$G^2$ cubic transition between two circles with shape control

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Abstract

This paper describes a method for joining two circles with an S-shaped or with a broken back C-shaped transition curve, composed of at most two spiral segments. In highway and railway route design or car-like robot path planning, it is often desirable to have such a transition. It is shown that a single cubic curve can be used for blending or for a transition curve preserving $G^2$ continuity with local shape control parameter and more flexible constraints. Provision of the shape parameter and flexibility provide freedom to modify the shape in a stable manner which is an advantage over previous work by Meek, Walton, Sakai and Habib.

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1. Introduction

Fair path planning is one of the fundamental problems, with numerous applications in the fields of science, engineering and technology such as highway or railway designing, networks, robotics, GIS, navigation, CAD systems, collision detection and avoidance, animation, environmental design, communications and other disciplines. One of the main approaches to path planning is through the use of cubic spline functions.

Parametric cubic curves are popular in CAD applications because these are the lowest degree curves that allow inflection points (where curvature is zero). Such curves are suitable for the composition of $G^2$ blending curves. The Bézier form of a parametric cubic curve is mostly used in CAD and CAGD applications because of its geometric and numerical properties. Many authors have advocated their use in different applications like data fitting and font design. A fair curve should have curvature extrema only where explicitly desired by the designer. The importance of using fair curves in the design process is well documented in the literature [1–5].

Consumer products such as ping-pong paddles can be designed by blending circles [6]. For applications in the design of highway or railway routes, or trajectories of mobile robots, it is desirable that the transitions be fair. In the discussion about geometric design standards in AASHO (American Association of State Highway Officials), Hickerson [7] (p. 17) states that “Sudden changes between curves of widely different radii or between long tangents and sharp curves should be avoided by the use of curves of gradually increasing or decreasing radii and, at the same
time, introducing an appearance of forced alignment”. The importance of this design feature which is highlighted in [8] links vehicle accidents to inconsistency in highway geometric design.

Cubic curves, although smoother, are not always helpful since they might have unwanted cusps, loops, up to two inflection points, and curvature extrema [2,9–12]. According to Farin [1], curvature extrema of a fair curve “should only occur where explicitly desired by the designer”. B-splines and Bézier curves do not normally allow this. However, it can be accomplished when designing with spiral segments. A spiral is free of local curvature extrema, making spiral design an interesting mathematical problem with importance for both physical and aesthetic applications [13–20]. The clothoid, Cornu or non-polynomial cubic spiral has been used in highway designing and robot path planning for many years [21–24]. A major drawback in using this spiral is that the spiral segment currently used is neither polynomial nor rational. It is thus not easily incorporated in CAD/CAM/CAGD packages mostly based on NURBS (Non-Uniform Rational B Splines).

Walton and Meek [6] considered planar $G^2$ transition between two circles with a single fair cubic Bézier curve. They showed that there is no curvature extremum in the case of an S-shaped transition, and that there is a curvature extremum in the case of C-shaped transition. Use of a single curve rather than two segments has the benefit that designers have fewer entities to be concerned with. Habib and Sakai [25] simplified the analysis of Walton and Meek [6] and provided a less restrictive ratio of the larger to the smaller radii of the given circular arcs.

The objectives of this paper are to

- Further simplify and extend the analysis in [6,25].
- Obtain a fair $G^2$ cubic transition between two circles with more flexible constraints than in [6].
- Introduce a parameter to control the transition curve while preserving its important shape features.
- Visualize the relationship between shape control parameter and numerical value of the arc-length.

Provision of flexibility in the selection of radii of circular arcs and local shape control of the transition curve is certainly an advantage over previous work in [6,25].

2. Background

2.1. Notation and conventions

The usual Cartesian coordinate system is presumed. Bold face is used for points and vectors, e.g.,

$$a = \left( \begin{array}{c} a_x \\ a_y \end{array} \right).$$

The Euclidean norm or length of a vector $a$ is denoted by the notation

$$\|a\| = \sqrt{a_x^2 + a_y^2},$$

and $a \parallel b$ means the vector $a$ is parallel to vector $b$. The positive angle of a vector $a$ is the counterclockwise angle from the vector $(1, 0)$ to $a$. The derivative of a function $f$ is denoted by $f'$. To aid concise writing of mathematical expressions, the symbol $\times$ is used to denote the signed $z$-component of the usual three-dimensional cross-product of two vectors in the $xy$ plane, e.g.,

$$a \times b = a_x b_y - a_y b_x = \|a\| \|b\| \sin \theta,$$

where $\theta$ is the counterclockwise angle from $a$ to $b$, $a \cdot b$ denotes the usual inner product.

