Shear viscosity to entropy density ratio of a relativistic Hagedorn resonance gas

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The new state of matter produced at Relativistic Heavy Ion Collider reveals a strongly coupled quark–gluon plasma with an extremely small shear viscosity to entropy density ratio \( \eta/s \). We calculate the \( \eta/s \) of an equilibrated hadron matter characterized by a relativistic hadron resonance gas with a Hagedorn mass spectrum that grows exponentially with the hadron mass. We find with increase in temperature of the system the \( \eta/s \) value decreases due to rapid increase in the multiplicity of massive resonances. In the vicinity of the critical temperature for deconfinement transition, the minimum value of \( \eta/s \) in the Hagedorn resonance gas is found to be consistent with the current estimates for a strongly coupled quark–gluon plasma.

Heavy ion collisions at the BNL Relativistic Heavy Ion Collider (RHIC) have revealed a new state of matter [1–4] comprising of strongly interacting quarks and gluons (sQGP) [5]. This conclusion is based on viscous hydrodynamic model analysis of elliptic flow that requires a very small shear viscosity to entropy density is based on viscous hydrodynamic model analysis of elliptic flow that requires a very small shear viscosity to entropy density ratio \( \eta/s \). The main uncertainty in these estimates stems from the equation of state and the initial conditions employed. This value is remarkably similar to the lower bound \( \eta/s \geq 1/4\pi \) obtained by Kovtun–Son–Starinets (KSS) [9] for an infinitely coupled supersymmetric Yang–Mills gauge theory based on the AdS/CFT duality conjecture. A recent lattice calculation [10] of gluonic plasma do support the current estimates. While leading order perturbative calculations result in a significantly large value of \( \eta/s \approx 0.8 \) for \( \alpha_s \approx 0.3 \) [11].

It has been also argued [12,13] that with increasing temperature in the hadronic phase \( \eta/s \) decreases and reaches a minimum at or near the critical temperature \( T_c \) and increases thereafter in the deconfined phase. Indeed this behavior has been observed in several substances in nature all of which satisfy the KSS bound suggesting the bound could be universal. However, no consensus has yet been reached [14] on the physical mechanism that accounts for the thermodynamic and transport properties of the system that leads to the minimal viscosity. Understanding the transport coefficients of the matter formed at RHIC is thus very challenging. Since in a heavy ion collision the system evolves from a QGP phase to the confined hadrons (low temperature phase of QCD), it is important to understand systematically the transport properties of a hadronic system in order to ascertain the sQGP properties with minimal uncertainty. In this Letter, we investigate the transport properties of an infinite equilibrated hadron matter comprising of massive hadronic resonances – the Hagedorn states [15–17] and all the low-lying observed hadrons within a Monte Carlo sequential binary emission model [18,19].

Several calculations of \( \eta/s \) in the hadronic phase have been performed with various techniques such as the chiral perturbation theory, linearized Boltzmann equation [20–22], and microscopic transport theory [23]. However, most of these calculations employ at most two component system or mesonic gas with a wide variation in the estimate of \( \eta/s \approx 0.08–1 \) at \( \sim T_c \). A recent microscopic transport calculation [24] within the UrQMD model accounted all the measured hadron species of mass \( m \lesssim 2 \text{ GeV} \) yields a minimum \( \eta/s \approx 0.9 \) at zero baryon chemical potential. This is significantly higher than in the viscous hydrodynamic estimate. Clearly it indicates the importance of massive resonances on the transport properties of hadron gas in the strong interaction domain of QCD matter especially near the critical temperature \( T_c = 196 \text{ MeV} \) predicted in lattice simulations [25].

On the other hand, it was proposed [15] that the density of hadronic states grows exponentially with resonance mass \( m, \rho_{\text{HS}}(m) \sim m^{-\alpha_s} \exp(m/T_H) \), where \( T_H \sim 150–200 \text{ MeV} \) is the Hagedorn temperature. The measured hadronic states up to \( m \sim 2 \text{ GeV} \) are indeed qualitatively consistent with the Hagedorn mass spectrum [26]. In absence of hadronic interaction, the energy density for such a system will diverge leading to a maximum/limiting temperature \( T_H \) for the hadronic matter. In the infinite mass limit, the thermodynamical quantities for the Hagedorn resonance gas would thus show critical behavior as it crosses \( T_H \approx T_c \) which may be associated with deconfining transition [27]. It has been also argued that the idea of Hagedorn states (HS) arise naturally in QCD at large \( N_c \) [28] and their indirect evidence can be found in lattice studies of thermodynamic properties of hadron matter above the...
deconfinement temperature [29]. For a hadron resonance gas with Hagedorn states the reduction of $\eta/s$ in the vicinity of $T_c$ has been demonstrated [30] within the rate equation approach.

