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The Isomorphism Theorem for Bernoulli Flows*,†

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INTRODUCTION

We say that the flow S_t is a Bernoulli flow of finite entropy if, for each fixed t_0, S_{t_0} is a Bernoulli shift of finite entropy. We proved the existence of Bernoulli flows in [2]. By a trivial normalization (change the time scale by multiplying all t by a fixed constant; that is, speeding up or slowing down the flow by a fixed constant) we can assume that $E(S_1) = 1$. In this paper we will show that any two Bernoulli flows with the above normalization are isomorphic.

We say S_t acting on X is isomorphic to $\bar{S}_t$ acting on $\bar{X}$ if there is an invertible measure-preserving map φ of X onto $\bar{X}$ and, for each fixed t_0, $\bar{S}_{t_0}\varphi(x) = \varphi(S_{t_0}x)$ holds for almost all x. (If S_t and $\bar{S}_t$ are flows built under functions—the theorem of Ambrose and Kakutani implies that they are isomorphic to such flows—then there is one set E of measure 0 and if x is not in E then $\bar{S}_t\,\varphi(x) = \varphi\, S_t(x)$ for all t.)

In [2] we showed that a certain flow is a Bernoulli flow of finite entropy (Totoki had already shown that this flow was a K-flow). The flow can be described as follows: Let T be the 2-shift and let Y be the space T acts on. Let P_1 and P_2 be the two atoms of the independent generator for T. Let g be the function on Y that is 1 on P_1 and $\sqrt{2}$ on P_2. The flow will act on X when X is the part of $Y \times R$ (R being the reals) that lies below the graph of g. Each point (y, r) moves straight up at unit speed until it hits $(y, g(y))$; it then goes to $(Ty, 0)$ and continues to move up at unit speed. It is clear from the proof in [2] that we can replace g by any function that is constant on P_1 and P_2 and such that the value on P_1 divided by the value on P_2 is irrational. Our theorem implies that after a normalization any two of these flows are isomorphic (or if

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we fix $\int_g$ we do not have to normalize). Similarly, we could have replaced T by another Bernoulli shift and we would still (after a normalization) get the same flow.

Another application of our theorem is the following: There are some deep results of Sinai and Anosov that allow us to apply the criteria of "very weak Bernoulli" defined in [2] to geodesic flow on surfaces of negative curvature, showing that they are Bernoulli flows. The theorem of this paper then shows that geodesic flows on surfaces of negative curvature are isomorphic to the simple flow described in the privious paragraph. Geodesic flows on surfaces of negative curvature were shown to be ergodic, and mixing and even a K-flow by Hedlund, Hopf, Sinai, and Anosov. Interest in them derives from the fact that they are mathematically simpler versions of mechanical systems such as the hard sphere gas. Also there are surfaces in three-space and centers of attraction and repulsion near these surfaces that make the motion of a particle constrained to move on these surfaces, mathematically equivalent to geodesic flow on a surface of negative curvature. (This was pointed out by Kolmogorov.)

In the course of proving the isomorphism theorem we get a "Sinai type" theorem (see the corollary to the main lemma) that says that if S_t is a mixing flow of finite entropy then there is a Bernoulli flow embedded in it. (The flow restricted to an invariant σ-algebra is a Bernoulli flow.) The condition of mixing is not essential.

Examination of the proof in this paper yields that if S_1 is a Bernoulli shift then S_t is a Bernoulli shift for all t. (Because roots of Bernoulli shifts are Bernoulli shifts [2] we already knew that S_t was a Bernoulli shift for rational t.)

Preliminaries

We will need the extension to countable partitions of a theorem (Theorem 1 below) proved in "A Kolmogorov automorphism that is not a Bernoulli shift" [3]. Even though the change is minor we will reproduce the proof here. The proof of Theorem 1 depends on Lemma 1 below which is a minor variation on a lemma in "Bernoulli shifts with the same entropy are isomorphic" [1].

