Computers and Mathematics with Applications 56 (2008) 2686–2691

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Non-planar shock waves in a magnetic field

Rajan Arora *

DPT, IIT Roorkee, Saharanpur Campus, Saharanpur-247001, UP, India

ARTICLE INFO

Article history:
Received 21 February 2007
Received in revised form 6 March 2008
Accepted 25 March 2008

Keywords:
Weakly nonlinear hyperbolic waves
Resonance
Shock waves
Magnetic field
Asymptotic solution

ABSTRACT

The method of multiple time scale is used to obtain the asymptotic solution to the spherically and cylindrically symmetric flow into a perfectly conducting gas permeated by a transverse magnetic field. The transport equations for the amplitudes of resonantly interacting high frequency waves are also found. The evolutionary behavior of non-resonant wave modes culminating into shock waves is studied.

© 2008 Elsevier Ltd. All rights reserved.

1. Introduction

The occurrence of shock waves in gas dynamics has been well studied in the past. The propagation of shock waves under the influence of strong magnetic field constitutes a problem of great interest to researchers in many branches of science and astrophysics. The work of Manganaro and Oliveri [1], Kumar et al. [2], Sharma and Srinivasan [3], Hunter and Ali [4], Webb et al. [5], Tsiklauri [6], Gunderson [7], Whitham [8], Moodie et al. [9], He and Moodie [10], Sharma and Arora [11], and Arora and Sharma [12] is worth mentioning in the context of this paper.

We use the asymptotic method for the analysis of weakly nonlinear hyperbolic waves. Choquet-Bruhat [13] proposed a method to discuss shockless solutions of hyperbolic systems which depend upon a single phase function. Germain [14] has given the general discussion of single phase progressive waves. Hunter and Keller [15] established a general non-resonant multi-wave-mode theory which has led to several interesting generalizations by Majda and Rosales [16] and Hunter et al. [17] to include resonantly interacting multi-wave-mode features.

We use the resonantly interacting multi-wave theory to examine small amplitude high frequency asymptotic waves for one-dimensional unsteady non-planar flows of a general inviscid ideal gas permeated by a transverse magnetic field, where at the leading order many waves coexist and interact with one another resonantly, and obtain evolution equations which describe the resonant wave interactions inherent in the system.

2. Basic equations

Assuming the electrical conductivity to be infinite and the direction of the magnetic field orthogonal to the trajectories of the fluid particles, the basic equations for a one-dimensional non-planar motion can be written as [1,2]

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial u}{\partial x} + u \frac{\partial \rho}{\partial x} + \frac{m \rho u}{x} = 0,$$

* Tel.: +91 1322714006.
E-mail addresses: rajan_a100@yahoo.com, rajanfpt@iitr.ernet.in.

0898-1221/$ – see front matter © 2008 Elsevier Ltd. All rights reserved.
doi:10.1016/j.camwa.2008.03.056

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{1}{\rho}\left(\frac{\partial p}{\partial x}+\frac{\partial h}{\partial x}\right)=0,$$
$$\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}+\rho a^2\frac{\partial u}{\partial x}+\frac{m\gamma pu}{x}=0,$$
$$\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial x}+2h\frac{\partial u}{\partial x}+\frac{2mhu}{x}=0, \tag{1}$$

where u is the fluid velocity, p is the pressure, ρ is the density, $h=\mu H^2/2$ is the magnetic pressure with H as the transverse magnetic field strength and μ as the magnetic permeability, $a=(\gamma p/\rho)^{1/2}$ is the speed of sound with γ as the adiabatic exponent, t is the time, and x is the spatial coordinate; $m=1$ and 2 correspond, respectively, to cylindrical and spherical symmetry. Also, all the variables in system (1) are dimensionless.

