(\Delta t)^2 \int_0^{t^l} \|u_{tt}\|^2 dt \\ &\quad + C(\epsilon) \sum_{n=1}^{l} \|(u^n - u_H^n)^2\|^2 \Delta t + C(\epsilon) \sum_{n=1}^{l} \|\xi^n\|_1^2 \Delta t + C\epsilon \sum_{n=1}^{l} \|\partial_t \xi^n\|^2 \Delta t. \end{aligned} \tag{4.5}$$

Choosing proper ϵ, kicking the last term into the left side of (4.5), and using discrete Gronwall lemma, for small Δt we have

$$\begin{aligned} \|\xi^l\|_1^2 + \sum_{n=1}^{l} \|\partial_t \xi^n\|^2 \Delta t &\le C h^2 \left[\int_0^{t^l} \|u_t\|_2^2 dt + \sum_{n=1}^{l} \|u^n\|_2^2 \Delta t + \sum_{n=1}^{l} \|f^n\|_1^2 \Delta t \right] \\ &\quad + C(\Delta t)^2 \int_0^{t^l} \|u_{tt}\|^2 dt + C \sum_{n=1}^{l} \|(u^n - u_H^n)^2\|^2 \Delta t. \end{aligned} \tag{4.6}$$

For the last term of (4.6), we have

$$\begin{aligned} \|(u^n - u_H^n)^2\|^2 &\le \|u^n - u_H^n\|_{0,\infty}^2 \|u^n - u_H^n\|^2 \\ &\le (\|u^n - R_H u^n\|_{0,\infty} + \|R_H u^n - u_H^n\|_{0,\infty})^2 \|u^n - u_H^n\|^2, \end{aligned} \tag{4.7}$$

where R_H is defined in the same way as R_h is defined by (3.3). By Theorem 2, (3.6), (3.24) and the inverse estimate, we get

$$\begin{aligned} \|(u^n - u_H^n)^2\|^2 &\le \mathcal{C}(H|\ln H| + H^{-1}(\Delta t + H^2))^2 (\Delta t + H^2)^2 \\ &\le \mathcal{C}(H|\ln H|\Delta t + H^3|\ln H| + H^{-1}(\Delta t)^2 + 2H\Delta t + H^3)^2. \end{aligned} \tag{4.8}$$

We can choose H and Δt such that $H^{-1}\Delta t < C$, then we have

$$\|(u^n - u_H^n)^2\|^2 \le \mathcal{C}(\Delta t + H^3|\ln H|)^2, \tag{4.9}$$

with (4.6), we get

$$\|\xi^l\|_1 \le \mathcal{C}(\Delta t + h + H^3|\ln H|), \tag{4.10}$$

where $\mathcal{C} = C(\|u\|_{L^\infty(H^2)}, \|u\|_{L^\infty(W^{1,\infty})}, \|u_t\|_{L^2(H^2)}, \|u_{tt}\|_{L^2(L^2)})$ is independent of h and Δt. Together with (3.5) this yields (4.1). □

For the two-grid FVE method Algorithm 2, we can have a similar result.

Theorem 4. *Let u and u_h be the solutions of (1.1) and the two-grid FVE method Algorithm 2, respectively. Assume that (1.2) is satisfied, $u \in L^\infty(H^2(\Omega) \cap W^{1,\infty}(\Omega))$, $u_t \in L^2(H^2(\Omega))$, $u_{tt} \in L^2(L^2(\Omega))$, and the coarse-grid partition H and the time step Δt satisfy $H^{-1}\Delta t < C$. For Δt small enough, if $u_h^0 = R_h u_0$ with R_h defined by (3.3), then we have, for $t^n \le T$,*

$$\|u^n - u_h^n\|_1 \le \mathcal{C}(\Delta t + h + H^3|\ln H|), \tag{4.11}$$

where $\mathcal{C} = C(\|u\|_{L^\infty(H^2)}, \|u\|_{L^\infty(W^{1,\infty})}, \|u_t\|_{L^2(H^2)}, \|u_{tt}\|_{L^2(L^2)})$ is independent of h and Δt.

5. Conclusions

In this paper, we have presented and derived error estimates for two-grid finite volume element methods for a nonlinear parabolic equation. The theorems demonstrate a remarkable fact about two-grid FVE method: we can iterate on a very coarse grid T_H and still get good approximations by taking one iteration on the fine grid T_h. It is proved that the coarse grid can be much coarser than the fine grid ($h \ll H$). We can achieve asymptotically optimal approximation in H^1-norm error estimate as long as the mesh sizes satisfy $h = O(H^3|\ln H|)$.