LEMMA 1. *Let I be an abstract partition; then given ϵ, there is a δ such that if T is a mixing transformation with $E(T) \geqslant E(I)$ and P a* *partition satisfying* (1) $| d(P) - d(I)| < \delta$, (2) $| E(P, T) - E(I)| < \delta$, *then there is a partition* $\hat{P}$ *such that* :(1) $d(\hat{P}) = d(I)$, (2) $T^i\hat{P}$ *are independent,* (3) $| \hat{P} - P | < \epsilon$ ($d(P)$ *stands for* dist (P) *as in the definitions in* [1]).

In [1] we proved the following (Lemma 5):

LEMMA. *Let I be an abstract partition. Let T be a mixing transformation on X such that* $E(T) - E(I)$. *Given* ϵ, *we can find a* δ *such that if P is a partition of X satisfying* (1) $|$ dist $P -$ dist $I | < \delta$ (2) $0 < E(T) - E(P, T) < \delta$, *then there is a partition* $\hat{P}$ *of X such that* (1) dist $\hat{P} =$ dist I, (2) $T^i\hat{P}$ *are independent,* (3) $| \hat{P} - P | < \epsilon$.

Remark. (a) In the above lemma, δ depends on ϵ and I but not on T. (This is not clear from the statement, but since Lemma 5 of [1] is a corollary of Lemma 4, all we need show is that in Lemma 4, $h(\epsilon)$ and $g(\epsilon)$ are independent of T. This in turn is clear from the proof of Lemma 1.)

(b) We only need $E(T) \geqslant E(I)$ and can replace (2) in the above lemma by $| E(P, T) - E(I)| < \delta$. (To see this, note that if $E(P, T) < E(I)$, then we could change T by restricting it to an inviariant sub-σ-algebra containing P so that $E(T) = E(I)$. If $E(P, T) \geqslant E(I)$, then we could continuously deform P through P_t, $0 \leqslant t \leqslant 1$, such that $P_0 = P$, $d(P_1) = d(I)$, $| P_t \rightarrow P | < \delta$, and $| d(P_t) - d(I)| < \delta$ $E(P_1, T) \leqslant E(I)$. If we have equality, P_1 is the $\hat{P}$ we want. If we have $<$, then we could find P_t such that (1) and the replacement for (2) holds, and $E(P, T) < E(I)$. We then have first situation.)

DEFINITION. Let P and P' be partitions on the same space. $e(P, P')$ will denote the entropy of the partition whose atoms are $P_i \cap P_j'$, $i \neq j$ and $\bigcup_{i=1}^{\infty} P_i \cap P_i'$. We will define $\bar{e}(P, P')$ (now P and P' may be on different spaces) to be the $\inf_{\bar{P}}$ of $e(\bar{P}, P')$ where $\bar{P}$ is a partition with the same distribution as P and is defined on the same space as P'.

THEOREM 1. *Let T be a Bernoulli shift and P a countable partition such that* $E(P) < \infty$ and $E(P, T) = E(T)$. *Then given* ϵ, *there is a* δ *and u such that if* $\bar{T}$ *is mixing,* $E(\bar{T}) \geqslant E(T)$, $\bar{P}$, $\bar{T}$ *satisfies* (0)$\bar{e}(\bar{P}, P) < \delta$, (1) $| E(\bar{P}, \bar{T}) - E(P, T)| < \delta$, (2) $| d(\bigvee_0^u T^iP) - d(\bigvee_0^u \bar{T}^i\bar{P})| < \delta$. *Then there is a* $\hat{P}$ *such that* (1) $| \hat{P} - \bar{P} | < \epsilon$, (2) $d(\bigvee_0^n T^iP) = d(\bigvee_0^n \bar{T}^i\hat{P})$ for *all n.*

Before giving the proof we will need the following:

DEFINITION. Let L be a partition of the set of sequences $\{\alpha_i\}_{i=-K}^{K}$, $1 \leqslant \alpha_i \leqslant k$. Let P be a partition with k atoms. There is a $1-1$ correspondence between the atom in $\bigvee_{-K}^{K} T^i P$ and sequences $\{\alpha_i\}_{i=-K}^{K}$. (Each atom in $\bigvee_{-K}^{K} T^i P$ has the form $\bigcap_{-K}^{K} T^i P_{\alpha_i}$.) L_P is the partition (of which $\bigvee_{-K}^{K} T^i P$ is a refinement) whose i-th atom consists of the atoms in $\bigvee_{-K}^{K} T^i P$ whose corresponding sequences are in the i-th atom of L.