The Eq. (1) may be written in the matrix form as

$$\frac{\partial U}{\partial t}+A\frac{\partial U}{\partial x}+B=0 \tag{2}$$

where U and B are the column vectors defined by

$$U=(\rho,u,p,h)^{\mathrm{tr}},\qquad B=\left(\frac{m\rho u}{x},0,\frac{m\gamma pu}{x},\frac{2mhu}{x}\right)^{\mathrm{tr}}, \tag{3}$$

A is the 4×4 matrix having the components A^{ij}, the non-zero ones are as follows

$$A^{11}=A^{22}=A^{33}=A^{44}=u,$$
$$A^{12}=\rho,\qquad A^{23}=A^{24}=1/\rho,\qquad A^{32}=\rho a^2,\qquad A^{42}=2h. \tag{4}$$

System (2) being strictly hyperbolic admits four families of characteristics, among them two represent waves propagating in $\pm x$ directions with the speed $u\pm c$, where $c=(a^2+b^2)^{1/2}$ represents the magneto-acoustic speed with $b=(2h/\rho)^{1/2}$ as the Alfvén speed. The remaining two families form a set of double characteristics representing entropy waves or particle paths propagating with velocity u. We consider waves propagating into a initial background state $U_0=(\rho_0,0,p_0,h_0)^{\mathrm{tr}}$. The characteristic speeds at $U=U_0$ are given by $\lambda_1=0$, $\lambda_2=0$, $\lambda_3=c_0$ and $\lambda_4=-c_0$. The subscript 0 refers to evaluation at $U=U_0$, and is synonymous with a state of equilibrium.

3. Interaction of High Frequency Waves

We denote the left and right eigenvectors of A_0 associated with the eigenvalue λ_i by $L^{(i)}$ and $R^{(i)}$. These eigenvectors satisfy the normalization condition $L^{(i)}R^{(j)}=\delta_{ij}$, $1\le i,j\le 4$, where δ_{ij} represents the Kronecker delta. These eigenvectors are given as

$$L^{(1)}=\left(1,0,-\frac{1}{c_0^2},-\frac{1}{c_0^2}\right),\qquad R^{(1)}=(1,0,a_0^2,-a_0^2)^{\mathrm{tr}},$$
$$L^{(2)}=\left(-a_0^2,0,1,0\right),\qquad R^{(2)}=(0,0,1,-1)^{\mathrm{tr}},$$
$$L^{(3)}=\left(0,\frac{\rho_0}{2c_0},\frac{1}{2c_0^2},\frac{1}{2c_0^2}\right),\qquad R^{(3)}=\left(1,\frac{c_0}{\rho_0},a_0^2,b_0^2\right)^{\mathrm{tr}}. \tag{5}$$
$$L^{(4)}=\left(0,-\frac{\rho_0}{2c_0},\frac{1}{2c_0^2},\frac{1}{2c_0^2}\right),\qquad R^{(4)}=\left(1,-\frac{c_0}{\rho_0},a_0^2,b_0^2\right)a^{\mathrm{tr}}.$$

We look for asymptotic solution for (2) as $\epsilon\to 0$ of the form

$$U\sim U_0+\epsilon U_1(x,t,\tilde{\theta})+\epsilon^2U_2(x,t,\tilde{\theta})+O(\epsilon^3), \tag{6}$$

where U_1 is a smooth bounded function of its arguments and U_2 is bounded in (x,t) in a certain bounded region of interest having at most sub-linear growth in θ as $\theta\to\pm\infty$. Here $\tilde{\theta}=(\theta_1,\theta_2,\theta_3,\theta_4)$ represents the "fast variables" characterized by the functions ϕ_i as $\theta_i=\phi_i/\epsilon$, where ϕ_i, $1\le i\le 4$, is the phase of the ith wave associated with the characteristic speed λ_i. Now we use (6) in (2), expand A and B in Taylor's series in powers of ϵ about $U=U_0$, replace the partial derivatives $\frac{\partial}{\partial X}$ (X being either x or t) by $\frac{\partial}{\partial X}+\epsilon^{-1}\sum_{i=1}^{4}\frac{\partial\phi_i}{\partial X}\frac{\partial}{\partial\theta_i}$, and equate to zero the coefficients of ϵ^0 and ϵ^1 in the resulting expansions, to obtain

$$O(\epsilon^0):\sum_{i=1}^{4}\left(I\frac{\partial\phi_i}{\partial t}+A_0\frac{\partial\phi_i}{\partial x}\right)\frac{\partial U_1}{\partial\theta_i}=0 \tag{7}$$

$$O(\epsilon^1):\sum_{i=1}^{4}\left(I\frac{\partial\phi_i}{\partial t}+A_0\frac{\partial\phi_i}{\partial x}\right)\frac{\partial U_2}{\partial\theta_i}=-\frac{\partial U_1}{\partial t}-A_0\frac{\partial U_1}{\partial x}-(U_1.\nabla B)_0-\sum_{i=1}^{4}\frac{\partial\phi_i}{\partial x}(U_1.\nabla A)_0\frac{\partial U_1}{\partial\theta_i} \tag{8}$$

