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Computing the spectrum of non-self-adjoint Sturm–Liouville problems with parameter-dependent boundary conditions

B. Chanane*

Mathematical Science Department, K.F.U.P.M., Dhahran 31261, Saudi Arabia

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Abstract

This paper deals with the computation of the eigenvalues of non-self-adjoint Sturm–Liouville problems with parameter-dependent boundary conditions using the *regularized sampling method*.

A few numerical examples among which singular ones will be presented to illustrate the merit of the method and comparison made with the exact eigenvalues when they are available.

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Keywords: Sturm–Liouville problems; Non-self-adjoint eigenvalue problems; Singular Sturm–Liouville problems; Shannon's sampling theory; Regularized sampling method; Whittaker–Shannon–Kotel'nikov theorem

1. Introduction

Non-self-adjoint eigenvalue problems arise, as is well known, in hydrodynamic and magnetohydrodynamic stability [9,8,11,17] while self-adjoint problems arise mostly in quantum mechanics [13]. The lack of oscillation theorems in the non-self-adjoint case makes any computation of the spectrum a very difficult task [10]. In fact, the eigenvalues are scattered over the complex plane and we need first to determine the regions which contain them. A method that finds the eigenvalues in a rectangle and in a left half plane has been introduced in [15]. It is based on the argument principle with compound matrix method using Magnus expansion. In [4] the authors report on a method that provides bounds for the eigenvalues of singular Sturm–Liouville problems over $[0, \infty)$ with a complex potential. The method consists in obtaining first a floating point approximation to the desired eigenvalue by truncating the infinite interval then use interval arithmetic to localize the eigenvalue. For more on singular problems see [3] and [14] for example. In [1], the author uses the sampling method introduced in [2] to compute the eigenvalues of non-self-adjoint Sturm–Liouville problems.

For the mathematical foundation one may consult [12,18,13]. On the numerical side [19,16] summarize most of the available software dealing with the computation of the eigenvalues of Sturm–Liouville problems.

* Tel.: +966 3 8602741.

E-mail addresses: chanane@kfupm.edu.sa, bchanane@yahoo.com.

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In [5], this author introduced the *regularized sampling method*; a method which is based on Shannon's sampling theory but applied to regularized functions. Hence avoiding any (multiple) integration and keeping the number of terms in the Cardinal series manageable. It has been demonstrated that the method is capable of delivering higher order estimates of the eigenvalues at a very low cost. The purpose in this paper is to extend the domain of application of this method to the problem at hand.

2. Main results

Consider the following non-self-adjoint Sturm–Liouville problem with non-separated parameter-dependent boundary conditions,

$$\begin{cases} -y'' + q(x)y = \mu^2 y \quad x \in [0, 1], \\ A(y(0), y'(0), y(1), y'(1))^{\mathrm{T}} = 0, \end{cases} \tag{2.1}$$

where the matrix

$$A(\mu) = \begin{pmatrix} a_{11}(\mu) & a_{12}(\mu) & a_{13}(\mu) & a_{14}(\mu) \\ a_{21}(\mu) & a_{22}(\mu) & a_{23}(\mu) & a_{24}(\mu) \end{pmatrix}$$

has rank 2, and q is a complex-valued function satisfying $q \in L^1_{\mathrm{loc}}(0, 1)$. We shall not make any assumption on the analyticity of A nor on the growth of its components.

The purpose in this paper is to compute the eigenvalues of (2.1) with the minimum of effort and a greater precision using the newly introduced *regularized sampling method* [5], an improvement on the method based on sampling theory introduced in [2]. We note here that the analyticity of A and the conditions on the growth of its components imposed in [5] are not necessary for the computation of the eigenvalues as shall be seen in the sequel. In fact all what is needed is the recovery of certain entire functions h_{kl} associated with some base problems defined below.

It is well known that the spectrum is discrete and scattered over the complex plane which makes difficult its computation. Also, there is no result about the distribution nor the multiplicity of the eigenvalues.