Proof. Let B be a finite partition such that $T^i B$ are independent and generate. Pick K so that $\bigvee_{-K}^{K} T^i B \supset^{1/10\epsilon} P$. This implies that there is an L as in the above definition such that $|L_B - P| < 1/10\,\epsilon$. Pick δ' (using our previous lemma) such that if $|E(B',\bar{T}) - E(B, T)| < \delta'$ and $|d(B') - dB| < \delta'$, then there is a $\bar{B}$ such that $|\bar{B} - B'| < \epsilon/100K$, $d(\bar{B}) = d(B)$ and $\bar{T}^i\bar{B}$ are independent. There exists M depending on P such that if $\bar{e}(\bar{P}, P) < (1/200)\delta'$ and if we let $\bar{P}_M = \{\bar{P}_1, ..., \bar{P}_M, \bigcup_{M+1}^{\infty} \bar{P}_i\}$, then $|E(\bar{P}_M, \bar{T}) - E(\bar{P}, \bar{T})| < (1/100)\delta'$. (We can define P_M in a similar way and can assume $|P_M - P| < (1/10)\epsilon$.) Pick γ, $\gamma < (1/100)\epsilon$, $\gamma < (1/100)\delta'$ such that if P' has M atoms and if $|P' - \bar{P}_M| < \gamma$, then $|E(P', \bar{T}) - E(\bar{P}_M, \bar{T})| < (1/10)\delta'$. Now pick $K' > K$ so that $\bigvee_{-K'}^{K'} T^i B \supset^{1/10\gamma} P$. Pick n so that

$$\frac{K'}{n} < \frac{1}{100}\gamma \quad \text{and} \quad \left|\frac{1}{n}\log\frac{1}{n} + \left(1-\frac{1}{n}\right)\log\left(1-\frac{1}{n}\right)\right| < \frac{1}{10}\delta'.$$

Note that so far our choice of n and γ depended only on T, P, and B. We now choose u and δ so that $u - n$, $\delta < (\gamma\alpha/100)$, where α is the measure of the smallest atom in $\bigvee_0^n T^i P_M$.

Now apply Rochlin's theorem to find a set F such that $T^i F$, $0 \leqslant i \leqslant n-1$, are disjoint, and $m[X - \bigcup_{i=0}^{n-1} T^i F] < (1/100)\gamma$. ($X$ is the space on which T acts.) If we replace F by $T^l F$, the above properties still hold, and we can assume that $d(\bigvee_0^{n-1} T^{-i}(P_M \vee B)/F)$ is as close as we want to $d(\bigvee_0^{n-1} T^{-i}(P \vee B))$. Therefore, by removing an arbitrarily small set from $T^l F$ (and calling the new set F), we can assume (1) $d(\bigvee_0^{n-1} T^{-i}(P \vee B)/F) = d(\bigvee_0^{n-1} T^{-i}(P \vee B))$.

Applying the same reasoning again, we can find a set $\bar{F}$ such that $\bar{T}^i\bar{F}$, $0 \leqslant i \leqslant n-1$, are disjoint $m(\bar{X} - \bigcup_{i=0}^{n-1} \bar{T}^i\bar{F}) < (\gamma/100)$, and (2) $d(\bigvee_0^{n-1} \bar{T}^{-i}\bar{P}_M/\bar{F}) = d(\bigvee_0^{n-1} \bar{T}^{-i}\bar{P}_M)$. Now because of hypothesis (2) and our choice of u and δ, we could (by removing a small part of $\bar{F}$) assume that we have instead of (2) the following:

$$(3)\quad d\left(\bigvee_0^{n-1} \bar{T}^i\bar{P}_M/\bar{F}\right) = d\left(\bigvee_0^{n-1} T^{-i}P_M\right).$$

