Journal of Algebra 323 (2010) 766–778

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

An algebraic proof of the cyclic sum formula for multiple zeta values

Tatsushi Tanaka *, Noriko Wakabayashi

Faculty of Mathematics, Kyushu University, 744, Motooka, Nishiku, Fukuoka City, 819-0395, Japan

ARTICLE INFO

Article history:
Received 19 January 2009
Communicated by Michel Van den Bergh

Keywords:
Multiple zeta values
Multiple zeta-star values
Cyclic sum formula
Derivation relation

ABSTRACT

We introduce an algebraic formulation of the cyclic sum formulas for multiple zeta values and for multiple zeta-star values. We also present an algebraic proof of cyclic sum formulas for multiple zeta values and for multiple zeta-star values by reducing them to Kawashima's relation.

1. Introduction

The only proof of the cyclic sum formula (abbreviated as CSF) for multiple zeta values (resp. for multiple zeta-star values) which has been known so far is by using partial fraction expansions, which appears in Hoffman and Ohno [4] (resp. Ohno and Wakabayashi [9]). We are interested in other proofs of the CSF, especially by discussing relationships between the CSF and the other known relations for multiple zeta values such as regularized double shuffle relation [6], associator relation [2], etc. With such a motivation, we introduce herein an algebraic formulation of the CSF for multiple zeta values and the CSF for multiple zeta-star values, and give an algebraic proof of the CSF by reducing it to (the linear part of) Kawashima's relation [7].

For $k_1 > 1$ and $k_2, \ldots, k_n \geqslant 1$, the multiple zeta value (abbreviated as MZV) is a real number defined by the convergent series

$$\zeta(k_1, k_2, \ldots, k_n) = \sum_{m_1 > m_2 > \cdots > m_n > 0} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_n^{k_n}},$$

* Corresponding author.
E-mail addresses: t.tanaka@math.kyushu-u.ac.jp (T. Tanaka), noriko@math.kyushu-u.ac.jp (N. Wakabayashi).

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2009.11.016

and the multiple zeta-star value (abbreviated as MZSV) is defined by the convergent series

$$\zeta^\star(k_1,k_2,\dots,k_n)=\sum_{m_1\geqslant m_2\geqslant\cdots\geqslant m_n>0}\frac{1}{m_1^{k_1}m_2^{k_2}\cdots m_n^{k_n}}.$$

We call the number $k_1+\cdots+k_n$ weight and n depth. If $n=1$, MZV and MZSV coincide and are known as the Riemann zeta value.

Throughout the present paper, we employ the algebraic setup introduced by Hoffman [3]. Let $\mathfrak{H}=\mathbb{Q}\langle x,y\rangle$ denote the non-commutative polynomial algebra over the rational numbers in two indeterminates x and y, and let $\mathfrak{H}^1$ and $\mathfrak{H}^0$ denote the subalgebras $\mathbb{Q}+\mathfrak{H}y$ and $\mathbb{Q}+x\mathfrak{H}y$, respectively. We define the $\mathbb{Q}$-linear map $Z:\mathfrak{H}^0\to\mathbb{R}$ by $Z(1)=1$ and

$$Z\bigl(x^{k_1-1}yx^{k_2-1}y\cdots x^{k_n-1}y\bigr)=\zeta(k_1,k_2,\dots,k_n).$$

We also define the $\mathbb{Q}$-linear map $\overline{Z}:\mathfrak{H}^0\to\mathbb{R}$ by $\overline{Z}(1)=1$ and

$$\overline{Z}\bigl(x^{k_1-1}yx^{k_2-1}y\cdots x^{k_n-1}y\bigr)=\zeta^\star(k_1,k_2,\dots,k_n).$$

The degree (resp. degree with respect to y) of a word is the weight (resp. the depth) of the corresponding MZV or MZSV. Let $d:\mathfrak{H}^1\to\mathfrak{H}^1$ be a $\mathbb{Q}$-linear map defined by

$$d(wy)=\gamma(w)y$$

for $w\in\mathfrak{H}$, where γ is the automorphism on $\mathfrak{H}$ given by

$$\gamma(x)=x,\qquad \gamma(y)=x+y.$$

It is well known that the identity $\overline{Z}=Zd$ holds.

For $k_1,\dots,k_l\geqslant 1$ with some $k_q>1$, the CSF for MZV's

$$\sum_{j=1}^{l}\sum_{i=1}^{k_j-1}\zeta(k_j-i+1,k_{j+1},\dots,k_l,k_1,\dots,k_{j-1},i)=\sum_{j=1}^{l}\zeta(k_j+1,k_{j+1},\dots,k_l,k_1,\dots,k_{j-1})\tag{1}$$

is proven in Hoffman and Ohno [4] by means of partial fraction expansions and the CSF for MZSV's

$$\sum_{j=1}^{l}\sum_{i=1}^{k_j-1}\zeta^\star(k_j-i+1,k_{j+1},\dots,k_l,k_1,\dots,k_{j-1},i)=k\zeta(k+1),\tag{2}$$

where $k=k_1+\cdots+k_l$, in Ohno and Wakabayashi [9] in a similar way. Hoffman and Ohno also introduced in [4] an algebraic expression of the CSF for MZV's. They formulated the CSF using two cyclic derivatives C_w and $\overline{C}_w$ on $\mathfrak{H}$ as follows.