Let $y_c(x, \mu)$ and $y_s(x, \mu)$, be the solutions of the base problems

$$\begin{cases} -y'' + q(x)y = \mu^2 y, & x \in [0, 1], \\ y(0) = 1 & y'(0) = 0 \end{cases} \tag{2.2}$$

and

$$\begin{cases} -y'' + q(x)y = \mu^2 y, & x \in [0, 1], \\ y(0) = 0, & y'(0) = 1, \end{cases} \tag{2.3}$$

respectively. Then the general solution of the differential equation in (2.1) and its derivative are

$$y(x, \mu) = c_1 y_c(x, \mu) + c_2 y_s(x, \mu),$$

$$y'(x, \mu) = c_1 y_c'(x, \mu) + c_2 y_s'(x, \mu).$$

The boundary condition gives after separating c_1 and c_2,

$$c_1 A w_1 + c_2 A w_2 = 0, \tag{2.4}$$

where

$$w_1 = (1, 0, y_c(1, \mu), y_c'(1, \mu))^{\mathrm{T}},$$

$$w_2 = (0, 1, y_s(1, \mu), y_s'(1, \mu))^{\mathrm{T}}.$$

Thus, a necessary and sufficient condition for $\lambda = \mu^2$ to be an eigenvalue is that μ satisfies the characteristic equation $B(\mu) = 0$, where B is the characteristic function $B(\mu) = \det(Aw_1|Aw_2) = \det[A(w_1|w_2)]$, that is,

$$\begin{aligned} B(\mu) = {} & (a_{11}(\mu) + a_{13}(\mu) y_c(1, \mu) + a_{14}(\mu) y_c'(1, \mu))(a_{22}(\mu) + a_{23}(\mu) y_s(1, \mu) + a_{24}(\mu) y_s'(1a, \mu)) \\ & - (a_{21}(\mu) + a_{23}(\mu) y_c(1, \mu) + a_{24}(\mu) y_c'(1, \mu))(a_{12}(\mu) + a_{13}(\mu) y_s(1, \mu) + a_{14}(\mu) y_s'(1, \mu)). \end{aligned}$$

We shall need the following well known results,

Lemma 2.1 (*Chanane* [7]). $\sin z/z$ *and* $\cos z$ *are entire as functions of* z *and satisfy the estimates*

$$|\sin z/z| \leqslant \beta_0 e^{|\mathrm{Im}\, z|}/(1+|z|) \quad \textit{and} \quad |\cos z| \leqslant e^{|\mathrm{Im}\, z|},$$

where $\beta_0 = 1.72$.

Using the above lemma one can show the following result to hold.

Theorem 2.2 (*Chanane* [7]). $y_c(x,\mu)$, $y_s(x,\mu)$, $y_c'(x,\mu)$ *and* $y_s'(x,\mu)$ *are entire as functions of* μ *for each fixed* $x \in (0,1]$ *and satisfy the growth conditions*,

$$|y_c(x,\mu)-\cos(\mu x)|, \quad \left|y_s(x,\mu)-\frac{\sin(\mu x)}{\mu}\right|, \quad |y_c'(x,\mu)+\mu\sin(\mu x))|, \quad |y_s'(x,\mu)-\cos(\mu x)| \leqslant \beta_1 e^{x|\mathrm{Im}\,\mu|}$$

for some positive constant β_1.

In [7,6] we have obtained much higher estimates of the eigenvalues than those presented in Theorem 2.2 above, at the expense of subtracting terms involving multiple integrals. Here and as in [5], we shall stick with the estimates given in Theorem 2.2, avoiding any (multiple) integration. We shall show by the same token that we can get a higher order estimate of the eigenvalues of the problem at hand at a very low cost. In fact we do not have even to keep on increasing the number of sampling points.