where I is the 4×4 unit matrix and ∇ is the gradient operator with respect to the dependent variable U. Now since the phase functions ϕ_i, $1 \le i \le 4$, satisfy the eikonal equation

$$\mathrm{Det}\left(I\frac{\partial \phi_i}{\partial t} + A_0\frac{\partial \phi_i}{\partial x}\right) = 0, \tag{9}$$

we choose the simplest phase function of this equation, namely

$$\phi_i(x,t) = x - \lambda_i t, \quad 1 \le i \le 4. \tag{10}$$

It follows from (5) that for each phase ϕ_i, $\frac{\partial U_1}{\partial \theta_i}$ is parallel to the right eigenvector $R^{(i)}$ of A_0 and thus

$$U_1 = \sum_{i=1}^{4} \sigma_i(x,t,\theta_i)R^{(i)}, \tag{11}$$

where $\sigma_i = (L^{(i)} \cdot U_1)$ is a scalar function called the wave amplitude, that depends only on the ith fast variable θ_i. We assume that $\sigma_i(x,t,\theta_i)$ has zero mean value with respect to the fast variable θ_i, that is,

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \sigma_i(x,t,\theta_i)\mathrm{d}\theta_i = 0. \tag{12}$$

We then use (11) in (8) and solve for U_2. To begin with we write

$$U_2 = \sum_{j=1}^{4} m_j R^{(j)},$$

substitute this value in (8), and premultiply the resulting equation by $L^{(i)}$ to obtain the system of decoupled inhomogeneous first order partial differential equations:

$$\sum_{j=1}^{4}(\lambda_i - \lambda_j)\frac{\partial m_i}{\partial \theta_j} = -\frac{\partial \sigma_i}{\partial t} - \lambda_i\frac{\partial \sigma_i}{\partial x} - L^{(i)}(U_1 \cdot \nabla B)_0 - \sum_{j=1}^{4} L^{(i)}(U_1 \cdot \nabla A)_0\frac{\partial U_1}{\partial \theta_j}, \quad 1 \le i \le 4. \tag{13}$$

The characteristic ODEs for the ith equation in (13) are given by

$$\dot{\theta}_j = \lambda_i - \lambda_j \quad \text{for } j \ne i, \qquad \dot{\theta}_i = 0, \qquad \dot{m}_i = H_i, \tag{14}$$

where $H_i(x,t,\theta_1,\theta_2,\theta_3,\theta_4) = -\frac{\partial \sigma_i}{\partial t} - \lambda_i\frac{\partial \sigma_i}{\partial x} - L^{(i)}(U_1 \cdot \nabla B)_0 - \sum_{j=1}^{4} L^{(i)}(U_1 \cdot \nabla A)_0\frac{\partial U_1}{\partial \theta_j}$.

We asymptotically average (13) along the characteristics and appeal to the sub-linearity of U_2 in θ, which ensures that the expression (6) does not contain secular terms. The constancy of θ_i along the characteristics and the vanishing asymptotic mean value of $\dot{m}_i$ along the characteristics imply that the wave amplitudes σ_i, $1 \le i \le 4$, satisfy the following system of coupled integro-differential equations

$$\frac{\partial \sigma_i}{\partial t} + \lambda_i\frac{\partial \sigma_i}{\partial x} + a_i\sigma_i + \Gamma^i_{ii}\sigma_i\frac{\partial \sigma_i}{\partial \theta_i} + \sum_{i\ne j\ne k}\Gamma^i_{jk}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\sigma_j(\theta_i + (\lambda_i - \lambda_j)s)\acute{\sigma}_k(\theta_i + (\lambda_i - \lambda_k)s)\mathrm{d}s = 0, \tag{15}$$

where $\acute{\sigma}_k = \frac{\partial \sigma_k}{\partial \theta_k}$ and the coefficients a_i and Γ^i_{jk} are given by

$$a_i = L^{(i)}(R^{(i)} \cdot \nabla B)_0, \qquad \Gamma^i_{jk} = L^{(i)}(R^{(j)} \cdot \nabla A)_0 R^{(k)}. \tag{16}$$