The signed curvature of a parametric curve $P(t)$ in the plane is

$$\kappa(t) = \frac{P'(t) \times P''(t)}{\|P'(t)\|^3},$$

(2.1)

when $\|P'(t)\|$ is non-zero. Positive curvature has the center of curvature on the left as one traverses the curve in the direction of increasing parameter. For non-zero curvature, the radius of curvature, positive by convention, is $1/|\kappa(t)|$. 
The derivative $\kappa'(t)$ of the curvature in (2.1) yields
\[
\kappa'(t) = \frac{\phi(t)}{||P'(t)||^3},
\]
where
\[
\phi(t) = ||P'(t)||^2 \frac{d}{dt} \{P'(t) \times P''(t)\} - 3\{P'(t) \times P''(t)\} \{P'(t) \cdot P''(t)\}.
\]

The term ‘spiral’ refers to a curved line segment whose curvature varies monotonically with constant sign. A $G^2$ point of contact of two curves is a point where the two curves meet and where their unit tangent vectors and signed curvatures match.

On the basis of Kneser’s theorem [26], any circle of curvature of a spiral encloses all smaller circles of curvature and is enclosed by all larger circles of curvature. So we cannot find the transition curve with a single spiral segment between the two tangent circles or the intersecting circles.

2.2. Cubic Bézier curve

The problem of finding a fair parametric transition curve between two circles $\Omega_0$, $\Omega_1$ with centers $C_0$, $C_1$, radii $r_0$, $r_1$ respectively and distance between the centers of two circles given as
\[
r = ||C_1 - C_0||,
\]
with $\sqrt{r_1/r_0} = \lambda$, $0 < \lambda \leq 1$, can be solved in a Hermite-like manner.

Here we consider the following problems.

1. Find a family of S-shaped transition curves between two non-intersecting circles $\Omega_0$ and $\Omega_1$, satisfying $r_0 + r_1 < r$, which guarantee the absence of interior curvature extrema.
2. Find a family of C-shaped transition curves between two circles $\Omega_0$ and $\Omega_1$, which have just one curvature extremum.

A planar cubic Bézier curve $z(t) = (x(t), y(t))$ is given by Farin [1] as
\[
z(t) = (1 - t)^3 P_0 + 3t(1 - t)^2 P_1 + 3t^2(1 - t)P_2 + t^3 P_3, \quad 0 \leq t \leq 1.
\]
Assume that the control points $P_i$, $i = 0, \ldots, 3$, are distinct. Also assume that the initial curvature of the curve is positive, for if it is negative then a reflection about the $x$-axis will make it positive. Without any loss of generality we can translate, rotate and, if necessary, reflect the shape to normalize it as shown in Figs. 1 and 2 so that the given starting point $P_0$ is at the origin, $P_1$ is on the positive $x$-axis, the given center $C_0 (= (0, r_0))$ of the larger circle is on the positive $y$-axis, the ending point $P_3$ is above the $x$-axis and, for an S-shaped transition, $P_3 - P_2$ is parallel to the $x$-axis. Let the control polygon in (2.4) be represented by
\[
\{g, h, k\} = \{||P_0P_1||, ||P_1P_2||, ||P_2P_3||\}.
\]
and $\theta$ be the angle from $P_1 - P_0$ to $P_2 - P_1$ where $0 < \theta < \pi/2$. Important parameters used to solve the problems are mentioned in the captions of Figs. 5(a) and 6(a).