Let PW_σ denote the Paley–Wiener space [20]

$$PW_\sigma = \left\{ f \,\text{entire}, \, |f(\mu)| \leqslant \beta e^{\sigma|\mathrm{Im}\,\mu|}, \int_{\mathrm{R}} |f(\mu)|^2 \, d\mu < \infty \right\}.$$

Let h_{kl} be defined by

$$\begin{cases} h_{11}(\mu) = \left(\dfrac{\sin\theta\mu}{\theta\mu}\right)^m (y_c(1,\mu)-\cos\mu), \\ h_{12}(\mu) = \left(\dfrac{\sin\theta\mu}{\theta\mu}\right)^m \left(y_s(1,\mu)-\dfrac{\sin\mu}{\mu}\right), \\ h_{21}(\mu) = \left(\dfrac{\sin\theta\mu}{\theta\mu}\right)^m (y_c'(1,\mu)+\mu\sin\mu), \\ h_{22}(\mu) = \left(\dfrac{\sin\theta\mu}{\theta\mu}\right)^m (y_s'(1,\mu)-\cos\mu). \end{cases}$$

Then we rewrite $y_c(1,\mu)$, $y_c'(1,\mu)$, $y_s(1,\mu)$ and $y_s'(1,\mu)$ as

$$\begin{cases} y_c(1,\mu) = h_{11}(\mu)\left(\dfrac{\sin\theta\mu}{\theta\mu}\right)^{-m} + \cos\mu, \\ y_s(1,\mu) = h_{12}(\mu)\left(\dfrac{\sin\theta\mu}{\theta\mu}\right)^{-m} + \dfrac{\sin\mu}{\mu}, \\ y_c'(1,\mu) = h_{21}(\mu)\left(\dfrac{\sin\theta\mu}{\theta\mu}\right)^{-m} - \mu\sin\mu, \\ y_s'(1,\mu) = h_{22}(\mu)\left(\dfrac{\sin\theta\mu}{\theta\mu}\right)^{-m} + \cos\mu. \end{cases}$$

Theorem 2.3. *Let ϑ be a positive constant and m be a positive integer* $(m \geqslant 2)$. *The functions* h_{kl}, $(k, l = 1, 2)$ *belong to the Paley space* PW_σ *with* $\sigma = 1 + m\theta$ *and satisfy the estimates*

$$|h_{kl}(\mu)| \leqslant \frac{\beta_2}{(1+\theta|\mu|)^m} e^{\sigma|\mathrm{Im}\,\mu|}$$

$k, l = 1, 2$ *for some positive constant* β_2.

Proof. That h_{kl} are entire and satisfy the given estimates is a direct consequence of Theorem 2.2 and the fact that $\sin\theta\mu/\theta\mu$ is an entire function of μ and satisfy the estimate in Lemma 2.1. □

Since the $h_{kl}(\mu)$ belong to the Paley–Wiener space PW_σ for each $k, l = 1, 2$, they can be recovered from their values at the points $\mu_j = j(\pi/\sigma)$, $j \in Z$, using the following celebrated theorem:

Theorem 2.4 (*Whitaker–Shannon–Kotel'nikov (Zayed [20])*). *Let* $h \in PW_\sigma$, *then*

$$h(\mu) = \sum_{j=-\infty}^{\infty} h(\mu_j) \frac{\sin\sigma(\mu-\mu_j)}{\sigma(\mu-\mu_j)}$$

$\mu_j = j(\pi/\sigma)$. *The series converges absolutely and uniformly on compact subsets of C and in* $L^2_{d\mu}(R)$.

For all practical purposes, we consider finite summations, therefore we need to approximate h_{kl} by a truncated series $h_{kl}^{[N]}$. The following lemma gives an estimate for the truncation error.