In the present study of the properties of an equilibrated Hagedorn gas in the binary emission model [18,19], we include the Hagedorn states (mass $\gtrsim 2$ GeV) and all the low-lying measured hadrons available in the Particle Data Book. Detail description of the decay widths of HS and other resonances and their formation cross section in binary collisions during dynamic evolution of the system can be found in Ref. [18, 19]. For brevity, we mention that depending on the mass, a HS may undergo binary decay into one of the three possible channels (i) two observed discrete hadrons (DH) of $m < 2$ GeV, (ii) a DH and a HS, (iii) two HSS. Based on Hagedorn hypothesis the density of massive states are assumed to be

$$\rho(m, q) = \frac{m_{g} f(m-m_{g})}{(m-m_{g}) f(m-m_{g})}$$

in the usual notation [18,19]. Here $m_{g}$ = 0.5 GeV, and a HS of mass $m$ is characterized by its baryon, strangeness, spin and isospin quantum numbers $q = \{B, S, J, I, I_{c}\}$ with a ground state mass $m_{g} = q_{0}(\max\{3B + S, 2I\} + q_{s}S)$. The parameters $(\alpha_q, \alpha_s)$ are determined empirically from the measured smallest masses. This model was found to successfully describe the stable and resonant particle densities and their spectra at RHIC. We consider here the Hagedorn temperature $T_H$ = 196 MeV consistent with the critical temperature $T_c$ in lattice QCD prediction of a crossover transition at vanishing baryon chemical potential [25]. A resulting exponent $\alpha = 2.44$ is then estimated by comparing the theoretical and experimental cumulants of the spectrum [26,18].

To generate the equilibrated matter we initiate a system of Hagedorn states in a box with periodic boundary conditions in configuration space. Subsequent binary decay and their regeneration in binary collision during time evolution enforce the system to thermodynamic equilibrium. Chemical equilibrium is determined from the saturation of various particle densities when their production and annihilation rates become identical. While kinetic equilibrium occurs when the system approaches momentum isotropization. By fitting the particle energy spectra with a Boltzmann distribution $dN/d\eta \sim \exp(-E/\beta^*T^*)$, thermal equilibrium is verified when all the particle species at later times can be described by the same slope temperature $T^*$. For point-particles employed in our simulation, the pressure can be evaluated from the virial theorem as

$$P_{pt}(T^*) = \frac{1}{3V} \sum_{i=1}^{N_{max}} |p_i|^2 p_i^0,$$

where $p_i$ and $p_i^0$ are the momenta and energy of the $i$th particle. While the inclusion of Hagedorn states in the system provides an attractive interaction, use of point particles in the simulation ignores the repulsive interactions among the finite size hadrons especially the massive HS [15–17] that would drive the system to deconfinement at $T_c$. The effects of repulsive interaction can be included via excluded-volume approach [16] where the volume occupied by $i$th hadron in the relativistic approximation is $V_i = p_i^3/4B$, where $B$ is the effective MIT bag constant. The excluded-volume temperature and pressure are related to their point-particle analogs as [16]

$$T^* = \frac{P_{pt}(T^*)}{1 - P_{pt}(T^*)/4B}, \quad P(T) = \frac{P_{pt}(T^*)}{1 - P_{pt}(T^*)/4B}.$$

The other thermodynamic quantities can be obtained using the thermodynamic identities. We consider $B^{1/4}$ = 0.40 GeV that results in $P(T_c) < 4B$ with a corresponding limiting temperature $T_{lim} > T_c$ [16] as evident from Eq. (3). We restrict our calculations to $T < T_c$ in the confined hadronic system.