If we let $\mathcal{J}$ be the gadget formed by partitioning T^iF by P_M and $\bar{\mathcal{J}}$ the gadget formed by partitioning the $\bar{T}^i\bar{F}$ by $\bar{P}_M$, then $\mathcal{J}$ and $\bar{\mathcal{J}}$ are isomorphic because of (1) and (3). Let $\mathcal{G}'$ be the gadget formed by partitioning T^iF by $P_M \vee B$. Choose a partition B' of $\bigcup_{i=0}^{n-1} \bar{T}^i\bar{F}$ such that if we form the gadget $\bar{\mathcal{J}}'$ by partitioning the $\bar{T}^i\bar{F}$ by $\bar{P}_M \vee B'$, then $\bar{\mathcal{J}}'$ is isomorphic to $\mathcal{J}'$. (Now extend B' to the rest of the space in any way.)

Because $\bar{\mathcal{J}}'$ and $\mathcal{J}'$ are isomorphic and fill up most of the space, because $(K/n) < (1/10)\gamma$, and because $|L_B - P_M| < (2/10)\epsilon$, we get that (4) $|L_{B'} - \bar{P}_M| < (3/10)\epsilon$. (This can be seen as follows: To each x, we associate a sequence $\{\alpha_i(x)\}_{i=-K}^{K}$ where $T^ix \in P_{\alpha_i}$. $|L_B - P_M|$ is the measure of the set of x such that $x \in P_j$ and $\{\alpha_i(x)\}_{-K}^{K}$ not in L_j(L_j the j-th atom of L). Call such an x a bad x. Then $|L_B - P_M| <$ $(1/10)\epsilon$ implies that the measure of the bad x in $\bigcup_{i=K}^{n-1-K} T^iF$ is $< (1/10)\epsilon$ since $\bar{\mathcal{J}}'$ and $\mathcal{J}'$ are isomorphic. The measure of the bad x in $\bigcup_{i=K}^{n-1-K}\bar{T}^i\bar{F}$ is $< (1/10)\epsilon$. Therefore, the measure of the bad x is $<(1/10)\epsilon +$ $(2/100)\gamma + (\gamma/100) < (2/10)\epsilon$.)

The same argument shows that $\bigvee_{-K'}^{K'} T^iB' \supset^{2/10\gamma} \bar{P}_M$, and hence by our choice of γ, (5) $|E(B', \bar{T}) - E(\bar{P}, \bar{T})| < (1/10)\delta'$. Furthermore, since $\bar{\mathcal{J}}'$ and $\mathcal{J}'$ are isomorphic and fill up most of the space, we get that (6) $|d(B') - d(B)| < \delta'$. (5), (6), and our lemma imply that there is a $\bar{B}$ such that (7) $(\bar{B} - B'| < (\epsilon/100\text{K})$, (8) $d(\bar{B}) = d(B)$, (9) $\bar{T}^i\bar{B}$ are independent. Because of (4) and (7), we get that (10) $|L_{\bar{B}} - \bar{P}_M| <$ $(2/10)\epsilon$. Because of (8) and (9), $\bar{T}$ acting on $\bigvee_{-\infty}^{\infty} \bar{T}^i\bar{B}$ is isomorphic to T under and isomorphism φ, which takes B onto $\bar{B}$. Let $\hat{P}$ be $\varphi(P)$. Then $d(\bigvee_0^n T^iP) = d(\bigvee_0^n \bar{T}^i\hat{P})$ for all n. Again, because of the isomorphism and because L was defined so that $|L_B - P| < (1/10)\epsilon$, we get that (11) $|L_{\bar{B}} - \hat{P}| < (1/10)\epsilon$. (10) and (11) imply $|\bar{P}_M - \hat{P}| < \frac{1}{2}\epsilon$, and hence $|\bar{P} - \hat{P}| < \epsilon$. (By our choice of M, $|P - P_M| < (1/10)\epsilon$, and because of hypothesis (2) $|\bar{P} - \bar{P}_M| < (2/10)\epsilon$.)