The interaction coefficients Γ^i_{jk}, which are asymmetric in j and k, denote the strength of coupling between the jth and kth wave modes ($j \ne k$) that can generate an ith wave ($i \ne j \ne k$). The coefficients Γ^i_{ii} which refer to the nonlinear self-interaction, are non-zero for genuinely nonlinear waves and zero for linearly degenerate waves. It is also interesting to note that if all the coupling coefficients Γ^i_{jk} ($i \ne j \ne k$) are zero or the integral in (15) vanishes, the waves do not resonate and (15) reduces to a system of uncoupled Burgers' equations. The coefficients a_i, Γ^i_{ii} and Γ^i_{jk}, given by (15), provide a picture of the nonlinear interaction process present in the system under consideration, and can be easily determined in the following form; the non-zero ones being:

$$a_3 = \frac{mc_0}{2x}, \qquad a_4 = -\frac{mc_0}{2x},$$

$$\Gamma^1_{23} = -\Gamma^1_{24} = \frac{(2-\gamma)}{\rho_0 c_0},$$

$$\Gamma^1_{34} = -\Gamma^1_{43} = \frac{c_0^2 + a_0^2(\gamma - 2)}{\rho_0 c_0},$$

$$\Gamma^2_{13} = \Gamma^2_{43} = -\Gamma^2_{14} = -\Gamma^2_{34} = (\gamma - 1)a_0^2 \frac{c_0}{\rho_0},$$
$$\Gamma^3_{14} = -\Gamma^4_{13} = -\frac{c_0^2 + a_0^2(\gamma - 2)}{2\rho_0 c_0} = \beta_1, \tag{17}$$
$$\Gamma^3_{24} = -\Gamma^4_{23} = \frac{(2-\gamma)}{2\rho_0 c_0} = \beta_2.$$

The resonance equation (15) can now be written as

$$\frac{\partial \sigma_1}{\partial t} = 0, \qquad \frac{\partial \sigma_2}{\partial t} = 0 \tag{18}$$
$$\frac{\partial \sigma_3}{\partial t} + c_0 \frac{\partial \sigma_3}{\partial x} + \frac{mc_0}{2x}\sigma_3 + \Gamma \sigma_3 \frac{\partial \sigma_3}{\partial \theta_3} - \lim_{P\to\infty} \frac{1}{2P}\int_{-P}^{P} K\left(x, t, \frac{\theta_3 + \phi}{2}\right)\sigma_4(x, t, \phi)\mathrm{d}\phi = 0,$$
$$\frac{\partial \sigma_4}{\partial t} - c_0 \frac{\partial \sigma_4}{\partial x} - \frac{mc_0}{2x}\sigma_4 - \Gamma \sigma_4 \frac{\partial \sigma_4}{\partial \theta_4} + \lim_{P\to\infty} \frac{1}{2P}\int_{-P}^{P} K\left(x, t, \frac{\theta_4 + \phi}{2}\right)\sigma_3(x, t, \phi)\mathrm{d}\phi = 0,$$

where $\Gamma = \Gamma^3_{33} = -\Gamma^4_{44} = [3c_0^2 - (2-\gamma)a_0^2]/(2c_0\rho_0)$, and the kernel K is defined as

$$K\left(x, t, \frac{\theta + \phi}{2}\right) = \frac{\beta_1}{2}\frac{\partial \sigma_1}{\partial \theta_1}\left(x, t, \frac{\theta + \phi}{2}\right) + \frac{\beta_2}{2}\frac{\partial \sigma_2}{\partial \theta_2}\left(x, t, \frac{\theta + \phi}{2}\right). \tag{19}$$

Let the initial value of σ_j be $\sigma_j|_{t=0} = \sigma_j^0(x, \theta_j)$. Hence $(18)_{1,2}$ gives $\sigma_1(x, t, \theta_1) = \sigma_1^0(x, \theta_1)$ and $\sigma_2(x, t, \theta_2) = \sigma_2^0(x, \theta_2)$, and subsequently the system (18) reduces to a pair of equations for the wave fields σ_3 and σ_4 coupled through the linear integral operator involving the kernel

$$K(x, t, \theta) = \frac{\beta_1}{2}\frac{\partial \sigma_1^0}{\partial \theta_1}(x, \theta) + \frac{\beta_2}{2}\frac{\partial \sigma_2^0}{\partial \theta_2}(x, \theta). \tag{20}$$