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In Fig. 2 (left panel) we present the equation of state, i.e. pressure versus density, of an equilibrated hadron matter at zero net baryon density. With increasing energy density when the temperature $T \sim T_c$, enhanced particle production of especially heavy resonances and HS causes softening of the equation of state. Consequently the speed of sound $c_s^2 = (\partial P/\partial s)_V$ in the medium drops gradually from $c_s^2 \approx 0.18$ at $T \approx 100$ MeV to about 0.09 at $T_c$. The estimated minimum speed of sound is consistent with the lattice data [25]. In contrast, the UrQMD model hadron resonance gas (without HS) yields a constant value of $c_s^2 = 0.18$ [24].

To gauge the dynamics of hadron–hadron interaction, we also investigate the thermodynamic properties in an independent statistical model for a hadron resonance gas in the grand canonical ensemble. The particle densities in the statistical model is given by [31]

$$\rho_i = \frac{g_i}{2\pi^3} \int_{0}^{\infty} \frac{4\pi P_i^2 dp}{\exp(P_i - \mu_i)/T^*} \pm 1,$$

where the $\pm$ sign refers to fermions/bosons. For the $i$th species, $g_i$ is the spin–isospin degeneracy and $\mu_i = \mu_B B_i + \mu_S S_i$ is the chemical potential with $\mu_B$ and $\mu_S$ the baryon and strangeness chemical potentials, respectively. For the observed resonances, $\rho_i(m)$ is taken as Breit–Wigner mass distribution while HS is replaced by $\rho_{H\Sigma}$ of Eq. (1). The results are found to be rather insensitive to the choice of upper mass limit of integration for HS $m_{max} \gtrsim 40$ GeV. In the limit $m_{max} \to \infty$, as the estimated expo-
ent $\alpha = 2.44$ in Eq. (1), the point-particle energy density, pressure and entropy density of an ideal classical (Boltzmann) Hagedorn gas remain finite at $T \leq T_{H}$ and diverge at $T > T_{H}$ exhibiting critical behavior [27]; albeit in the excluded-volume approach, the corresponding quantities evaluated at $T \leq T_{H} = T_{c}$ would remain finite.

In Fig. 2 (left panel) we also show the equation of state obtained in the statistical model. Compared to a pion gas (stars), inclusion of measured discrete resonances up to mass $\sim 2$ GeV (triangles) results in a considerable decrease in the system pressure. The interaction measure $\Theta = (\epsilon - 3P)/T^{4}$, then gradually increases as $T \to T_{c}$. Further inclusion of Hagedorn states in the statistical model opens up additional degrees of freedom and provides an attractive interaction that slows the pressure increase at $\epsilon \geq 1.5$ GeV/fm$^{3}$. The trace anomaly $\Theta$ then increases dramatically near $T = T_{H}$. The speed of sound in the model with HS turns out to be rather small $c_{s}^{2} \approx 0.07$ at $T_{c}$. Comparison of this curve with sequential binary emission model result provides a measure of the dynamics of repulsive interaction in the medium. The interaction effect starts at temperature above the pion mass and becomes significant at high energies where massive resonances are produced.

The point-particle entropy density of the system can be computed using the Gibbs formula

$$s_{pt}(T^{*}) = \left( \frac{\epsilon_{pt} + P_{pt} - \sum_{i} \mu_{i}\rho_{i}}{T^{*}} \right),$$

$$= (\epsilon_{pt} + P_{pt} - \mu_{B}\rho_{B})/T^{*}, \quad (5)$$

where $\rho_{B} = \sum_{i} B_{i}\rho_{i}$ is the net baryon density and the (initial) net strangeness density $\rho_{S}$ is set to zero in our calculation. The baryon chemical potential $\mu_{B}$ is evaluated from the particle yield ratios at equilibrium. Fig. 2 (right panel) shows the excluded-volume entropy density [16] as a function of temperature at zero baryon chemical potential. In the statistical model, compared to the pion gas, larger degrees of freedom in the hadron (and Hagedorn) resonance gas at the same energy density lowers its temperature which results in higher entropy [32]. In the sequential emission model the increase (or slope) of entropy density with temperature near $T_{c}$ is somewhat less dramatic and captures the trend associated with that of a crossover transition obtained in the lattice data [25].