A cyclic derivative $\phi_\bullet:\mathfrak{H}\to\mathrm{End}(\mathfrak{H})$ is defined by a $\mathbb{Q}$-linear map with the property

$$\phi_{w_1w_2}(W)=\phi_{w_1}(w_2W)+\phi_{w_2}(Ww_1)$$

for any $w_1,w_2,W\in\mathfrak{H}$. Such a cyclic derivative is uniquely determined by ϕ_x and ϕ_y. Let $C_\bullet$ (resp. $\overline{C}_\bullet$) be the cyclic derivative defined by $C_x=0$, $C_y=L_xR_y$ (resp. $\overline{C}_x=L_xR_y$, $\overline{C}_y=0$), where, for $w\in\mathfrak{H}$, the map L_w (resp. R_w) is a $\mathbb{Q}$-linear map, called left- (resp. right-) multiplication, defined by $L_w(W)=wW$ (resp. $R_w(W)=Ww$) for any $W\in\mathfrak{H}$. Let $\check{\mathfrak{H}}^1$ be a subvector space of $\mathfrak{H}^1$ generated by words of $\mathfrak{H}^1$ except for powers of y. Then:

Theorem 1.1 *(Hoffman–Ohno). For any word $w \in \check{\mathfrak{H}}^1$, we have*

$$Z\big((\bar{C}_w - C_w)(1)\big) = 0.$$

2. Main results

In this section, we give an algebraic formulation of the CSF for MZV's, which the authors have been inspired by works for double Poisson algebras introduced in [1]. The method of the formulation is a little different from Hoffman–Ohno's. Then we present an algebraic proof of the CSF for MZV's by reducing it to (the linear part of) Kawashima's relation.

Let $n \geqslant 1$. We denote an action of $\mathfrak{H}$ on $\mathfrak{H}^{\otimes(n+1)}$ by "$\diamond$", which is defined by

$$a \diamond (w_1 \otimes \cdots \otimes w_{n+1}) = w_1 \otimes \cdots \otimes w_n \otimes a w_{n+1},$$
$$(w_1 \otimes \cdots \otimes w_{n+1}) \diamond b = w_1 b \otimes w_2 \otimes \cdots \otimes w_{n+1}.$$

The action $\diamond$ gives a $\mathfrak{H}$-bimodule structure on $\mathfrak{H}^{\otimes(n+1)}$.

Let $z = x + y$. We define the $\mathbb{Q}$-linear map $\mathcal{C}_n : \mathfrak{H} \to \mathfrak{H}^{\otimes(n+1)}$ by

$$\mathcal{C}_n(x) = x \otimes z^{\otimes(n-1)} \otimes y, \qquad \mathcal{C}_n(y) = -\big(x \otimes z^{\otimes(n-1)} \otimes y\big)$$

and

$$\mathcal{C}_n\big(ww'\big) = \mathcal{C}_n(w) \diamond w' + w \diamond \mathcal{C}_n\big(w'\big) \tag{3}$$

for any $w, w' \in \mathfrak{H}$. The map $\mathcal{C}_n$ is well defined because of the identities

$$a \diamond (b \diamond w) = ab \diamond w, \qquad (w \diamond a) \diamond b = w \diamond ab,$$

where $a, b \in \mathfrak{H}$, $w \in \mathfrak{H}^{\otimes(n+1)}$. We also find that $\mathcal{C}_n(1) = 0$ by putting $w = w' = 1$ in (3).

Let $M_n : \mathfrak{H}^{\otimes(n+1)} \to \mathfrak{H}$ denote the multiplication map, i.e.,

$$M_n(w_1 \otimes \cdots \otimes w_{n+1}) = w_1 \cdots w_{n+1},$$

and let $\rho_n = M_n \mathcal{C}_n$ $(n \geqslant 1)$. Then our main theorem is the following.

Theorem 2.1. *For $n \geqslant 1$, we have $\rho_n(\check{\mathfrak{H}}^1) \subset \ker Z$.*

The theorem contains Theorem 1.1 because of the following proposition.

Proposition 2.2. *For any $w \in \mathfrak{H}$, we have $\rho_1(w) = (\bar{C}_w - C_w)(1)$.*

To prove the proposition, we firstly show the following lemma.

Lemma 2.3. *For cyclically equivalent words $w, w' \in \mathfrak{H}$, we have $\rho_1(w) = \rho_1(w')$.*

Proof. Let $u_1, \dots, u_l \in \{x, y\}$ and $\mathrm{sgn}(u) = 1$ or -1 according to $u = x$ or y. Since

$$\mathcal{C}_1(u) = \mathrm{sgn}(u)(x \otimes y)$$

for $u \in \{x, y\}$, we have

$$\mathcal{C}_1(u_1\cdots u_l)=\sum_{j=1}^{l}u_1\cdots u_{j-1}\diamond\mathcal{C}_1(u_j)\diamond u_{j+1}\cdots u_l$$
$$=\sum_{j=1}^{l}\operatorname{sgn}(u_j)(xu_{j+1}\cdots u_l\otimes u_1\cdots u_{j-1}y),$$

where we assume $u_1\cdots u_{j-1}=1$ if $j=1$ and $u_{j+1}\cdots u_l=1$ if $j=l$. Therefore we have

$$\rho_1(u_1\cdots u_l)=\sum_{j=1}^{l}\operatorname{sgn}(u_j)xu_{j+1}\cdots u_lu_1\cdots u_{j-1}y.$$

Since the right-hand side does not change under the cyclic permutations of $\{u_1,\dots,u_l\}$, we conclude the lemma. □

Proof of Proposition 2.2. Let $z_k=x^{k-1}y$ $(k\geqslant 1)$. It suffices to show the identity for words $w=z_{k_1}\cdots z_{k_l}$ and x^q $(q\geqslant 1)$ because of Lemma 2.3.