If the initial data $\sigma_j^0(x, \theta)$ are 2π periodic functions of the phase variable θ, then the pair of resonant asymptotic equations in system (18) becomes

$$\frac{\partial \sigma_3}{\partial t} + c_0 \frac{\partial \sigma_3}{\partial x} + \frac{mc_0}{2x}\sigma_3 + \Gamma \sigma_3 \frac{\partial \sigma_3}{\partial \theta_3} - \frac{1}{2\pi}\int_{-\pi}^{\pi} K\left(x, t, \frac{\theta_3 + \phi}{2}\right)\sigma_4(x, t, \phi)\mathrm{d}\phi = 0, \tag{21}$$
$$\frac{\partial \sigma_4}{\partial t} - c_0 \frac{\partial \sigma_4}{\partial x} - \frac{mc_0}{2x}\sigma_4 - \Gamma \sigma_4 \frac{\partial \sigma_4}{\partial \theta_4} + \frac{1}{2\pi}\int_{-\pi}^{\pi} K\left(x, t, \frac{\theta_4 + \phi}{2}\right)\sigma_3(x, t, \phi)\mathrm{d}\phi = 0,$$

where K is given by (20).

4. Nonlinear geometrical acoustics solution

The asymptotic solution (6) of hyperbolic system (2) satisfying small amplitude oscillating initial data

$$U(x, 0) = U_0 + \epsilon U_1^0(x, x/\epsilon) + O(\epsilon^2), \tag{22}$$

is non-resonant if $U_1^0(x, x/\epsilon)$ are smooth functions with a compact support [16]. The expansion (6) is uniformly valid to the leading order till shock waves have formed in the solution.

The characteristic equations are

$$\frac{\mathrm{d}\theta_j}{\mathrm{d}x} = \frac{\Gamma \sigma_j}{c_0}, \qquad \frac{\mathrm{d}t}{\mathrm{d}x} = \frac{e_j}{c_0}, \tag{23}$$

where

$$e_j = \begin{cases} +1, & \text{if } j = 3, \\ -1, & \text{if } j = 4. \end{cases}$$

In terms of the characteristic equations, the decoupled equations $(21)_2$ and $(21)_3$ can be written as

$$\frac{\mathrm{d}\sigma_j}{\mathrm{d}x} = -\frac{m\sigma_j}{2x}, \tag{24}$$

which yields on integration

$$\sigma_j = \sigma_j^0(s_j, \xi_j)(x/s_j)^{-m/2} \tag{25}$$

along the rays $s_j = x - e_j c_0 t =$ constant, where the function σ_j^0 is obtained from the initial condition (22), and the fast variable ξ_j parametrizes the set of characteristic curves $(23)_1$.

Thus, we obtain from (23)

$$\xi_j = \theta_j - e_j \Gamma \sigma_j^0(s_j, \xi_j) I_j^{(m)}(t), \tag{26}$$

where $I_j^{(m)}(t) = \int_0^t \left(1 + \frac{e_j c_0}{s_j}\tilde{t}\right)^{-m/2} \mathrm{d}\tilde{t}$.

The solution of (2), satisfying (22), where $U_1^0(x, x/t)$ has compact support, is obtained as

$$\begin{aligned}
\rho(x,t) &= \rho_0 + \epsilon\sigma_1^0(x, x/\epsilon) + \epsilon x^{-\frac{m}{2}} \left((x - c_0 t)^{\frac{m}{2}} \sigma_3^0(s_3, \xi_3) + (x + c_0 t)^{\frac{m}{2}} \sigma_4^0(s_4, \xi_4)\right) + O(\epsilon^2), \\
u(x,t) &= \epsilon \frac{c_0}{\rho_0} x^{-\frac{m}{2}} \left((x - c_0 t)^{\frac{m}{2}} \sigma_3^0(s_3, \xi_3) - (x + c_0 t)^{\frac{m}{2}} \sigma_4^0(s_4, \xi_4)\right) + O(\epsilon^2), \\
p(x,t) &= p_0 + \epsilon a_0^2 \sigma_1^0(x, x/\epsilon) + \epsilon\sigma_2^0(x, x/\epsilon) + \epsilon a_0^2 x^{-\frac{m}{2}} \left((x - c_0 t)^{\frac{m}{2}} \sigma_3^0(s_3, \xi_3) + (x + c_0 t)^{\frac{m}{2}} \sigma_4^0(s_4, \xi_4)\right) + O(\epsilon^2), \\
h(x,t) &= h_0 - \epsilon a_0^2 \sigma_1^0(x, x/\epsilon) - \epsilon\sigma_2^0(x, x/\epsilon) + \epsilon b_0^2 x^{-\frac{m}{2}} \left((x - c_0 t)^{\frac{m}{2}} \sigma_3^0(s_3, \xi_3) + (x + c_0 t)^{\frac{m}{2}} \sigma_4^0(s_4, \xi_4)\right) + O(\epsilon^2)
\end{aligned} \tag{27}$$

