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On compact fibered spaces ☆

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Abstract

A space X is called *fibered* if there exists a countable family γ of sets closed in X such that $\gamma(x) = \bigcap\{F\colon\ x \in F \in \gamma\}$ is metrizable for each $x \in X$. In the paper we answer two problems of Tkachuk raised in [Topology Proc. 19 (1994) 321–334] about compact fibered spaces. © 2002 Elsevier Science B.V. All rights reserved.

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A space is *metrizably fibered* if it can be mapped onto a separable metrizable space with metrizable fibers (i.e., the inverse image of any point is metrizable). See Tkachuk's paper [3] for a proof that a Tychonoff space X is metrizably fibered iff there is a countable family γ of zero sets in X such that

$$\gamma(x) = \bigcap\{F\colon\ x \in F \in \gamma\}$$

is metrizable for each point $x \in X$.

This characterization justifies the following definition: a space X is said to be *fibered* if there is a countable family of closed sets γ in X such that

$$\gamma(x) = \bigcap\{F\colon\ x \in F \in \gamma\}$$

is metrizable for each point $x \in X$. Our terminology differs from that of Tkachuk, he calls these spaces weakly metrizably fibered. The same class of spaces was defined also by Tkachenko [2]; he called these spaces metrizable-approximable.

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In his paper [3] Tkachuk proves that a compact fibered space has countable tightness and shows by an example that it is not necessarily Frechet. Hence he raises the following natural question (Problem 3.7): is a compact fibered space sequential? We show that the answer is yes, but first we give the following result.

Lemma 1. *A countably compact fibered space is compact.*

Proof. Let X be a countably compact fibered space, γ be a countable family of closed subsets with $\gamma(x) = \bigcap\{F\colon x \in F \in \gamma\}$ metrizable for $x \in X$ and choose an open cover $\mathcal{G}$ of the space X. For each $x \in X$ the subspace $\gamma(x)$ is metrizable and countably compact, hence it is compact and so there exists a finite $\mathcal{G}_x \subset \mathcal{G}$ covering $\gamma(x)$. By countable compactness, there exists a finite $\gamma_x \subset \gamma$ with $x \in \bigcap \gamma_x \subset \bigcup \mathcal{G}_x$.

Hence the family of those finite intersections of members of γ which can be covered with finitely many members of $\mathcal{G}$, form a cover of X. As γ has only countably many finite subsets, X can be covered with a countable subfamily of $\mathcal{G}$. The countable compactness of X now implies that there is also a finite subcover of $\mathcal{G}$. □

The following lemma is taken from [3], it is included here only to make the paper self-contained.

Lemma 2 (V. Tkachuk). *Let X be a fibered compact Hausdorff space. Then X has a point of countable character.*

Proof. Let $\gamma = \{F_n\colon n \in \omega\}$ be a fibering of X. It is easy to choose a sequence of nonempty open sets $\{U_n\colon n \in \omega\}$ with $\overline{U_{n+1}} \subset U_n$ such that $U_n \subset F_n$ or $U_n \cap F_n = \emptyset$ for $n \in \omega$. Let $H = \bigcap\{U_n\colon n \in \omega\}$. Then H is a non-empty G_δ subset of X and H is metrizable, because $H \subset \gamma(x)$ for any $x \in H$, hence any point of H is a G_δ-point in X. □

Proposition 1. *A compact Hausdorff fibered space X is sequential.*

Proof. Let $H \subset X$ be sequentially closed; we have to prove that it is also closed in X. We can suppose that H is not countably compact because otherwise it would be compact by Lemma 1. Choose a countably infinite subset $D \subset H$ closed discrete in H and let F be the set of all cluster points of D in X. Then F is nonempty and there is a point $x \in F$ which is a G_δ-point in F by Lemma 2. But F is also a G_δ-set in $\overline{D} = D \cup F$ hence the point x is a G_δ-point in the compact set $\overline{D}$. Consequently x is a point of countable character in $\overline{D}$ and so there is a subsequence of D converging to the point $x \notin H$, contradicting that H is sequentially closed. □

Another problem mentioned in [3, Problem 3.8]: Is the Helly space (i.e., the subspace of I^I with the topology of pointwise convergence which consists of the monotone functions) (metrizably) fibered? The answer is affirmative.

Proposition 2. *The Helly space H is metrizably fibered.*

Proof. We prove that H can be mapped into the separable metrizable space I^ω in such a way that the inverse image of any point is metrizable.

Let Q denote the set of the rationals in I and let π be the projection of H onto I^Q. The projection $\pi(f)$ of a function $f \in I^I$ is just the restriction of the function f to Q. Observe now that for any monotone function $g \in I^Q$, for all but countably many $x \in I$ the limits

$$g(x-0) = \lim_{\substack{q \to x-0 \\ q \in Q}} g(q),$$
$$g(x+0) = \lim_{\substack{q \to x+0 \\ q \in Q}} g(q)$$

are equal. If now $S = \{x \in I\colon\ g(x-0) \neq g(x+0)\}$ and $I_x = [g(x-0), g(x+0)]$ for $x \in S$ then

$$\pi^{-1}(g) = \prod_{x \in S} I_x \times \prod_{x \in I-S} \{g(x)\}$$

is homeomorphic to the separable metrizable space I^S. □

The last theorem of the paper is connected with the Galvin–Telgarsky game [1]. Let X be a topological space and consider the following two-person game on X: White (**W**) chooses a point $x_0 \in X$ then Black (**B**) selects an open set G_0 with $x_0 \in G_0$. In the nth turn of the play **W** chooses a point $x_n \in X$ and **B** answers with a neighbourhood G_n of x_n and so on. **W** wins if the family $\{G_n\}$ is a cover of X, otherwise **B** wins. Although originally only the ω-length game was considered, we can continue it through the ordinal numbers: the game ends if the selected open sets form a cover of the space. Call a space X *winnable for* **W** *in countably many steps* if **W** has a strategy such that any play ends at some countable ordinal. Observe that a hereditarily Lindelöf-space is winnable in countably many steps: if **W** always chooses a new point (i.e., a not yet covered one) then the points chosen form a right separated subspace so it has to be countable. As far as we know, the following problem is open:

Problem 1. Is every compact first countable space winnable in countably many steps?

Proposition 3. *Every compact fibered space is winnable for* **W** *in countably many steps.*

Proof. We prove that if X is compact, γ is a countable system of closed sets in X such that $\gamma(x) = \bigcap\{F\colon\ x \in F \in \gamma\}$ is winnable for **W** for any $x \in X$ then also X is winnable for **W**.

Take any $x_0 \in X$ and win the subspace $\gamma(x_0)$ in countably many steps. Let the open set G_0 be the union of the answers of **B**. In the αth turn choose a point x_α not covered by the previous G_ξ's and let **W** win the subspace $\gamma(x_\alpha)$ in countably many steps. We have to prove that the play ends at some countable ordinal.

As $\gamma(x_\xi) \subset G_\xi$ for each ξ considered, the compactness of X implies that there exists a set H_ξ which is a finite intersection of members of γ and $\gamma(x_\xi) \subset H_\xi \subset G_\xi$. Note that these H_ξ's are all different: if $\xi < \eta$ then $x_\eta \notin G_\xi$ hence $x_\eta \in H_\eta - H_\xi$. But γ is countable so it has only countably many finite subsets. □

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