If $w=x^q$, we easily calculate

$$C_w(1)=0,\qquad \bar{C}_w(1)=\rho_1(w)=qz_{q+1},$$

and hence the proposition holds.

When $w=z_{k_1}\cdots z_{k_l}$, Hoffman and Ohno showed in [4] that

$$(\bar{C}_w-C_w)(1)=\sum_{j=1}^{l}\sum_{i=1}^{k_j-1}z_{k_j-i+1}z_{k_{j+1}}\cdots z_{k_l}z_{k_1}\cdots z_{k_{j-1}}z_i-\sum_{j=1}^{l}xz_{k_{j+1}}\cdots z_{k_l}z_{k_1}\cdots z_{k_j}. \tag{4}$$

To prove the proposition, we show that $\rho_1(w)$ equals the right-hand side of this identity.

By the definition of $\mathcal{C}_1$, we calculate

$$\mathcal{C}_1(z_k)=\sum_{j=1}^{k-1}z_{k-j+1}\otimes z_j-x\otimes z_k.$$

Therefore we have

$$\mathcal{C}_1(w)=\sum_{j=1}^{l}\sum_{i=1}^{k_j-1}z_{k_j-i+1}z_{k_{j+1}}\cdots z_{k_l}\otimes z_{k_1}\cdots z_{k_{j-1}}z_i-\sum_{j=1}^{l}xz_{k_{j+1}}\cdots z_{k_l}\otimes z_{k_1}\cdots z_{k_j},$$

and hence, $\rho_1(w)$ equals the right-hand side of (4). □

We find that the operator ρ_1 induces the CSF for MZV's because of Proposition 2.2 and Theorem 1.1. According to Lemma 2.3 and Proposition 2.2, we also find that Theorem 1.1 holds if $w\in\check{\mathfrak{H}}$, where $\check{\mathfrak{H}}$ is a subvector space of $\mathfrak{H}$ generated by words of $\mathfrak{H}$ except for powers of x and powers of y.

To prove our main theorem (Theorem 2.1), we use the linear part of Kawashima's relation introduced in [7], which is stated as follows. Let φ be the automorphism on $\mathfrak{H}$ given by

$$\varphi(x)=x+y,\qquad \varphi(y)=-y.$$

The product "$*$" stands for the harmonic product on $\mathfrak{H}^1$ introduced in Hoffman [3], which is known to be associative and commutative. Under these notations, the linear part of Kawashima's relation for MZV's is the following.

Theorem 2.4 *(Kawashima).* $L_x\varphi(\mathfrak{H}y * \mathfrak{H}y) \subset \ker Z$.

The theorem is used in [12] to prove the quasi-derivation relation for MZV's. To prove Theorem 2.1 by reducing the CSF for MZV's to Theorem 2.4, we should show

Proposition 2.5. *For* $n \geqslant 1$*, we have* $\rho_n(\check{\mathfrak{H}}^1) \subset L_x\varphi(\mathfrak{H}y * \mathfrak{H}y)$.

This proposition holds because of the Key Proposition and the lemma below. Let $A_0 = 1$ and $A_j = z^{j-1}y$ for $j \geqslant 1$, where $z = x + y$.

Proposition 2.6. *For* $k_1 \geqslant n, k_2, \ldots, k_l \geqslant 1$*, we have*

$$\varphi L_x^{-1}\rho_n(A_{k_1+\cdots+k_l-n+1} - A_{k_1-n+1}A_{k_2}\cdots A_{k_l}) = \sum_{m=2}^{l}\frac{(-1)^{l-m}}{m}\sum_{j=1}^{l}\sum_{\substack{\alpha_1+\cdots+\alpha_m=l\\ \alpha_1,\ldots,\alpha_m\geqslant 1}} H(j,\alpha_1,\ldots,\alpha_m).$$

Here, $H(j,\alpha_1,\ldots,\alpha_m)$ *is given by*

$$H(j,\alpha_1,\ldots,\alpha_m) = (z_{k_j}\cdots z_{k_{\alpha_1+j-1}}) * (z_{k_{\alpha_1+j}}\cdots z_{k_{\alpha_1+\alpha_2+j-1}}) * \cdots * (z_{k_{\alpha_1+\cdots+\alpha_{m-1}+j}}\cdots z_{k_{\alpha_1+\cdots+\alpha_m+j-1}}),$$

where the subscripts of k's of the right-hand side are viewed as numbers modulo l $(\in \{1,\ldots,l\})$.

Proof. First we give some notations. Let $U = U(k_1,\ldots,k_l)$ be the set of tuples consisting of at most l components, each of which is the sum of some of $k_1,\ldots,k_l$, such that each k_i occurs in exactly one of the components. For example,

$$\begin{aligned} U(k_1,k_2) &= \bigl\{(k_1,k_2),(k_2,k_1),(k_1+k_2)\bigr\},\\ U(k_1,k_2,k_3) &= \bigl\{(k_1,k_2,k_3),(k_1,k_3,k_2),(k_2,k_1,k_3),(k_2,k_3,k_1),(k_3,k_1,k_2),\\ &\qquad (k_3,k_2,k_1),(k_1+k_2,k_3),(k_2+k_3,k_1),(k_3+k_1,k_2),(k_1,k_2+k_3),\\ &\qquad (k_2,k_3+k_1),(k_3,k_1+k_2),(k_1+k_2+k_3)\bigr\}, \end{aligned}$$

and so on. For $1 \leqslant i \leqslant l$, let $I_i = I_i(k_1,\ldots,k_l)$ be the set of tuples in U such that k_{i-1} occurs in a component that lies to the left of the one containing k_i. For example,