where the fast variables ξ_j $(1 \le j \le 4)$ are given in (26), and the initial values for σ_i, $(1 \le i \le 4)$ are obtained from the solution (27) specified at $t = 0$ as

$$\begin{aligned}
\sigma_1^0(x, x/\epsilon) &= \rho_1^0(x, x/\epsilon) - \left(\frac{1}{c_0^2}\right) \left(p_1^0(x, x/\epsilon) + h_1^0(x, x/\epsilon)\right) \\
\sigma_2^0(x, x/\epsilon) &= -a_0^2 \rho_1^0(x, x/\epsilon) + p_1^0(x, x/\epsilon) \\
\sigma_3^0(x, \xi_3) &= \left(\frac{\rho_0}{2c_0}\right) u_1^0(x, \xi_3) + \left(\frac{1}{2c_0^2}\right) \left(p_1^0(x, \xi_3) + h_1^0(x, \xi_3)\right), \\
\sigma_4^0(x, \xi_4) &= -\left(\frac{\rho_0}{2c_0}\right) u_1^0(x, \xi_4) + \left(\frac{1}{2c_0^2}\right) \left(p_1^0(x, \xi_4) + h_1^0(x, \xi_4)\right).
\end{aligned} \tag{28}$$

This is the complete solution of (2) and (22); any multi-valued overlap in this solution is resolved by introducing shocks into the solution.

5. Shock waves

Following [15] it can be shown that the shock location θ_j^s satisfies the relation

$$\frac{\mathrm{d}\theta_j^s}{\mathrm{d}t} = \frac{1}{2} \Lambda_{jj}^j \left(\sigma_j^{(-)} + \sigma_j^{(+)}\right), \quad j = 3, 4 \tag{29}$$

which is the shock speed in the $\theta_j - t$ plane. Here $\sigma_j^{(-)}$ and $\sigma_j^{(+)}$, respectively, are the values of σ_j just ahead and behind the shock. We have $\sigma_j^{(-)} = 0$ for the undisturbed region ahead of the shock. Now we use (25) and drop the superscripts on θ_j^s and σ_j^+ to obtain

$$\frac{\mathrm{d}\theta_j}{\mathrm{d}t} = \frac{\Gamma}{2} e_j \sigma_j^0(s_j, \xi_j) \left(\frac{x}{s_j}\right)^{-\frac{m}{2}}. \tag{30}$$

Using Eqs. (30) and (26) we obtain the following relation between ξ_j and t on the shock

$$I_j^m(t) = -\left(\frac{2e_j}{(\sigma_j^0)^2 \Gamma}\right) \int_0^{\xi_j} \sigma_j^0(\tilde{t}) \mathrm{d}\tilde{t}. \tag{31}$$

Now using (31) and (26) we obtain the following equation which determines the shock path parametrically,

$$\theta_j = \xi_j - \frac{2}{\sigma_j^0} \int_0^{\xi_j} \sigma_j^0(\tilde{t}) \mathrm{d}\tilde{t}. \tag{32}$$

If $\sigma_j^0 \neq 0$ then the shock forms right at the origin.