PART I

Because of a theorem of Ambrose and Kakutani [6, 7] we can assume that our flow is a flow built under a function. By this we mean the following: We start with a measure space Y and a transformation T on Y. Let g be an integrable nonnegative function on Y. The space X on which the flow acts is the part of $Y \times R$ that lies below the graph of g. Each point (y, r) in X moves straight up at unit speed until it hits

$(y, g(y))$. It then goes to $(Ty, 0)$ and continues to move up at unit speed. ($S_t(y, r) = (y, r + t)$ if $r + t < g(y)$. $S_t(y, r) = (Ty, r + t - g(y))$ if $g(y) < r + t$ and $r + t - g(y) < g(Ty)$, etc.)

We will refer to Y as the "base" of the flow when represented in this way.

We will always assume that S_t is mixing for each t. We will also assume that $m(X) = 1$, and that $E(S_t) < \infty$.

DEFINITION. The continuous P-name of x (of length N) will mean the interval $(0, N)$ partitioned by the P_i where t will be considered in P_i if $S_t(x) \in P_i$.

DEFINITION. $\bar{d}(\{S_tP\}_a^b, \{\bar{S}_t\bar{P}\}_a^b)$ or "the distance between the distributions of the continuous names of P and $\bar{P}$" will be defined as follows: Let φ be an invertible, measure-preserving map of X onto $\bar{X}$. Let $\bar{d}(\varphi) = (1/b - a) \int_{a<t<b} \int_{\bar{x}} | \bar{S}_t\bar{P} - \varphi(S_tP)|$. $\bar{d}$ will be the inf $\bar{d}(\varphi)$ over all φ. ($|P_1 - P_2|$ is the measure of the symmetric difference between P_1 and P_2.)

DEFINITION. We will call a countable partition $P = \{P_i\}_1^\infty$ good if (1) $E(P) < \infty$, (2) for each P_i there is a $\beta_i > 0$ such that if $x \in P_i$, then there is some interval I containing 0 of length β_i and $S_t(x) \in P_i$ for all t in I. (If we have a good partition P we will define the number α_i to be the sup of the possible β_i.) (3) For each x and finite interval I, $S_t(x)$, $x \in I$ will intersect only a finite number of the P_i.

LEMMA 0. *Let S_t and $\bar{S}_t$ be flows acting on Lebesque spaces X and $\bar{X}$. Let P and $\bar{P}$ be good partitions for S_t and $\bar{S}_t$, where $\bar{P}$ generates under $\bar{S}_t$. Let X_p be X with the measure algebra generated by S_tP. We will also assume that the continuous P (or $\bar{P}$) name of each x will be partitioned into intervals that contain their left end points and not their right end points. (This could be accomplished by changing P and $\bar{P}$ on a set of measure 0). We then have that $\bar{d}(\{S_tP\}_0^n, \{\bar{S}_t\bar{P}\}_0^n) = 0$ for all n implies that there is a 1-1 invertible, measure preserving φ mapping $\bar{X}$ onto X_p and a set E of measure 0 such that if $x \notin E$ then $d\bar{S}_t(x) = S_td(x)$ for all t.*

Proof. (1) It is easy to see that dist $\bigvee_{i=1}^k S_{t_i}P =$ dist $\bigvee_{i=1}^k \bar{S}_{t_i}\bar{P}$.

(2) There is an invertible measure preserving of mapping $\bar{X}$ onto X_p and a set E of measure 0 such that if $x \notin E$ then the continuous $\bar{P}$-name of x and the continuous P-name of $d(x)$ agree on the rationals.

(We get (2) by a standard argument since X and $\bar{X}$ are Lebesque spaces). Our lemma follows immediately from (2).

We will prove a flow version of Rochlin's theorem.

LEMMA 1. *Given N and ϵ, we can represent the flow on a flow built under a function, f, where $f \leqslant N$ and $f = N$ except on a set of measure $< \frac{1}{2}\epsilon$.*

Proof. Using the Kakutani–Ambrose representation theorem [6, 7] we can represent the flow S_t as a flow built under a function where Y is the base space, T a transformation on Y, and g a function on Y. It is easy to see that we can assume the measure of Y to be as small as we want (by taking a subset of Y of small measure and using that for the base instead of Y). Now let A_j be the part of Y where $jN \leqslant g < (j+1)N$. Let $\bar{Y} = \bigcup_{j=0}^{\infty} \bigcup_{i=0}^{i=j} S_{iN} A_j$. We can now represent our flow as a flow built under a function with base $\bar{Y}$ and function $\bar{g}$. Clearly $\bar{g} \leqslant N$ and $\bar{g} = N$ except on a set of measure $< m(Y)$.