$$\begin{aligned} I_1(k_1,k_2) &= \bigl\{(k_2,k_1)\bigr\},\\ I_2(k_1,k_2) &= \bigl\{(k_1,k_2)\bigr\},\\ I_1(k_1,k_2,k_3) &= \bigl\{(k_3,k_1,k_2),(k_3,k_2,k_1),(k_2,k_3,k_1),(k_3,k_1+k_2),(k_3+k_2,k_1)\bigr\},\\ I_2(k_1,k_2,k_3) &= \bigl\{(k_1,k_2,k_3),(k_1,k_3,k_2),(k_3,k_1,k_2),(k_1,k_2+k_3),(k_1+k_3,k_2)\bigr\},\\ I_3(k_1,k_2,k_3) &= \bigl\{(k_2,k_3,k_1),(k_2,k_1,k_3),(k_1,k_2,k_3),(k_2,k_3+k_1),(k_2+k_1,k_3)\bigr\}, \end{aligned}$$

and so on. We also define $W : U \to \mathfrak{H}$ by

$$W(k_1, \dots, k_l) = z_{k_1} \cdots z_{k_l}$$

for $k_1, \dots, k_l \geqslant 1$. Then we find

$$\sum_{\mathbf{k} \in U} W(\mathbf{k}) = z_{k_1} * \cdots * z_{k_l} \tag{5}$$

and

$$U \backslash \{(k_1 + \cdots + k_l)\} = \bigcup_{i=1}^{l} I_i. \tag{6}$$

Since

$$\varphi L_x^{-1} \rho_n (A_{k_1+\cdots+k_l-n+1} - A_{k_1-n+1} A_{k_2} \cdots A_{k_l}) = z_{k_1+\cdots+k_l} + (-1)^l \sum_{j=1}^{l} z_{k_j} \cdots z_{k_l} z_{k_1} \cdots z_{k_{j-1}},$$

it suffices to show

$$z_{k_1+\cdots+k_l} + (-1)^l \sum_{j=1}^{l} z_{k_j} \cdots z_{k_l} z_{k_1} \cdots z_{k_{j-1}} = \sum_{m=2}^{l} \frac{(-1)^{l-m}}{m} \sum_{j=1}^{l} \sum_{\substack{\alpha_1+\cdots+\alpha_m=l \\ \alpha_1,\dots,\alpha_m \geqslant 1}} H(j, \alpha_1, \dots, \alpha_m), \tag{7}$$

where the subscripts of k's of the right-hand side are viewed as numbers modulo l ($\in \{1, \dots, l\}$).

Put $N^{(l)} = \{n \in \mathbb{N} \mid 1 \leqslant n \leqslant l\}$, $N_j^{(l)} = N^{(l)} \backslash \{j\}$ $(1 \leqslant j \leqslant l)$ and $A = \{\alpha_1 + \cdots + \alpha_s + j \mid 1 \leqslant s < m\}$ for a fixed $(\alpha_1, \dots, \alpha_m)$ with $\alpha_1 + \cdots + \alpha_m = l$, $\alpha_1, \dots, \alpha_m \geqslant 1$. Expanding the harmonic products, we have

$$H(j, \alpha_1, \dots, \alpha_m) = \sum_{\mathbf{k} \in \bigcap_{r \in N_j^{(l)} \backslash A} I_r} W(\mathbf{k}).$$

Hence, we obtain

$$\begin{aligned}\sum_{\substack{\alpha_1+\cdots+\alpha_m=l \\ \alpha_1,\dots,\alpha_m \geqslant 1}} H(j, \alpha_1, \dots, \alpha_m) &= \sum_{\substack{\alpha_1+\cdots+\alpha_m=l \\ \alpha_1,\dots,\alpha_m \geqslant 1}} \sum_{\mathbf{k} \in \bigcap_{r \in N_j^{(l)} \backslash A} I_r} W(\mathbf{k}) \\ &= \sum_{\substack{S \subset N_j^{(l)} \\ |S|=m-1}} \sum_{\mathbf{k} \in \bigcap_{r \in N_j^{(l)} \backslash S} I_r} W(\mathbf{k}).\end{aligned}$$

Adding up from $j = 1$ to l, we obtain

$$\begin{aligned}\sum_{j=1}^{l} \sum_{\substack{S \subset N_j^{(l)} \\ |S|=m-1}} \sum_{\mathbf{k} \in \bigcap_{r \in N_j^{(l)} \backslash S} I_r} W(\mathbf{k}) &= m \sum_{\substack{T \subset N^{(l)} \\ |T|=m}} \sum_{\mathbf{k} \in \bigcap_{r \in N^{(l)} \backslash T} I_r} W(\mathbf{k}) \\ &= m \sum_{0 < i_1 < \cdots < i_{l-m} \leqslant l} \sum_{\mathbf{k} \in \bigcap_{p=1}^{l-m} I_{i_p}} W(\mathbf{k}).\end{aligned}$$