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References

[1] V. Thomée, Galerkin Finite Element Methods for Parabolic Problems, Springer-Verlag, Berlin, 1997.
[2] Z. Cai, On the finite volume element methods, Numer. Math. 58 (1991) 713–735.
[3] Z. Cai, S. McCormick, On the accuracy of the finite volume element method for diffusion equations on composite grids, SIAM J. Numer. Anal. 27 (1990) 636–655.
[4] R.E. Ewing, R.D. Lazarov, Y.P. Lin, Finite volume element approximations of nonlocal reactive flows in porous media, Numer. Methods Partial Differential Equations 16 (2000) 285–311.
[5] R.E. Ewing, T. Lin, Y.P. Lin, On the accuracy of the finite volume element method based on piecewise linear polynomials, SIAM J. Numer. Anal. 39 (6) (2002) 1865–1888.
[6] R.E. Bank, D.J. Rose, Some error estimates for the box method, SIAM J. Numer. Anal. 24 (1987) 777–787.
[7] W. Hackbusch, On first and second order box schemes, Computing 41 (1989) 277–296.
[8] I.D. Mishev, Finite volume methods on Voronoi meshes, Numer. Methods Partial Differential Equations 14 (1998) 193–212.
[9] R. Li, Z. Chen, W. Wu, Generalized Difference Methods for Differential Equations Numerical Analysis of Finite Volume Methods, Marcel Dekker Inc., New York, 2000.
[10] P. Chatzipantelidis, R.D. Lazarov, V. Thomée, Error estimate for a finite volume element method for parabolic equations in convex polygonal domains, Numer. Methods Partial Differential Equations 20 (2004) 650–674.
[11] P. Chatzipantelidis, Finite volume methods for elliptic PDE's: A new approach, M2AN Math. Model. Numer. Anal. 36 (2002) 307–324.
[12] J. Xu, A novel two-grid method for semilinear elliptic equations, SIAM J. Sci. Comput. 15 (1994) 231–237.
[13] J. Xu, Two-grid discretization techniques for linear and nonlinear PDEs, SIAM J. Numer. Anal. 33 (1996) 1759–1777.
[14] C.N. Dawson, M.F. Wheeler, Two-grid methods for mixed finite element approximations of nonlinear parabolic equations, Contemp. Math. 180 (1994) 191–203.
[15] C.N. Dawson, M.F. Wheeler, C.S. Woodward, A two-grid finite difference scheme for nonlinear parabolic equations, SIAM J. Numer. Anal. 35 (1998) 435–452.
[16] L. Wu, M.B. Allen, A two-grid method for mixed finite-element solutions of reaction-diffusion equations, Numer. Methods Partial Differential Equations 15 (1999) 589–604.
[17] Y. Chen, Y. Huang, D. Yu, A two-grid method for expanded mixed finite-element solution of semilinear reaction-diffusion equations, Internat. J. Numer. Methods Engrg 57 (2003) 139–209.
[18] C. Bi, V. Ginting, Two-grid finite volume element method for linear and nonlinear elliptic problems, Numer. Math. 108 (2007) 177–198.
[19] Yu.A. Kuznetsov, New algorithm for approximate realization of implicit difference scheme, Sov. J. Numer. Anal. math. Modeling. 3 (1988) 95–114.
[20] Yu.A. Kuznestsov, Domain decomposition methods for unsteady convection-diffusion problems, In: Proc. 9-th Int. Conf. Computing Methods in Appl. Sci. Eng., 1990. pp. 211–227.
[21] R. Rannacher, G. Zhou, Analysis of a domain-splitting method for non-stationary convection-diffusion problems, East-West J. Numer. Math. 2 (1994) 151–172.
[22] S.C. Brenner, L.R. Scott, The Mathematical Theory of Finite Element Methods, Springer, Verlag, New York, 1994.
[23] T.F. Russell, Time stepping along characteristics with incomplete iteration for a Galerkin approximation of miscible displacement in porous media, SIAM J. Numer. Anal. 22 (5) (1985) 970–1013.
[24] S.H. Chou, Q. Li, Error estimates in L^2, H^1 and L^∞ in covolume methods for elliptic and parabolic problems: A unified approach, Math. Comp. 69 (2000) 103–120.