Lemma 2.5 (*Truncation error*). *Let* $h_{kl}^{[N]}(\mu) = \sum_{j=-N}^{N} h_{kl}(\mu_j)(\sin\sigma(\mu-\mu_j))/(\sigma(\mu-\mu_j))$ *denote the truncation of* $h_{kl}(\mu)$. *Then, for* $|\mu| < N\pi/\sigma$,

$$|h_{kl}(\mu) - h_{kl}^{[N]}(\mu)| \leqslant \frac{|\sin\mu|\beta_3}{\pi(\pi/\sigma)^{m-1}\sqrt{1-4^{-m+1}}} \left[\frac{1}{\sqrt{(N\pi/\sigma)-\mu}} + \frac{1}{\sqrt{(N\pi/\sigma)+\mu}}\right] \frac{1}{(N+1)^{m-1}},$$

where $\beta_3 = \|\mu^{m-1}h_{kl}(\mu)\|_2$.

Proof. Since $\mu^{m-1}h_{kl}(\mu) \in L^2(-\infty, \infty)$, Jagerman's result (see [20, Theorem 3.21, p. 90]) is applicable and yields the given estimate for the h_{kl}, $k, l = 1, 2$. □

An approximation B_N to the characteristic function B is provided by replacing the h_{kl} by its approximation $h_{kl}^{[N]}$, and we obtain at once,

Lemma 2.6. *The approximate characteristic function* B_N *satisfies the estimate*,

$$|B(\mu) - B_N(\mu)| \leqslant \left|\frac{\sin\theta\mu}{\theta\mu}\right|^{-m} \frac{|\sin\mu|\beta_4}{\pi(\pi/\sigma)^{m-1}\sqrt{1-4^{-m+1}}} \left[\frac{1}{\sqrt{(N\pi/\sigma)-\mu}} + \frac{1}{\sqrt{(N\pi/\sigma)+\mu}}\right] \frac{1}{(N+1)^{m-1}},$$

for some positive constant β_4.

We claim the following:

Theorem 2.7. *Let $\overline{\mu}^2$ be an exact eigenvalue of B of multiplicity n and denote by μ_N^2 the corresponding approximation of a square of a zero of B_N. Then, for $|\mu_N| < N\pi/\sigma$, we have,*

$$|\mu_N - \overline{\mu}| \leqslant \left(\frac{m!}{\inf |B^{(m)}(\widetilde{\mu})|} \left| \frac{\sin \theta\mu_N}{\theta\mu_N} \right|^{-m} \frac{|\sin \mu_N|\beta_4}{\pi(\pi/\sigma)^{m-1}\sqrt{1-4^{-m+1}}} \right)^{1/n} \times \left[\frac{1}{\sqrt{(N\pi/\sigma)-\mu_N}} + \frac{1}{\sqrt{(N\pi/\sigma)+\mu_N}} \right]^{1/n} \frac{1}{(N+1)^{(m-1)/n}},$$

where the inf *is taken over a ball centered at μ_N with radius $|\mu_N - \overline{\mu}|$ and not containing a multiple of π/θ.*

Proof. Since $\overline{\mu}$ is a zero of B with multiplicity n, then

$$B(\overline{\mu}) - B(\mu_N) = \frac{(\overline{\mu}-\mu_N)^n}{n!} B^{(n)}(\widetilde{\mu})$$

for some $\widetilde{\mu}$. Thus,

$$\begin{aligned} |\overline{\mu} - \mu_N|^n &= \frac{m!|B(\overline{\mu}) - B(\mu_N)|}{|B^{(m)}(\widetilde{\mu})|} \\ &\leqslant \frac{m!}{\inf |B^{(m)}(\widetilde{\mu})|} \left| \frac{\sin \theta\mu_N}{\theta\mu_N} \right|^{-m} \frac{|\sin \mu_N|\beta_4}{\pi(\pi/\sigma)^{m-1}\sqrt{1-4^{-m+1}}} \\ &\quad \times \left[\frac{1}{\sqrt{(N\pi/\sigma)-\mu_N}} + \frac{1}{\sqrt{(N\pi/\sigma)+\mu_N}} \right] \frac{1}{(N+1)^{m-1}}, \end{aligned}$$

where the inf is taken over a ball centered at μ_N with radius $|\mu_N - \overline{\mu}|$ and not containing a multiple of π/θ. Thus, the result. □