LEMMA 2. *Given t_0, there exists a good partition P such that P generates under S_{t_0}.*

Proof. (1) If P is any countable partition of finite entropy, then, for any ϵ, we can find a good finite partition P', such that $E(P') < 2E(P)$, $| P - P' | < \epsilon$ and there is an R, $E(R) < \epsilon$ and $P' \vee R \supset P$. (This follows almost immediately from the Ambrose–Kakutani representation).

(2) Fix a countable partition P of finite entropy and fix ϵ. Choose ϵ' so that if $| Q^* - P | < \epsilon'$. Then there is an R such that $Q^* \vee R \supset P$ and $E(R) < \epsilon$. Choose N so that the Shannon–McMillan–Breiman theorem implies that there is an $X_1 \supset X$, $m(X - X_1) < (1/10)\epsilon'$ and the number of different P-names relative to S_{t_0} of length $2N$ in X_1 is $< 2^{4E(P)N}$ (see footnote 1). Apply Lemma 1 and represent the flow as built under a function g defined on Y where $g \leqslant 2N t_0$ and $g = 2Nt_0$ on A and where $m(Y - A) < (\epsilon'/5Nt_0$. Let F' be the set of points in X, (x, y) where $x \in A$ and $N \leqslant y \leqslant N + t_0)$. Let $F = S_t F'$. If t is large enough, we can assume that the fraction of F not in X_1 is less than $\epsilon'/2$. Let Q be the partition of $F \cap X$, into P-names under S_{t_0} of length $2N$ (one of the atoms of Q is the complement of $F \cap X_1$). Then $\bigvee_{-N}^{N} S_{t_0} Q \supset^{\epsilon'} P$. Because Q partitions a set of measure $< (2N)^{-1}$ into $< 2^{4N \cdot E(P)}$ sets, we have $E(Q) < 2\, E(P)$. By our choice of ϵ' we can find a partition P_1, $E(P_1) < \epsilon$ and $\bigvee_{-\infty}^{\infty} S_{t_0}^i (Q \vee P_1) \supset P$. Because of

[1] These names will be taken from $-N$ to N.

(1) we can assume that $\underset{\sim}{Q}$ is also good and is a partition of F. (Change $\underset{\sim}{Q}$ by so little that it still picks up P very well and the new P_1 will still have small entropy).

(3) We will now repeat (2) using P_1 instead of P (and N_1 and ϵ_1' instead of N and ϵ'). We will end up with a set F_1 and a partition of it, $\underset{\sim}{Q}_1$. The measure of the points z, in $(F \cup S_{t_0} F \cup S_{-t_0} F) \cap F_1 < 4m(F)m(F_1)$. (This is so because the flow is mixing and because F_1 was chosen as $S_t F_1'$ for some arbitrarily large t). This implies that the set of z in F, such that there is a z_1 in $F \cap F_1$ and $z_1 = S_t z$, where $| t | \leqslant t_0$ has measure $< 10m(F)m(F_1)$. $m(F_1)$ can be taken to be as small as we want by making N_1 large. We can thus change $\underset{\sim}{Q}$ on a small set so that $\underset{\sim}{Q}$ and $\underset{\sim}{Q}_1$ partition disjoint subsets and both are good and we now have that $E(\underset{\sim}{Q} \vee \underset{\sim}{Q}_1) < 2E(P) + 2\epsilon$ and there is a P_2, $E(P_2) < \epsilon_1$ and $\bigvee_{-\infty}^{\infty} S_{t_0}(\underset{\sim}{Q} \vee \underset{\sim}{Q}_1 \vee P_2) \supset P$.

(4) Continuing this process proves the lemma.