3. Description of method

The steps of our analysis are first to derive the conditions for no curvature extrema and a single curvature extremum for an S-shaped and a C-shaped transition respectively, and then find the angle $\theta$ for the given distance $r$ between two circles. The results are expressed as Theorems 3.1 and 3.2. The two cases of an S-shaped and a C-shaped transition curve are then considered separately in the following sections.
3.1. S-shaped transition

Here we consider an S-shaped transition curve \( z(t) \) of the form (2.4). The curvature of the transition curve should change sign to allow an inflection point. Let the angle from \( P_3 - P_2 \) to \( P_2 - P_1 \) be \( \theta \). Recall that \( P_1 - P_0 \) and \( P_3 - P_2 \) are parallel to the \( x \)-axis, i.e., the tangents at the end points \( z'(0) \) and \( z'(1) \) are parallel to \((1,0)\). Referring to Fig. 1, we summarize the above discussion as

\[
\begin{align*}
P_0 &= (0, 0), \\
P_1 &= (0, g), \\
P_2 &= (g + h \cos \theta, h \sin \theta), \\
P_3 &= (g + h \cos \theta + k, h \sin \theta).
\end{align*}
\] (3.1)

To simplify the analysis, let \( h = p^2 r_0 \). Then the end point curvature condition

\[
(\kappa(0), \kappa(1)) = \left( \frac{1}{r_0}, -\frac{1}{r_1} \right),
\] (3.2)

yields

\[
(g, k) = pr_0 \sqrt{\frac{2 \sin \theta}{3}} (1, \lambda).
\] (3.3)
Therefore, from (2.4), we have a family of cubic transition curves

\[
\begin{align*}
x(t) &= r_0 p^2 t^2 (3 - 2t) \cos \theta + pr_0 t \left(3(1 - t) + (1 + \lambda)t^2\right) \sqrt{\frac{2 \sin \theta}{3}}, \\
y(t) &= r_0 p^2 (3 - 2t)t^2 \sin \theta.
\end{align*}
\]  

(3.4)

The following theorem guarantees the fairness of the desired single S-shaped transition curve with shape control parameter \(m\) on choosing the constraint as

\[p = m \sec \theta \sqrt{\frac{8 \sin \theta}{27}},\]

with \(m \geq 1\) and \(\theta \in (0, \pi/2)\).

**Theorem 3.1.** If \(r > r_0 + r_1\) and \(1/6 \leq \lambda (= \sqrt{r_1/r_0}) \leq 1\), then each value of \(m \geq 1\) determines a \(G^2\) cubic S-shaped transition curve \(z(t)\) of the form (3.4), defined by (2.4), which joins the two given circles with \(G^2\) contact and has no interior curvature extrema. Moreover, for a given distance \(r\) between the centers of two circles, we have a unique positive solution.

**Proof.** The proof of \(G^2\) contact of an S-shaped cubic transition curve with given circles follows immediately from the above discussion. For the absence of interior curvature extrema we need to show that the derivative of curvature, as defined in (2.2), does not change sign on \([0, 1]\). The derivative of curvature in (2.2), for \(t = s/(1 + s), s \geq 0\), is

\[
\kappa'(t) = \frac{-4096m^5 r_0^4 Q \sec^2 \theta \tan^5 \theta}{2187(1 + s)^5 \left\{x'(t)^2 + y'(t)^2\right\}^{5/2}},
\]

for

\[Q = 3 \cos^2 \theta f_1(s) + 4m^2sf_2(s),\]

where

\[
\begin{align*}
f_1(s) &= 3(m - \lambda)s^2 + (6 - 5m)s^4 + (3\lambda + 10m)s^3 + (3 + 10m)s^2 + (6\lambda - 5m)s - 3(1 - m), \\
f_2(s) &= 3 - s - \lambda s^2 + 3\lambda s^3 \\
&\geq 0 \quad \text{for} \quad \frac{253 - 80\sqrt{10}}{3} \leq \lambda \leq 1.
\end{align*}
\]

The parameter \(s\) has been introduced to simplify the analysis and the lower bound of \(\lambda\) is derived by a function InequalitySolve[] of a symbolic manipulator MATHEMATICA 5.