6. Conclusion and discussion

The method of multiple scales is used to obtain small amplitude high frequency asymptotic solution to the basic equations governing one-dimensional cylindrically and spherically symmetric flow in an ideal gas, where it is assumed that the electrical conductivity is infinite, and the direction of the magnetic field is orthogonal to the trajectories of the fluid particles. Weakly nonlinear geometrical acoustics theory is used to analyze the resonant wave interaction. We derived the transport equations for the wave amplitudes along the rays of the governing system; these transport equations constitute a system of inviscid Burger's' equations with quadratic nonlinearity coupled through linear integral operators with a known kernel. The coefficients appearing in the transport equations provide a measure of coupling between the various modes and set a qualitative picture of the interaction process involved therein. It is observed that the wave fields associated with the particle paths do not interact with each other; however they do interact with an acoustic wave field to produce resonant contribution towards the other acoustic field. The acoustic wave fields may or may not interact, but in either case their net contribution, which is directed towards the entropy field, is always zero. In our analysis the governing system of Euler equations reduces to a pair of resonant asymptotic equations for the acoustic wave fields. For a non-resonant multi-wave-mode case, proposed by Hunter and Keller [15], the reduced system of transport equations gets decoupled with vanishing integral average terms, and the occurrences of shocks in the acoustic wave fields are analyzed.

It is found that in a contracting piston motion having spherical symmetry, a shock is always formed before the formation of a focus no matter how small the initial wave amplitude may be; this is in contrast with the corresponding cylindrical situation where a shock forms before the focus only if the initial amplitude exceeds a critical value.

It is also found that cylindrical and spherical shock waves decay like $t^{-3/4}$ and $t^{-1}(\log t)^{-1/2}$, respectively; these results are in agreement with earlier results [8].

References

[1] N. Manganaro, F. Oliveri, Group Analysis approach in magnetohydrodynamics: Weak discontinuity propagation in a non-constant state, Meccanica 24 (1989) 71–78.
[2] A. Kumar, V.D. Sharma, B.D. Pandey, On the formation of secondary shocks in a self-similar magnetogasdynamic flow, Canadian Journal of Physics 68 (1990) 1203–1206.
[3] V.D. Sharma, Gopala Krishna Srinivasan, Wave interaction in a non-equilibrium gas flow, International Journal of Non-Linear Mechanics 40 (2005) 1031–1040.
[4] J.K. Hunter, G. Ali, Wave interactions in magnetohydrodynamics, Wave Motion 27 (1998) 257–277.
[5] G.M. Webb, A.R. Zakharian, M. Brio, G.P. Zank, Nonlinear and three-wave resonant interactions in magnetohydrodynamics, Journal of Plasma Physics 63 (2000) 393–445.
[6] D. Tsiklauri, T.D. Arber, V.M. Nakariakov, A weakly nonlinear Alfvenic pulse in a transversely inhomogeneous medium, Astronomy and Astrophysics 379 (2001) 1098–1105.
[7] R.M. Gunderson, Linearized analysis of one-dimensional magneto-hydrodynamic flows, Springer Tracts in Natural Philosophy 1 (1964).
[8] G.B. Whitham, Linear and Nonlinear Waves, Wiley, New York, 1974.
[9] T.B. Moodie, Y. He, D.W. Barclay, Wavefront expansions for nonlinear hyperbolic waves, Wave Motion 14 (1991) 347–367.
[10] Y. He, T.B. Moodie, Geometrical optics and post shock behaviour for nonlinear conservation laws, Applicable Analysis 57 (1995) 145–176.
[11] V.D. Sharma, Rajan Arora, Similarity solutions for strong shocks in an ideal gas, Studies in Applied Mathematics 114 (2005) 375–394.
[12] Rajan Arora, V.D. Sharma, Convergence of strong shock in a Van der Waals gas, SIAM Journal of Applied Mathematics 66 (2006) 1825–1837.
[13] V. Choquet-Bruhat, Ondes asymptotique et approchees pour systemes d'equations aux derivees partielles nonlineaires, Journal of Mathématiques Pures et Appliquies 48 (1969) 119–158.
[14] P. Germain, Progressive waves, in: 14th Prandtl Memorial Lecture, Jarbuch, der DGLR, 1971, pp. 11–30.
[15] J.K. Hunter, J. Keller, Weakly nonlinear high frequency waves, Communications on Pure and Applied Mathematics 36 (1983) 547–569.
[16] A. Majda, R. Rosales, Resonantly interacting weakly nonlinear hyperbolic waves, Studies in Applied Mathematics 71 (1984) 149–179.
[17] J.K. Hunter, A. Majda, R. Rosales, Resonantly interacting weakly nonlinear hyperbolic waves II. Several space variables, Studies in Applied Mathematics 75 (1986) 187–226.