Definition. We will define a "continuous gadget" as follows: Let S_t be a flow on X and let X_1 be a subset of X and P a partition of X_1 with the following properties: S_t can be represented as a flow built under a function with base Y and function F. f is a constant C on a subset Y_1 of Y, and X_1 is the part of X that lies above Y_1. We will call C the height of the continuous gadget. We will say that the continuous gadgets X_1, P and $\bar{X}_1$, $\bar{P}$ are isomophic if they both have the same height C and the distribution of continuous P-names in Y_1 of legnth C is equal to the distribution of continuous $\bar{P}$-names in $\bar{Y}_1$ of length C.

Main Lemma. *We are given $\bar{S}_t$, $\bar{P}$ where $\bar{P}$ is good and generates under some $\bar{S}_{t_1}$ and $\bar{S}_t$ is a Bernoulli flow. Let S_t be a flow such that S_t is mixing for each t, and $E(S_1) = E(\bar{S}_1)$.*

Given ϵ, there exists $t_0 (t_0 = (1/M)t_1$, M an integer), N, and $\delta > 0$ such that if P is a partition satisfying

(1) P is good and $\alpha_i > \frac{1}{2} \bar{\alpha}_i$, for all of the $P_i \in P$ except for a collection of P_i the measure of whose union is $< \delta(\alpha_i$ and α_i for P and $\bar{P}$ are defined on p. 10);

(2) $\bar{e}(P, \bar{P}) < \delta$;

(3) $\bar{d}(\{S_t, P\}_0^N, \{\bar{S}_t, \bar{P}\}_0^N) < \delta$;

(4) $| E(P, S_{t_0}) - E(\bar{P}, \bar{S}_{t_0})| < \delta$;

then, given $\bar{t}_0$, $\bar{N}$, and $\bar{\delta}$ (assume $\bar{t}_0$ is such that t_0 is an integer multiple of $\bar{t}_0$), there is a $\tilde{P}$ such that

(0) $| \tilde{P} - P | < \epsilon$;

($\bar{1}$) $\tilde{P}$ is good and $\tilde{\alpha}_i > \frac{1}{2}\bar{\alpha}_i$ for all of the $\tilde{P}_i$ except for a collection of $\tilde{P}_i$ the measure of whose union is $< \bar{\delta}$;

($\bar{2}$) $\bar{e}(\tilde{P}, \bar{P}) < \bar{\delta}$;

($\bar{3}$) $\bar{d}(\{S_t, \tilde{P}\}_0^{\bar{N}}, \{\bar{S}_t, \bar{P}\}_0^{\bar{N}}) < \bar{\delta}$;

($\bar{4}$) $| E(\tilde{P}, S_{\bar{t}_0}) - E(\bar{P}, \bar{S}_{\bar{t}_0})| < \bar{\delta}$.

Proof. Choice of t_0, N, *and* δ.

(1) Choose K such that $m(\bigcup_K^\infty \bar{P}_i) < (1/100)\epsilon$. (If δ is small enough, $m(\bigcup_K^\infty P_i) < (2/100)\epsilon$).

(2) Choose t_0 such that if the continuous S_t, P-name of x and the continuous $\bar{S}_t$, $\bar{P}$-name of $\bar{x}$ (assuming the names have the same length and that P satisfies hypothesis (1)) have the property that the percentage of $\bigcup_K^\infty P_i$ and $\bigcup_K^\infty \bar{P}_i$ is less than $(3/100)\epsilon$ in each of the above names, and if the names restricted to multiples of t_0 disagree in less than $(1/100)\epsilon$ percent, then the continuous names disagree in less than $(1/10)\epsilon$ percent.

To see this let α be the minimum of the α_i, $\bar{\alpha}_i$, $i \leqslant K$. If the continuous names disagreed in more than ϵ percent, then at least $\frac{1}{2}\epsilon$ of the disagreement would be due to P_i intervals intersecting $\bar{P}_j$ intervals, $i \neq j$, $i < K$, $j < K$, and the length of the intersection is greater than $(1/100)\epsilon\alpha$. Now choose $t_0 < (1/1000)\epsilon\alpha$.