3. Numerical examples

In this section, we shall present a few examples to illustrate our method. We have taken $\theta = 1/(N-m)$ in order to avoid the first singularity of $(\sin \theta\mu_N/\theta\mu_N)^{-1}$. The sampling values were obtained using the Fehlberg 4–5 order Runge–Kutta method. The first two problems are taken from [4] in which the authors use interval arithmetic to localize the eigenvalues of singular Sturm–Liouville problems with complex potentials. The third problem, taken from [1], shows that the *regularized sampling method* provides much better results than the sampling method without regularization. The last example demonstrates that our method can estimate the eigenvalues with a great precision even in situation where other methods might introduce spurious eigenvalues and/or miss some of them. We shall mention however that we shall not make use of the error estimate given above for the time being. The method consists first in the recovery of the entire functions h_{kl} with great precision, then use the boundary conditions to determine the characteristic function. The zeros of this characteristic function are the square roots of the sought eigenvalues. We shall denote $\imath = \sqrt{-1}$.

Example 3.1 (*Taken from [4]*). Consider the singular Sturm–Liouville problem

$$\begin{cases} -y''(x) + 10\imath \sin x e^{-x} y(x) = \lambda y(x), & 0 \leqslant x < \infty, \\ y(0) = 0. \end{cases}$$

We shall use interval truncation and compute the eigenvalues of

$$\begin{cases} -y_\gamma''(x) + 10\imath \sin x e^{-x} y_\gamma(x) = \mu^2 y_\gamma(x), & 0 \leqslant x \leqslant \gamma, \\ y_\gamma(0) = 0, & y_\gamma'(\gamma) = \imath\mu y_\gamma(\gamma) \end{cases}$$

Table 1
Approximation of an eigenvalue for different values of m in Example 3.1

m	Approximate eigenvalue
5	$1.604391348283 + 1.797884747658\imath$
10	$1.604391251270 + 1.797884973775\imath$
15	$1.604391251323 + 1.797884973746\imath$

Table 2
Approximation of an eigenvalue for different values of m in Example 3.2

m	Approximate eigenvalue
5	$2.812264032443898167911 + 2.172237236667318523531\imath$
10	$2.812267289462846945426l + 2.172238191264223861\imath$
15	$2.812267288417814133626 + 2.172238191179093864\imath$

and as in [4] we shall take $\gamma = 10$. The second boundary condition has been obtained by considering the Jost solution $y = e^{\imath\mu x}$ and its derivative $y' = \imath\mu e^{\imath\mu x}$, thus, $y'(\gamma) = \imath\mu y(\gamma)$. In [4] the authors obtained an eigenvalue lying in $1.6043912_{58}^{64} + 1.7978849_{67}^{81}\imath$ where the notation 2.1_6^4 stands for the interval $[2.14, 2.16]$. Taking $N = 40$, and for different values of m we obtained the results summarized in Table 1.

Example 3.2 (*Taken from [4]*). Consider the singular problem

$$\begin{cases} -y''(x) + 10\imath e^{-x} y(x) = \lambda y(x), & 0 \leqslant x < \infty, \\ y(0) = 0. \end{cases}$$

We shall use interval truncation and compute the eigenvalues of

$$\begin{cases} -y_\gamma''(x) + 10\imath e^{-x} y_\gamma(x) = \mu^2 y_\gamma(x), & 0 \leqslant x \leqslant \gamma, \\ y_\gamma(0) = 0, & y_\gamma'(\gamma) = \imath\mu y_\gamma(\gamma) \end{cases}$$

and as in [4] we shall take $\gamma = 10$. In [4] the authors obtained an eigenvalue lying in $2.8122672_{89}^{92} + 2.17223818_{78}^{99}\imath$. Taking $N = 40$, and for different values of m we obtained the results summarized in Table 2.