The reason why the middle term of the last equation is multiplied by m is because $m-1$ elements belonging to S have been already removed from $N_j^{(l)}$ at the third summation of the first term and there are m ways to get rid $N^{(l)}$ of the additional element in T. Owing to the above equation, we obtain

$$\begin{aligned}\sum_{m=1}^{l-1}\frac{(-1)^{l-m}}{m}\sum_{j=1}^{l}\sum_{\substack{\alpha_1+\cdots+\alpha_m=l\\ \alpha_1,\ldots,\alpha_m\geqslant 1}}H(j,\alpha_1,\ldots,\alpha_m)&=\sum_{m=1}^{l-1}(-1)^{l-m}\sum_{0<i_1<\cdots<i_{l-m}\leqslant l}\ \sum_{\mathbf{k}\in\bigcap_{p=1}^{l-m}I_{i_p}}W(\mathbf{k})\\ &=\sum_{m=1}^{l}(-1)^{m}\sum_{0<i_1<\cdots<i_m\leqslant l}\ \sum_{\mathbf{k}\in\bigcap_{p=1}^{m}I_{i_p}}W(\mathbf{k})\\ &=-\sum_{\mathbf{k}\in\bigcup_{i=1}^{l}I_i}W(\mathbf{k}).\end{aligned}$$

The last equality is by the inclusion–exclusion property. Because of (6) and (5), we obtain

$$\begin{aligned}-\sum_{\mathbf{k}\in\bigcup_{i=1}^{l}I_i}W(\mathbf{k})&=-\sum_{\mathbf{k}\in U\setminus\{(k_1+\cdots+k_l)\}}W(\mathbf{k})\\ &=z_{k_1+\cdots+k_l}-\sum_{\mathbf{k}\in U}W(\mathbf{k})\\ &=z_{k_1+\cdots+k_l}-z_{k_1}*\cdots*z_{k_l}.\end{aligned}$$

Therefore we conclude (7). □

We also need the following lemma.

Lemma 2.7. *The set* $\{A_{k_1+\cdots+k_l}-A_{k_1}\cdots A_{k_l}\mid k_1,\ldots,k_l\geqslant 1,\ l\geqslant 1\}$ *is a set of bases of* $\check{\mathfrak{H}}^1$.

Proof. Since the indeterminates y and $z(=x+y)$ can be generators of $\mathfrak{H}$, the set $X=\{A_{k_1}\cdots A_{k_l}\mid k_1,\ldots,k_l\geqslant 1\}$ is a set of bases of $\mathfrak{H}^1$. For each $k\geqslant 1$, the dimension of the space of weight k generated by the set $Y=\{A_{k_1}\cdots A_{k_l}-A_{k_1+\cdots+k_l}\mid k_1,\ldots,k_l\geqslant 1\}$ is one less than the space of weight k generated by X. Also we find that any power of y cannot be expressed by elements of Y. Therefore we conclude the lemma. □

Thus we obtain Proposition 2.5, and hence Theorem 2.1 because of Theorem 2.4.

3. For MZSV's

In Section 2, we exploited a new algebraic formulation to prove the CSF for MZV's by reducing it to Kawashima's relation. In this section, we describe an algebraic formulation and a proof of the CSF for MZSV's.

As in the previous sections, let z be $x+y$ and γ the automorphism on $\mathfrak{H}$ given by

$$\gamma(x)=x,\qquad \gamma(y)=x+y.$$

We notice that γ^{-1} is also the automorphism on $\mathfrak{H}$ given by

$$\gamma^{-1}(x)=x,\qquad \gamma^{-1}(y)=y-x.$$

We define the $\mathbb{Q}$-linear map $\bar{\mathcal{C}}_n : \mathfrak{H} \to \mathfrak{H}^{\otimes(n+1)}$ by

$$\bar{\mathcal{C}}_n(x) = x \otimes y^{\otimes n}, \qquad \bar{\mathcal{C}}_n(y) = -\big(x \otimes y^{\otimes n}\big)$$

and

$$\bar{\mathcal{C}}_n\big(ww'\big) = \bar{\mathcal{C}}_n(w) \diamond \gamma^{-1}\big(w'\big) + \gamma^{-1}(w) \diamond \bar{\mathcal{C}}_n\big(w'\big)$$

for any $w, w' \in \mathfrak{H}$. The map $\bar{\mathcal{C}}_n$ is well defined and $\bar{\mathcal{C}}_n(1) = 0$. Let $\bar{\rho}_n = M_n \bar{\mathcal{C}}_n$ $(n \geqslant 1)$. Then:

Lemma 3.1. *For any $n \geqslant 1$, we have $\rho_n = d\bar{\rho}_n$ on $\mathfrak{H}$.*

Proof. It suffices to show $\rho_n(w) = d\bar{\rho}_n(w)$ for $w = z_{k_1} \cdots z_{k_l} x^q$, where $q \geqslant 1$, $l \geqslant 0$ and $z_k = x^{k-1}y$ $(k \geqslant 1)$. By the definition of $\mathcal{C}_n$ and $\bar{\mathcal{C}}_n$, we calculate

$$\begin{aligned}
\mathcal{C}_n(w) = {} & \sum_{p=1}^{q} x^{q-p+1} \otimes z^{\otimes(n-1)} \otimes z_{k_1} \cdots z_{k_l} z_p \\
& + \sum_{j=1}^{l} \sum_{i=1}^{k_j-1} z_{k_j-i+1} z_{k_{j+1}} \cdots z_{k_l} x^q \otimes z^{\otimes(n-1)} \otimes z_{k_1} \cdots z_{k_{j-1}} z_i \\
& - \sum_{j=1}^{l} x z_{k_{j+1}} \cdots z_{k_l} x^q \otimes z^{\otimes(n-1)} \otimes z_{k_1} \cdots z_{k_j}
\end{aligned}$$

and

$$\begin{aligned}
\bar{\mathcal{C}}_n(w) = {} & \sum_{p=1}^{q} \gamma^{-1}\big(x^{q-p+1}\big) \otimes y^{\otimes(n-1)} \otimes \gamma^{-1}(z_{k_1} \cdots z_{k_l}) z_p \\
& + \sum_{j=1}^{l} \sum_{i=1}^{k_j-1} \gamma^{-1}\big(z_{k_j-i+1} z_{k_{j+1}} \cdots z_{k_l} x^q\big) \otimes y^{\otimes(n-1)} \otimes \gamma^{-1}(z_{k_1} \cdots z_{k_{j-1}}) z_i \\
& - \sum_{j=1}^{l} \gamma^{-1}\big(x z_{k_{j+1}} \cdots z_{k_l} x^q\big) \otimes y^{\otimes(n-1)} \otimes \gamma^{-1}(z_{k_1} \cdots z_{k_{j-1}}) z_{k_j}.
\end{aligned}$$