We now extract the shear viscosity for the dynamically evolving system of an equilibrated hadron resonance gas with Hagedorn states using the Kubo relation [33]. The transport coefficients determine the dynamics of fluctuations of dissipative fluxes about the equilibrated state. The Kubo formalism employs the linear response theory to relate the transport coefficients as correlations of dissipative fluxes by considering the dissipative fluxes as perturbations to local thermodynamic equilibrium [33–35]. The Green–Kubo relation for shear viscosity is

$$\eta = \frac{1}{T} \int d^{3}r \int_{0}^{\infty} dt \left\langle \pi_{xy}(0,0)\pi_{xy}(r,t) \right\rangle_{\text{eqlb}}, \quad (6)$$

where $t$ refers to time after the system equilibrates which is set at $t = 0$, and $\pi_{xy}$ is the shear component (traceless part) of the energy momentum tensor $\pi^{\mu\nu}$:

$$\pi_{xy} = \int d^{3}p \frac{p_{x}p_{y}}{p^{0}} f(x,p) = \frac{1}{V} \sum_{i=1}^{N_{\text{part}}} \frac{p_{x}^{i}p_{y}^{i}}{p_{0}^{i}}. \quad (7)$$

The equivalence accounts for the point particles used in our simulation. The averaging in Eq. (6) is over the ensemble of events generated in the simulation.

Fig. 3 displays the correlation function of the shear-stress tensor for a function of time. The correlation functions are found to damp exponentially with time, i.e. $\langle \pi_{xy}(0)\pi_{xy}(t) \rangle \sim \exp(-t/\tau_{\pi})$. Note at small temperatures (right panel) the correlations sustain for long times however show a power law tail. The relaxation times $\tau_{\pi}$ for the shear fluxes so obtained by fitting the correlation functions are found to decrease with increasing temperature. The shear viscosity is estimated from Eq. (6) by integrating the correlation function over time using the exponential decay found empirically.

The shear viscosity to entropy density ratio $\eta/s$ for the Hagedorn resonance gas in equilibrium at zero baryon chemical potential is presented in Fig. 4. Also shown in the figure are the $\eta/s$ values for hadron resonance gas without Hagedorn states in the UrQMD model [24]. Our results for $\eta/s$ are comparable to the UrQMD estimates at temperatures $T < 150$ MeV. At higher temperatures, massive resonances contribute dominantly resulting in gradual reduction of $\eta/s$ with temperature [30] and reaches a minimum of $(\eta/s)_{\text{min}} \approx 0.34$ at $T_{c} = 196$ MeV. In contrast, the UrQMD model calculation shows almost a constant value for $\eta/s \approx 0.9$ at $T \approx 125–166$ MeV. The reduction of $\eta/s$ with Hagedorn mass spectrum can be realized as primarily due to enhancement of massive resonances that leads to a decrease in the mean free path of
Fig. 4. The shear viscosity to entropy density ratio $\eta/s$ as a function of temperature for a hadron resonance gas with Hagedorn states in thermodynamic equilibrium at various baryon chemical potential $\mu_B$. The $\eta/s$ for a pure hadron resonance gas in the UrQMD model [24] is also shown (squares).

While weakly coupled perturbative QCD estimate of $\eta/s \approx 1$ at $T \gg T_c$, the QGP formed at RHIC in the region $T_c \leq T \lesssim 2T_c$ is thought to be strongly interacting. The current estimate of $\eta/s \approx 0.08-0.24$ [6–8,10,39] at RHIC may be related to a minimum at $T_c$. Our estimate of $\eta/s$ for the Hagedorn resonance gas suggests that, as the sQGP cools, the $\eta/s$ in deconfined phase makes a relatively continuous/smooth transition into the confined phase near $T_c$ without any discontinuity or a sharp increase proposed [24] in absence of Hagedorn states.

In summary, we have studied the thermodynamic and transport properties of an infinite equilibrated matter composed of interacting hadron resonance gas plus an exponentially increasing Hagedorn mass spectrum. At temperatures close to the Hagedorn temperature, $T_H = T_c = 196$ MeV, the particle densities of the massive resonances are enhanced significantly. This leads to a sharp rise in the trace anomaly and the entropy density of the system which captures the trend seen in the lattice data at vanishing baryon chemical potential. The shear viscosity to entropy density ratio of the Hagedorn resonance gas, both in and out of chemical equilibrium, decreases with temperature and baryon chemical potential and reaches a minimum of $(\eta/s)_{\text{min}} \approx 0.3$ at about $T_c$. The extracted minimum value is near the upper bound of current estimates for a strongly coupled quark–gluon plasma formed in ultra-relativistic heavy ion collisions at RHIC.

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References