Example 3.3 (*Taken from [1]*). Consider the non-self-adjoint problem

$$\begin{cases} -y''(x) + (3 - 2\imath)y(x) = \lambda y(x), & 0 \leqslant x \leqslant \pi, \\ y(0) = y(\pi) = 0. \end{cases}$$

The exact eigenvalues of the original problem are $\lambda_k = k^2 + 3 - 2\imath,\ k = 1, 2, \ldots$. Taking $N = 40$ and $m = 10$, we obtained the results summarized in Table 3.

Example 3.4. Consider now the following non-self-adjoint Sturm–Liouville problem with complex potential and parameter-dependent boundary condition,

$$\begin{cases} -y''(x) + e^{2\imath x} y(x) = \mu^2 y(x), & 0 \leqslant x \leqslant 1, \\ y(0) + \mu y(1) = 0, \\ y'(0) = 0. \end{cases}$$

Here again we are in a position to derive the exact characteristic function which in fact can be expressed in terms of Bessel functions. Indeed, let $\lambda = \mu^2$ and consider the change of variables $t = \mathbf{e}^{\imath x}$. The differential equation becomes

Table 3
Exact and approximate eigenvalues in Example 3.3

Index	Exact eigenvalue	Approximate eigenvalue	Absolute error
1	$4-2\imath$	$3.9999999999999289-1.9999999999998304519\imath$	7.30×10^{-14}
2	$7-2\imath$	$6.9999999999998187-1.99999999999984438039\imath$	2.38×10^{-13}
3	$12-2\imath$	$11.999999999999561-1.9999999999997718164\imath$	4.93×10^{-13}
4	$19-2\imath$	$18.999999999999172-2.0000000000000542265\imath$	8.29×10^{-13}
5	$28-2\imath$	$27.999999999999391-2.0000000000011521175\imath$	1.30×10^{-12}
6	$39-2\imath$	$39.000000000001586-2.0000000000029645542\imath$	3.36×10^{-12}
7	$52-2\imath$	$52.000000000005729-2.0000000000033954538\imath$	6.66×10^{-12}
8	$67-2\imath$	$67.000000000006628-1.9999999999977990747\imath$	6.98×10^{-12}
9	$84-2\imath$	$83.999999999993226-1.9999999999829522498\imath$	1.83×10^{-11}
10	$103-2\imath$	$102.999999999961-1.9999999999690674138\imath$	4.95×10^{-11}
11	$124-2\imath$	$123.999999999944-1.9999999999941154414\imath$	5.55×10^{-11}
12	$147-2\imath$	$147.000000000038-2.0000000001107296594\imath$	1.17×10^{-10}
13	$172-2\imath$	$172.000000000323-2.0000000002862574821\imath$	4.32×10^{-10}
14	$199-2\imath$	$199.000000000556-2.0000000001798594170\imath$	5.85×10^{-10}
15	$228-2\imath$	$227.999999999678-1.9999999989989010834\imath$	1.05×10^{-9}
16	$259-2\imath$	$258.999999996079-1.9999999963552374114\imath$	5.35×10^{-9}
17	$292-2\imath$	$291.999999991669-1.9999999961982129591\imath$	9.15×10^{-9}
18	$327-2\imath$	$327.000000004474-2.0000000150098733387\imath$	1.56×10^{-8}
19	$364-2\imath$	$364.000000082618-2.0000000798746168342\imath$	1.14×10^{-7}
20	$403-2\imath$	$403.000000232131-2.0000001291033812382\imath$	2.65×10^{-7}

the Bessel equation of order μ given by

$$t^2\frac{\mathrm{d}^2z}{\mathrm{d}t^2}+t\frac{\mathrm{d}z}{\mathrm{d}t}+(t^2-\mu^2)z=0$$

whose solution is

$$z(t)=c_1\mathscr{J}_\mu(t)+c_2\mathscr{J}_{-\mu}(t),$$

where $\mathscr{J}_\mu$ and $\mathscr{J}_{-\mu}$ are the Bessel functions of the first kind of order μ.