According to the definition of the map d, we conclude $\rho_n(w) = d\bar{\rho}_n(w)$. $\square$

We define $\alpha \in \mathrm{Aut}(\mathfrak{H})$ by

$$\alpha(x) = y, \qquad \alpha(y) = x,$$

and a $\mathbb{Q}$-linear map $\widetilde{\alpha} : \mathfrak{H}^1 \to \mathfrak{H}^1$ by

$$\widetilde{\alpha}(wy) = \alpha(w)y \quad (w \in \mathfrak{H}).$$

We easily find that

$$\varphi d = -d\widetilde{\alpha}. \tag{8}$$

Let $\bar{*}: \mathfrak{H}^1 \times \mathfrak{H}^1 \to \mathfrak{H}^1$ be the $\mathbb{Q}$-bilinear map defined by

(i) $1 \bar{*} w = w \bar{*} 1 = w$ for any $w \in \mathfrak{H}^1$,
(ii) $z_p w \bar{*} z_q w' = z_p(w \bar{*} z_q w') + z_q(z_p w \bar{*} w') - z_{p+q}(w \bar{*} w')$ for any $p, q \geqslant 1$ and any $w, w' \in \mathfrak{H}^1$,

which is known as an associative and commutative product on $\mathfrak{H}^1$, namely the harmonic product which is modeled by a series shuffle product rule of MZSV's. It is also known that the identity

$$d\big(w \bar{*} w'\big) = d(w) * d\big(w'\big) \tag{9}$$

holds for any $w, w' \in \mathfrak{H}^1$ (see [7,8] for example).

The linear part of Kawashima's relation for MZSV's proven in [7] is then stated as follows.

Theorem 3.2 *(Kawashima).* $L_x\widetilde{\alpha}(\mathfrak{H}y \bar{*} \mathfrak{H}y) \subset \ker \overline{Z}$.

We easily obtain the following equivalence.

Proposition 3.3. *For any $n \geqslant 1$, we have*

$$\overline{\rho}_n\big(\check{\mathfrak{H}}^1\big) \subset L_x\widetilde{\alpha}(\mathfrak{H}y \bar{*} \mathfrak{H}y) \quad \Longleftrightarrow \quad \rho_n\big(\check{\mathfrak{H}}^1\big) \subset L_x\varphi(\mathfrak{H}y * \mathfrak{H}y).$$

Proof. Assume that $\overline{\rho}_n(\check{\mathfrak{H}}^1) \subset L_x\widetilde{\alpha}(\mathfrak{H}y \bar{*} \mathfrak{H}y)$. Using Lemma 3.1,

$$\rho_n\big(\check{\mathfrak{H}}^1\big) = d\overline{\rho}_n\big(\check{\mathfrak{H}}^1\big) \subset dL_x\widetilde{\alpha}(\mathfrak{H}y \bar{*} \mathfrak{H}y).$$

Since the operators d and L_x commute,

$$\rho_n\big(\check{\mathfrak{H}}^1\big) \subset L_x d\widetilde{\alpha}(\mathfrak{H}y \bar{*} \mathfrak{H}y) = -L_x\varphi d(\mathfrak{H}y \bar{*} \mathfrak{H}y) = -L_x\varphi\big(d(\mathfrak{H}y) * d(\mathfrak{H}y)\big).$$

The first equality is by (8) and the second by (9). Therefore we have $\rho_n(\check{\mathfrak{H}}^1) \subset L_x\varphi(\mathfrak{H}y * \mathfrak{H}y)$ because of $d(\mathfrak{H}y) = \mathfrak{H}y$. In the same way, we can prove the reverse assertion. □

According to this proposition, we find that the CSF for MZV's is equivalent to that for MZSV's, which is also proven in Ihara, Kajikawa, Ohno and Okuda [5]. (Their proof is by direct calculation, which can also be applied to the q-analogue of MZV's.) Combining Theorem 3.2 with Proposition 3.3, we obtain

Corollary 3.4. *For any $n \geqslant 1$, we have $\overline{\rho}_n(\check{\mathfrak{H}}^1) \subset \ker \overline{Z}$.*

Therefore, the operator $\overline{\rho}_n$ induces relations among MZSV's.

4. Remarks

4.1. Special evaluations

We find that the following identities hold

$$\rho_1(z_{k_1} \cdots z_{k_l}) = \sum_{j=1}^{l} \sum_{i=1}^{k_j-1} z_{k_j-i+1} z_{k_{j+1}} \cdots z_{k_l} z_{k_1} \cdots z_{k_{j-1}} z_i - \sum_{j=1}^{l} x z_{k_{j+1}} \cdots z_{k_l} z_{k_1} \cdots z_{k_j},$$

$$\overline{\rho}_1\big(\gamma(z_{k_1}\cdots z_{k_l})-x^{k_1+\cdots+k_l}\big)=\sum_{j=1}^{l}\sum_{i=1}^{k_j-1} z_{k_j-i+1}z_{k_{j+1}}\cdots z_{k_l}z_{k_1}\cdots z_{k_{j-1}}z_i-kz_{k+1},$$

where $k=k_1+\cdots+k_l$. Evaluating them by Z and $\overline{Z}$ respectively, we obtain the CSF's (1) and (2).