Since

\[
\lim_{\theta \to \frac{\pi}{2}} Q = 4m^2sf_2(s),
\]

\[
\geq 0 \quad \text{for} \quad \frac{253 - 80\sqrt{10}}{3} \leq \lambda \leq 1,
\]

\[
\lim_{\theta \to 0} Q = \left(3 + 4ms + 3s^2\lambda\right) \left[mf_2(s) - 3 \left(1 - 2s\lambda - 2s^2\lambda + s^3\lambda^2\right)\right],
\]

\[
\geq 0 \quad \text{for} \quad \frac{1}{6} \leq \lambda \leq 1.
\]

(3.6)

and \(Q\) is linear with respect to \(\cos^2 \theta\), therefore \(Q \geq 0\) for \(1/6 \leq \lambda \leq 1\). Hence the derivative of curvature does not vanish on \([0, 1]\) so there are no interior curvature extrema in the cubic as defined.

For the existence and uniqueness of the solution we need to simplify the procedure by letting \(q = \tan^2 \theta\). For the given distance \(r\) between the centers

\[
C_0 = (x(0), y(0) + r_0), \quad C_1 = (x(1), y(1) - r_1),
\]
of two circles \( \Omega_0 \) and \( \Omega_1 \) respectively, the condition (2.3) determines a polynomial equation \( g(q) = 0 \) in \( q \) given by

\[
g(q) = \sum_{i=0}^{2} d_i q^i,
\]

where

\[
d_2 = 64m^4 r_0^2,
\]
\[
d_1 = 32m^2 r_0^2 \left( 2m^2 + 6m(1+\lambda) - 9 \left( 1 - \lambda + \lambda^2 \right) \right),
\]
\[
d_0 = -729 \left( r^2 - r_0^2 \left( 1 + \lambda^2 \right)^2 \right).
\]

Since \( r_0 + r_1 < r \), the coefficients of \((q^2, q, 1)\) are \((+, +, -)\), where “+” and “−” include “0”, and “?” means either “+” or “−”. Therefore, by the Descartes rule of signs ([27], pp. 439–443), we have just one positive value of angle \( \theta \) and therefore this ensures a unique positive solution which is determined by

\[
quation (3.7)
\]

\[
q = \tan^2 \theta = \frac{1}{8m^2 r_0} \left[ 2r_0 \left( 9 \left( 1 - \lambda + \lambda^2 \right) - 6m(1+\lambda) - 2m^2 \right) \right.
\]
\[
+ \sqrt{729r^2 + r_0^4(3 + 2m + 3\lambda)^2 \left( 4m^2 + 12m(1+\lambda) - 9 (5 - 2\lambda + 5\lambda^2) \right)} \right].
\]

\[
(3.8)
\]

\[\square\]

**Remark 3.1.** For \( m \geq 2 \), the terms inside the braces of (3.6) are greater than

\[3 + 2s(3\lambda - 1) + 4s^2\lambda + 3s^3(2 - \lambda)\lambda,\]

which is positive for \( 0.015046 \leq \lambda \leq 1 \), where the lower bound of \( \lambda \) is derived by a function InequalitySolve[] of a symbolic manipulator MATHEMATICA 5. Therefore the restriction on \( 1/6 \leq \lambda \leq 1 \), necessary for \( m \geq 1 \), could be relaxed by choosing a larger value of \( m \).

### 3.2. C-shaped transition

Here we consider the case of C-shaped transition curve \( z(t) \) of the form (2.4). Let the angle from \( P_2 - P_1 \) to \( P_3 - P_2 \), as shown in Fig. 2, be \( \theta \). Two sub-cases arise.

1. One circle is inside the other. In this case it is possible to have a transition curve with no interior curvature extrema.
2. One circle is not inside the other, i.e., \( r_0 - r_1 < r \).

The first sub-case has been discussed in [18,28,29] for cubic and Pythagorean hodograph (PH) quintic transitions. We examine the second sub-case here.

By Kneser’s theorem, the transition curve cannot be a “spiral” arc, so it has at least one interior curvature extremum. We prove that the transition curve has exactly one interior curvature extremum and does not have an inflection point, so the curvature has no change of sign.