Returning to the original variables, we obtain

$$y(x)=c_1\mathscr{J}_\mu(\mathbf{e}^{\imath x})+c_2\mathscr{J}_{-\mu}(\mathbf{e}^{\imath x}).$$

Taking into account the boundary conditions, we obtain the homogeneous system in c_1 and c_2

$$\begin{cases} c_1\mathscr{J}_\mu(1)+c_2\mathscr{J}_{-\mu}(1)+\mu(c_1\mathscr{J}_\mu(\mathbf{e}^\imath)+c_2\mathscr{J}_{-\mu}(\mathbf{e}^\imath))=0,\\ c_1\mathscr{J}'_\mu(1)+c_2\mathscr{J}'_{-\mu}(1)=0.\end{cases}$$

In order to have a non-trivial solution, a necessary and sufficient condition is to have $B_{\text{exact}}(\mu)=0$ where

$$B_{\text{exact}}(\mu)=\det\begin{pmatrix}\mathscr{J}_\mu(1)+\mu\mathscr{J}_\mu(\mathbf{e}^\imath) & \mathscr{J}_{-\mu}(1)+\mu\mathscr{J}_{-\mu}(\mathbf{e}^\imath)\\ \mathscr{J}'_\mu(1) & \mathscr{J}'_{-\mu}(1)\end{pmatrix}$$

is the characteristic function. Now, using the well-known result

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathscr{J}_\mu(x)=(\mathscr{J}_{-\mu-1}(x)-\mathscr{J}_{\mu+1}(x))/2,$$