4.2. An example

Here we introduce another way to show the CSF by means of the linear part of Kawashima's relation in a special case. We first prove a lemma.

Lemma 4.1. *For any $q\geqslant 1$, $w\in\mathfrak{H}y$, $w'\in\mathfrak{H}^1$, we have*

$$zw*z_qw'=z\big(w*z_qw'\big)+z_q\big(zw*w'\big),$$

where $z=x+y$ and $z_q=x^{q-1}y$ $(q\geqslant 1)$.

Proof. Let $w=z_pW$ $(p\geqslant 1)$. We see that

$$\begin{aligned}xw*z_qw'&=z_{p+1}W*z_qw'\\&=z_{p+1}\big(W*z_qw'\big)+z_q\big(z_{p+1}W*w'\big)+z_{p+q+1}\big(W*w'\big)\end{aligned}$$

and

$$\begin{aligned}x\big(w*z_qw'\big)&=x\big(z_pW*z_qw'\big)\\&=z_{p+1}\big(W*z_qw'\big)+z_{q+1}\big(z_pW*w'\big)+z_{p+q+1}\big(W*w'\big).\end{aligned}$$

By subtracting one from another, we obtain the identity

$$xw*z_qw'=x\big(w*z_qw'\big)-z_{q+1}\big(w*w'\big)+z_q\big(xw*w'\big).$$

We also know that

$$yw*z_qw'=y\big(w*z_qw'\big)+z_{q+1}\big(w*w'\big)+z_q\big(yw*w'\big).$$

Adding up these two identities, we have the lemma. □

As a method to derive the CSF from the linear part of Kawashima's relation, we need to write down $\rho_n(z_{k_1}\cdots z_{k_l})$ explicitly in terms of the harmonic product. On the way, we find that the following identity holds.

Proposition 4.2. *For any $n,k\geqslant 1$, we have*

$$\rho_n(z_k)=L_x\varphi(A_{k-1}*z_n).$$

Proof. Note that the automorphism φ is an involution. Since

$$\mathcal{C}_n(z_k)=\sum_{i=1}^{k-1} z_{k-i+1}\otimes z^{\otimes(n-1)}\otimes z_i-x\otimes z^{\otimes(n-1)}\otimes z_k,$$

we have

$$\varphi L_x^{-1}\rho_n(z_k)=\sum_{i=1}^{k}A_{k-i}x^{n-1}A_i.$$

Hence, it is enough to show

$$A_{k-1}*z_n=\sum_{i=1}^{k}A_{k-i}x^{n-1}A_i. \tag{10}$$

We prove (10) by induction on k. If $k=1$, it holds because both sides become z_n. If $k=2$,

$$\begin{aligned}\mathrm{LHS}&=y*z_n=yz_n+z_ny+z_{n+1},\\ \mathrm{RHS}&=A_1x^{n-1}A_1+x^{n-1}A_2=yx^{n-1}y+x^{n-1}(x+y)y.\end{aligned}$$

Therefore the identity (10) holds. Assume that (10) holds for $k-1$ ($k\geqslant 2$). By Lemma 4.1,

$$A_{k-1}*z_n=z(A_{k-2}*z_n)+z_nA_{k-1}.$$

By the induction hypothesis,

$$A_{k-2}*z_n=\sum_{i=1}^{k-1}A_{k-i-1}x^{n-1}A_i,$$

and hence, we have

$$A_{k-1}*z_n=z\sum_{i=1}^{k-1}A_{k-i-1}x^{n-1}A_i+z_nA_{k-1}.$$

Notice that $zA_j=A_{j+1}$ ($j\geqslant 1$) but $zA_0=A_1+y$. Then, we have (10) and the proposition. □

When $k\geqslant 2$, the right-hand side of the identity of Proposition 4.2 is an element of $L_x\varphi(\mathfrak{H}y*\mathfrak{H}y)$. Therefore we have

$$\rho_n(z_k)\in\ker Z$$

according to Theorem 2.4. As an easy application, we also obtain the following.

Corollary 4.3. *For any $n,k\geqslant 1$, we have*

$$\rho_n(yz_k)=L_x\varphi(A_{k-1}*z_{n+1}-A_k*z_n).$$

Proof. The corollary is proven by Proposition 4.2 and the identity

$$\rho_n(zw)=\rho_{n+1}(w)\quad(w\in\mathfrak{H}).\qquad\square \tag{11}$$

This corollary and Theorem 2.4 yield $\rho_n(yz_k)\in\ker Z$ when $k\geqslant 2$.

Table 1

Weight $d+n$	3	4	5	6	7	8	9	10	11	12	13	⋯
$n=1$	1	2	4	6	12	18	34	58	106	186	350	⋯
$n=2$		1	3	5	11	17	33	57	105	185	349	⋯
$n=3$			1	3	7	13	26	48	91	167	319	⋯
$n=4$				1	3	7	15	29	58	111	218	⋯
$n=5$					1	3	7	15	31	61	122	⋯
$n=6$						1	3	7	15	31	63	⋯
$n=7$							1	3	7	15	31	⋯
$n=8$								1	3	7	15	⋯
$n=9$									1	3	7	⋯
$n=10$										1	3	⋯
$n=11$											1	⋯

4.3. *Dimensions*

We denote by $\check{\mathfrak{H}}^1_{(d)}$ the degree-d homogeneous part of $\check{\mathfrak{H}}^1$. For $n, d \geqslant 1$, let

$$\mathrm{CSF}^n_d = \langle \rho_n(w) \mid w \in \check{\mathfrak{H}}^1_{(d)} \rangle_{\mathbb{Q}}, \qquad \mathrm{CSF}_d = \bigoplus_{n \geqslant 1} \mathrm{CSF}^n_d.$$

Then, we see the following filtration structure.