We require the angles between \( P_i - P_{i-1} \) and \( P_{i+1} - P_i \), \( 1 \leq i \leq 2 \), to be \( \theta \); refer to Fig. 2. Recall that \( P_1 - P_0 \) is parallel to the \( x \)-axis, i.e., the tangent at the starting point \( z'(0) \) is parallel to \((1,0)\). We summarize the above discussion as

\[
P_0 = (0, 0),
\]
\[
P_1 = (0, g),
\]
\[
P_2 = (g + h \cos \theta, h \sin \theta),
\]
\[
P_3 = (g + h \cos \theta + k \cos 2\theta, h \sin \theta + k \sin 2\theta).
\]

To simplify the analysis, again let \( h = p^2 r_0 \). Then the end point curvature condition

\[
(\kappa(0), \kappa(1)) = (1/r_0, 1/r_1),
\]

(3.10)
yields (3.3) and therefore, from (2.4), we have a family of cubic transition curves

\[
x(t) = \frac{pr_0 t}{3} \left[ 3pt(3 - 2t) \cos \theta + \left( 3 - 3t + t^2 + t^2 \cos 2\theta \right) \sqrt{6 \sin \theta} \right]
\]

\[
y(t) = \frac{pr_0 t^2}{3} \left[ 3p(3 - 2t) + 2t \lambda \cos \theta \sqrt{6 \sin \theta} \right] \sin \theta.
\]

(3.11)

The desired single C-shaped transition curve with shape control parameter \(m\) is obtained by choosing the constraint as

\[
p = m \sec \theta \sqrt{\frac{2 \sin \theta}{3}},
\]

with \(m > (1 + \sqrt{1 + 3\lambda})/3\) and \(\theta \in (0, \pi/2)\), ensuring fairness according to the following theorem.

**Theorem 3.2.** If \(r > r_0 - r_1\) and \(0 < \lambda = \sqrt{r_1/r_0} \leq 1\), then each value of \(m\) greater than \((1 + \sqrt{1 + 3\lambda})/3\) determines a \(G^2\) cubic C-shaped transition curve \(z(t)\) of the form (3.11), defined by (2.4), which joins the two given circles with \(G^2\) contact and has a single interior curvature extremum. Also, for a given distance \(r\) between the centers of two circles, we have a unique positive solution.

The curve begins with monotone decreasing curvature and ends with monotone increasing curvature.

**Proof.** The proof of \(G^2\) contact of a C-shaped cubic transition curve with given circles follows immediately from the above discussion. Next we need to show the presence of exactly one interior curvature extremum on \([0, 1]\), i.e., the derivative of curvature \(\kappa(t)\), defined in (2.2), has just one zero on \([0, 1]\).

Since the curve \(z(t)\) has starting monotone decreasing curvature and ending monotone increasing curvature, therefore \(\phi(t)\) in (2.2) is negative at \(t = 0\) and positive at \(t = 1\), i.e.,

\[
\phi(0) = -64m^4r_0^4 \tan^5 \theta \left( 3m^2 - 2m - \lambda \cos^2 \theta \right) < 0,
\]

\[
\phi(1) = 64m^4r_0^4 \lambda^2 \tan^5 \theta \left( 3m^2 - 2m\lambda - \lambda \cos^2 \theta \right) > 0,
\]

which implies that \(m > u\), where

\[
u = \frac{1 + \sqrt{1 + 3\lambda}}{3}.
\]

So \(\phi(t)\) has a zero in \([0, 1]\). To show that \(\phi(t)\) has just one zero, we note that the derivative of \(\phi(t)\) is

\[
\phi'(t) = - \left\{ x'(t)y''(t) - y'(t)x''(t) \right\} \left\{ 3x''(t)^2 + 3y''(t)^2 + 4x'(t)x^{(3)}(t) + 4y'(t)y^{(3)}(t) \right\}.
\]

(3.12)

To simplify the analysis, we consider \(t = s/(1 + s)\), \(s \geq 0\). Then the right hand side of (3.13) can be simplified using

\[
x'(t)y''(t) - y'(t)x''(t) = \frac{8m^2r_0^2 \tan^3 \theta \left( m + ms^2 \lambda + 2s \lambda \cos^2 \theta \right)}{(1 + s)^2} > 0,
\]