Table 4
Exact and approximate eigenvalues in Example 3.4

Index	Exact eigenvalue	Approximate eigenvalue	Absolute error	Relative error
1	$4.9685430929323576 + 0.3906545895360696\imath$	$4.9685430929323625 + 0.3906545895360721\imath$	5.549×10^{-15}	1.113×10^{-15}
2	$20.60271034889337 + 0.75023252353154\imath$	$20.60271034889340 + 0.75023252353155\imath$	3.393×10^{-14}	1.645×10^{-15}
3	$64.14038244804547 + 0.68422837531133\imath$	$64.14038244804526 + 0.68422837531099\imath$	3.977×10^{-13}	6.201×10^{-15}
4	$119.34792168887388 + 0.71497240479401\imath$	$119.34792168887345 + 0.71497240479334\imath$	8.004×10^{-13}	6.706×10^{-15}
5	$202.31443747778734 + 0.70057212586525\imath$	$202.31443747778739 + 0.70057212586545\imath$	2.064×10^{-13}	1.020×10^{-15}
6	$419.44558800598641 + 0.70446189520144\imath$	$419.44558800598892 + 0.70446189520528\imath$	4.582×10^{-12}	1.092×10^{-14}
7	$553.61789373762934 + 0.70954623577257\imath$	$553.61789373762976 + 0.70954623577282\imath$	4.969×10^{-13}	8.977×10^{-16}
8	$715.53365857906959 + 0.70595783818772\imath$	$715.53365857906140 + 0.70595783817453\imath$	1.553×10^{-11}	2.170×10^{-14}
9	$889.18520034251622 + 0.70898948206981\imath$	$889.18520034250143 + 0.70898948204681\imath$	2.734×10^{-11}	3.075×10^{-14}
10	$1090.57859485902126 + 0.70668585309098\imath$	$1090.57859485902214 + 0.70668585309385\imath$	3.00×10^{-12}	2.751×10^{-15}
11	$1303.70898166607058 + 0.70869788000992\imath$	$1303.70898166611992 + 0.70869788008925\imath$	9.341×10^{-11}	7.165×10^{-14}
12	$1544.58037965386611 + 0.70709389168016\imath$	$1544.58037965396658 + 0.70709389183628\imath$	1.856×10^{-10}	1.202×10^{-13}
13	$1797.18943505543540 + 0.70852627026801\imath$	$1797.18943505544546 + 0.70852627027458\imath$	1.201×10^{-11}	6.687×10^{-15}
14	$2077.53900632820814 + 0.70734525957323\imath$	$2077.53900632774381 + 0.70734525883073\imath$	8.757×10^{-10}	4.215×10^{-13}
15	$2369.6266391592291618 + 0.70841680475450\imath$	$2369.62663915816209 + 0.70841680308871\imath$	1.978×10^{-9}	8.348×10^{-13}
16	$2689.45447190894724 + 0.70751097714777\imath$	$2689.45447190899851 + 0.70751097732396\imath$	1.834×10^{-10}	6.822×10^{-14}
17	$3021.02063035583927 + 0.70834272705249\imath$	$3021.02063036245257 + 0.70834273761505\imath$	1.246×10^{-8}	4.125×10^{-12}
18	$3380.32677490847313 + 0.70762595333135\imath$	$3380.32677492777072 + 0.70762598345876\imath$	3.577×10^{-8}	1.058×10^{-11}
19	$3751.37142735725052 + 0.70829027475581\imath$	$3751.37142735927201 + 0.70829027606949\imath$	2.410×10^{-9}	6.426×10^{-13}
20	$4150.15591451714336 + 0.70770896968510\imath$	$4150.15591430123958 + 0.70770862603678\imath$	4.058×10^{-7}	9.778×10^{-11}
21	$4560.67904058883973 + 0.70825178027457\imath$	$4560.67903973138474 + 0.70825043937887\imath$	1.591×10^{-6}	3.489×10^{-10}
22	$4998.94189026423779 + 0.70777085938814\imath$	$4998.94189032279592 + 0.70777105963898\imath$	2.086×10^{-7}	4.173×10^{-11}
23	$5448.94347623327640 + 0.70822269647649\imath$	$5448.94349859286012 + 0.70825825178573\imath$	0.00004200	7.708×10^{-9}
24	$5926.68470186115217 + 0.70781822661900\imath$	$5926.68487236793343 + 0.70808488449668\imath$	0.0003165	5.340×10^{-8}
25	$6416.16473814590617 + 0.70820018792052\imath$	$6416.16478405538947 + 0.70823000684001\imath$	0.00005474	8.532×10^{-9}

we obtain

$$B_{\text{exact}}(\mu) = \det \begin{pmatrix} \mathscr{I}_{\mu}(1) + \mu \mathscr{I}_{\mu}(\mathbf{e}^{\imath}) & \mathscr{I}_{-\mu}(1) + \mu \mathscr{I}_{-\mu}(\mathbf{e}^{\imath}) \\ (\mathscr{I}_{-\mu-1}(1) - \mathscr{I}_{\mu+1}(1))/2 & (\mathscr{I}_{\mu-1}(1) - \mathscr{I}_{-\mu+1}(1))/2 \end{pmatrix}.$$

Taking $N = 40$, and $m = 10$ we obtained the results summarized in Table 4.

4. Conclusion

In this paper, we have used the *regularized sampling method* introduced recently [5] to compute the eigenvalues of non-self-adjoint Sturm–Liouville problems with non-separable parameter-dependent boundary conditions. We recall that this method constitutes an improvement upon the method based on Shannon's sampling theory introduced in [2] since it uses a regularization avoiding any multiple integration. The method allows us to get higher order estimates of the eigenvalues at a very low cost. We have presented a few examples, including singular ones, to illustrate the method and compared the computed eigenvalues with the exact ones when they are available.

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