Proposition 4.4. *For any $n, d \geqslant 1$, we have $\mathrm{CSF}^{n+1}_d \subset \mathrm{CSF}^n_{d+1}$.*

Proof. The proof is just due to the identity (11). □

We obtain Table 1 of dimensions of CSF^n_d by calculation using Risa/Asir, an open source general computer algebra system.

We also find that the sequence of $\dim_{\mathbb{Q}} \mathrm{CSF}_d$ corresponds to the sequence of $\dim_{\mathbb{Q}} \mathrm{CSF}^1_d$, which is the number of cyclically equivalent indices of weight k and depth $\leqslant k-1$ given by

$$-2 + \frac{1}{k} \sum_{m \mid k} \phi\left(\frac{k}{m}\right) 2^m,$$

where $k = d+1$ and $\phi(n) = |(\mathbb{Z}/n\mathbb{Z})^\times|$, Euler's totient function (see [10, Chap. 1, Ex. 27], [11, Chap. 7, Ex. 7.112] for example). We note that the same table can be obtained by considering $\mathbb{Q}$-vector spaces generated by the CSF for MZSV's instead of the CSF for MZV's.

4.4. *Algebraic formulations of the CSF and Derivation relation*

Let "$\cdot$" be an action of $\mathfrak{H}$ on $\mathfrak{H}^{\otimes(n+1)}$ defined by

$$a \cdot (w_1 \otimes \cdots \otimes w_{n+1}) = a w_1 \otimes w_2 \otimes \cdots \otimes w_{n+1},$$
$$(w_1 \otimes \cdots \otimes w_{n+1}) \cdot b = w_1 \otimes \cdots \otimes w_n \otimes w_{n+1} b.$$

The action "$\cdot$" gives a $\mathfrak{H}$-bimodule structure on $\mathfrak{H}^{\otimes(n+1)}$ called the outer bimodule structure. For $n \geqslant 1$, we define the $\mathbb{Q}$-linear map $\mathcal{D}_n : \mathfrak{H} \to \mathfrak{H}^{\otimes(n+1)}$ by

$$\mathcal{D}_n(x) = x \otimes z^{\otimes(n-1)} \otimes y, \qquad \mathcal{D}_n(y) = -\left(x \otimes z^{\otimes(n-1)} \otimes y\right)$$

and

$$\mathcal{D}_n(ww') = \mathcal{D}_n(w) \cdot w' + w \cdot \mathcal{D}_n(w')$$

for any $w, w' \in \mathfrak{H}$. We find that the map $\mathcal{D}_n$ is well defined. Let $\partial_n = M_n \mathcal{D}_n$. Then, we find that this ∂_n gives the derivation operator introduced in Ihara, Kaneko and Zagier [6], which induces Derivation relation for MZV's.

According to the above settings, we find that there is a nice resemblance between algebraic formulations of CSF and Derivation relation. The images of x and y of the operator $\mathcal{D}_n$ coincide with those of the operator $\mathcal{C}_n$. The only difference between $\mathcal{D}_n$ and $\mathcal{C}_n$ is the product rule appearing in their Leibniz rules.

Acknowledgments

The authors thank to Professor Masanobu Kaneko and Dr. Shingo Saito for many helpful comments and advice.

References

[1] M. Van den Bergh, Double Poisson algebras, Trans. Amer. Math. Soc. 360 (11) (2008) 5711–5769.
[2] V.G. Drinfel'd, On quasitriangular quasi-Hopf algebras and on a group that is closely connected with Gal($\overline{\mathbb{Q}}/\mathbb{Q}$), Leningrad Math. J. 2 (4) (1991) 829–860.
[3] M. Hoffman, The algebra of multiple harmonic series, J. Algebra 194 (1997) 477–495.
[4] M. Hoffman, Y. Ohno, Relations of multiple zeta values and their algebraic expression, J. Algebra 262 (2003) 332–347.
[5] K. Ihara, J. Kajikawa, Y. Ohno, J. Okuda, MZV vs. MZSV, preprint.
[6] K. Ihara, M. Kaneko, D. Zagier, Derivation and double shuffle relations for multiple zeta values, Compos. Math. 142 (2) (2006) 307–338.
[7] G. Kawashima, A class of relations among multiple zeta values, J. Number Theory 129 (4) (2009) 755–788.
[8] G. Kawashima, T. Tanaka, Newton series and extended derivation relations for multiple L-values, arXiv:0801.3062.
[9] Y. Ohno, N. Wakabayashi, Cyclic sum of multiple zeta values, Acta Arith. 123 (2006) 289–295.
[10] R.P. Stanley, Enumerative Combinatorics, vol. 1, Cambridge Stud. Adv. Math., vol. 49, Cambridge University Press, Cambridge, 1997.
[11] R.P. Stanley, Enumerative Combinatorics, vol. 2, Cambridge Stud. Adv. Math., vol. 62, Cambridge University Press, Cambridge, 1999.
[12] T. Tanaka, On the quasi-derivation relation for multiple zeta values, J. Number Theory 129 (2009) 2021–2034.