(3.14)

\[
3x''(t)^2 + 3y''(t)^2 + 4x'(t)x^{(3)}(t) + 4y'(t)y^{(3)}(t) = \left\{ 8m^2r_0^2 \sec^2 \theta \tan^2 \theta \right\} p(t),
\]

where

\[
p(t) = 5 \left\{ 1 + 8m^2 + 2d^2 \lambda + \lambda^2 - 4m(1 + \lambda) + d(1 - 4m + \lambda)(1 + \lambda) \right\} t^2
\]

\[
- 10 \left[ 1 + 4m^2 + d^2 \lambda - m(3 + \lambda) + d \{ 1 + \lambda - m(3 + \lambda) \} \right] t
\]

\[
+ 5 - 10m + 6m^2 + 2d^2 \lambda + d(5 - 10m + 2\lambda),
\]

(3.15)

with

\[d = \cos(2\theta).\]
Since $2/3 < u \leq 1$ for $0 < \lambda \leq 1$, and for $c = \cot^2 \theta$, $d = (c - 1)/(c + 1)$, the coefficient of $t^2$ in $(c + 1)^2 p(t)$ is

$$10 \left\{ 4u^2 + c^2 \left( 1 - 4u + 3u^2 \right)^2 + c \left( 1 + 14u^2 - 24u^3 + 9u^4 \right) \right\} + 40 \left\{ 2u + c(1 - u)(3u - 1) \right\} \left( c + 1 \right)(m - u) + 40(c + 1)^2 (m - u)^2,$$

which is positive for $m > u$.

If $p(t)$ has one or no zero then $\phi(t)$ has just one zero in $[0, 1]$. If $p(t)$ has two zeros in $[0, 1]$, i.e., $0 < \alpha, \beta < 1$, then since the coefficient of the leading term of $p(t)$ is positive, the signs of $\phi'(t)$ on $(0, \alpha)$, $(\alpha, \beta)$, and $(\beta, 1)$ are $-$, $+$, $-$, respectively. Hence, as $\phi(t)$ has just one zero on $[0, 1]$, the cubic Bézier curve defined in the statement of the theorem has exactly one curvature extremum for $0 \leq t \leq 1$.

For the existence and uniqueness of the solution we consider again $q = \tan^2 \theta$. Then the centers of two circles $\Omega_0$ and $\Omega_1$ are given by

$$C_0 = (x(0), y(0) + r_0) = (0, r_0),$$

$$C_1 = (x(1) - r_1 \sin 2\theta, y(1) + r_1 \cos 2\theta) = \left( x(1) - r_1 \frac{2\sqrt{q}}{1 + q}, y(1) + r_1 \frac{1 - q}{1 + q} \right).$$

For given distance $r$, the condition (2.3) determines a polynomial equation $h(q) = 0$ in $q$ given by

$$h(q) = \sum_{i=0}^{3} e_i q^i,$$

where

$$e_3 = 4m^4 r_0^2,$$

$$e_2 = 8m^2 r_0^2 \left\{ (m - 1)(\lambda + 1) + m^2 - \lambda^2 \right\},$$

$$e_1 = -9r^2 + r_0^2 \left\{ 9\lambda^4 - 2 \left( -9 + 12m + 4m^2 \right) \lambda^2 + 8m \left( -3 + m + m^2 \right) \lambda + 9 - 8m^2 + 8m^3 + 4m^4 \right\},$$

$$e_0 = -9 \left\{ r^2 - r_0^2 \left( \lambda^2 - 1 \right)^2 \right\}.$$ 

Since $r_0 - r_1 < r$, the coefficients of $(q^3, q^2, q, 1)$ are $(+, +, ?, -)$. Therefore, by the Descartes rule of signs, we have just one positive value of angle $\theta$ and so a unique positive solution exists.

As $m$ increases, the transition between two circles becomes tighter and tends to a straight line segment in the limit as $m \to \infty$.

The location of the end point of the transition curve (i.e., $z(1)$) and the smaller circle $\Omega_1$ also changes as a result, whereas the distance between two circles remains fixed. This behavior can be seen from Fig. 3. The start point of the transition curve is also fixed but it seems to be shifted in Figs. 5–8 due to the transformation. Fig. 4(a) and (b) show how the arc-length of the transition curve decreases as $m$ increases. Since the curvatures remain unchanged at the ends of the transition while the arc-length of the transition is reduced, the rate of change of the curvature increases; see the curvature plots of transition curves for different values of $m$ in Figs. 5(b) and 6(b) indicating the $G^2$ continuous joints of the circles with the transition curves.

On the basis of the above analysis, we have adopted the approach for constructing the $G^2$ cubic Bézier transition between two circles described in the following algorithm.

4. The algorithm

Given the radii $r_0, r_1$ and centers $C_0, C_1$ of the larger circle $\Omega_0$ and the smaller circle $\Omega_1$, respectively, the distance $r$ between two circles, and starting point $P_0$ on $\Omega_0$, we can find the S-shaped transition if the two circles do not intersect and $1/6 \leq \lambda \leq 1$. We can find the C-shaped transition if the smaller circle is not inside the larger circle. Next we proceed as follows:
1. Normalize both the circles by translation, rotation and, if necessary, by reflection according to Fig. 1 or 2 for the S-shaped or C-shaped transition, respectively.
2. Set the value of the shape control parameter \( m (\geq 1) \).
3. Find \( \theta \) from (3.8) or (3.18) for the S-shaped or C-shaped transition, respectively.
4. Find the required spiral transition from (2.4) by using (3.4) or (3.11) for the S-shaped or C-shaped transition, respectively.
5. If the shape of curve is not as required then go to step 2 and try for a different value of \( m \).
6. Do the reverse transformation to bring the shape back to its original location.

5. Examples

Fig. 5 shows an S-shaped cubic transition with no curvature extrema and Fig. 6 shows a C-shaped transition with a single curvature extremum. Figs. 5 and 6 indicate the \( G^2 \) continuous joints of the circles with the transition curves and show the effect of shape control parameter \( m \) on the arc-lengths of the transition curves.

Two examples from [6] are presented in Figs. 7 and 8 controlled by the shape parameter, having a family of transitions between two circles. Each transition is a single fair cubic Bézier curve, i.e., without extraneous curvature extrema, or inflection points. In the illustrations of the examples, the end points of the S-shaped and C-shaped transition curves are indicated with circles and disks, respectively. One example, shown in Fig. 7, represents the cross-section of a cam. It is composed of two circular arcs joined by an upper and a lower family of C-shaped curves. The other examples, shown in Fig. 8, represent the profile of a vase. The side and base of the vase are represented by families of S-shaped and C-shaped curves, respectively. The end points of the S-shaped and C-shaped transition curves are indicated with small circles and disks, respectively.
Fig. 5. S-shaped transitions with \((r_0, r, \lambda) = (2, 4, 1/\sqrt{2})\) for \(m = 1\) (thin) and 2 (bold).

(a) Circle to circle transition  
(b) Curvature plot of transition curves

Fig. 6. C-shaped transitions with \((r_0, r, \lambda) = (2, 4, 1/\sqrt{2})\) for \(m = 0.923\) (thin), 1.5 (bold), 2 (extra bold).

(a) Circle to circle transition.  
(b) Curvature plot of transition curves.

Fig. 7. Cam cross-section for \(m = 1\) (thin), 1.5 (bold).
6. Comparison and conclusion

We have extended the analysis of Walton and Meek [6] and Habib and Sakai [25] on the planar $G^2$ transition between two fixed circles with a fair single cubic Bézier curve, by introducing a tension control and shape preserving parameter $m \geq 1$. This parameter allows interactive alteration of the shape of the curve while preserving required geometric features as can be seen from Figs. 5–8. For $m = 1$, the results reduce to those of [6,25] which have no provision for shape control.

Since the end points of transition curve lie somewhere on the fixed circles with their given centers and radii, it is not so important for the designer to get the transition between two fixed points for most of the practical applications. Transition between two fixed points may also be achieved but it would be either at the cost of two cubic segments instead of a single one or with less flexible constraints.

By algebraic reorganization and manipulation we have produced more flexible constraints than in [6]. To guarantee the absence of an interior curvature extremum in an S-shaped transition curve, the ratio of the radii of the circles may be as big as 36 because $\lambda (= \sqrt{r_1/r_0}) \geq 1/6$, whereas in [6] the results were shown to be valid for a ratio of up to 9.

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References