(3) Applying Theorem 1 of Preliminaries we can choose N and δ, $\delta < (1/100)\epsilon$ such that [2, 4] in the statement of this lemma imply that there is a $\hat{P}$ such that $| \hat{P} - P | < ((1/100)\epsilon)^2$ and $\text{dist}(\bigvee_0^n S_{t_0}^i \hat{P}) = \text{dist}(\bigvee_0^n \bar{S}_{t_0}^i \bar{P})$ for all n. (Note that if [3] holds for large enough N, then

$$\left| d\left(\bigvee_0^u S_{t_0}^i P\right) - d\left(\bigvee_0^u \bar{S}_{t_0}^i \bar{P}\right)\right| < 2\delta.\Big)$$

We now assume we have a P satisfying the hypothesis of the lemma. Choose $\hat{P}$ satisfying (3).

(4) Choose Q to be a good finite partition and

$$| E(Q, S_{t_0}) - E(\bar{P}, \bar{S}_{t_0})| < (1/100)\bar{\delta}.$$

(5) Form $\bar{P}'$ by lumping all but a finite number of atoms in $\bar{P}$ together and so that $\bar{e}(\bar{P}, \bar{P}') < (1/100)\bar{\delta}$ and $|\bar{P}' - \bar{P}| < (1/100)\bar{\delta}$. Define P' and $\hat{P}'$ by lumping together corresponding sets. Do not lump $\bar{P}_i$, $i \leqslant K$. We can assume that the measure of the lumped atoms of $\bar{P}$ and P is $< (1/100)\bar{\delta}$.

(6) Pick $\gamma < \bar{\delta}$ such that if we change any partition with the same number of sets as $\underset{\sim}{Q}$, by less than γ then we change its entropy relative to S_{t_0} by less than $(1/100)\bar{\delta}$.

Pick $\xi < (1/100)\bar{\delta}$, $\xi < \gamma$, such that if R_1 and R_2 have the same number of atoms as $\bar{P}'$ and dist R_1 − dist $R_2 | < \xi$, then $\bar{e}(R_1, R_2) < (1/100)\bar{\delta}$.

(7) Pick L so that (a) if $|t| < t_0/L$, and if n is large enough then the $\bar{P}'$ names of x and $\bar{S}_t(x)$ under $\bar{S}_{t_0/L}$ (of length nL) agree to within $(1/100)\xi$; (b) if $|t| < t_0/L$, and if n is large enough then the, $\underset{\sim}{Q}$-, names of x and $S_t(x)$ under $S_{t_0/L}$ (of length nL) differ by $< (1/100)\xi$; (c) t_0/L less than (1/100) of the minimum of the α_i appearing in $\bar{P}'$. (*d*) if $|t| < t_0/L$ (and if n is large enough) then the continuous P' name of x and $S_t x$ (of length nt_0) differ by less than $(1/100)\epsilon$.

(8) We will now form a continuous gadget of height nt_0. *Choice of n. n will be chosen so that after removing a set of measure* $< (1/100)\xi$ *from* $\bar{X}$*(call what remains* $\bar{X}_1$*) we have*:

(*a*) suppose x is in $\bar{X}_1$ and i is an integer $< L$ and suppose we know the $\bar{P}'$-name under $\bar{S}_{t_0}$ (length n) of $\bar{S}_{it_0}(x)$. Then there are less than $2^{(1/100)\bar{\delta}n}$ possible $\bar{P}$-names under $\bar{S}_{t_0/L}$ (of length $(n+1)L^2$) that x can have. We can see this as follows: The size of most of the atoms in $\bigvee_{j=0}^{n} \bar{S}_{t_0}^{j} \bar{P}'$ will approach

$$E(\bar{P}', \bar{S}_{t_0})n$$

(as $n \to \infty$) $\frac{1}{2}$.

The size of most of the atoms of $\bigvee_{-L}^{nL} S_{t_0/L}$ will approach

$$\tfrac{1}{2}^{E(\bar{P}', \bar{S}_{t_0}/L)nL} > \tfrac{1}{2}^{E(\bar{S}_{t_0}/L)nL} = \tfrac{1}{2}^{E(\bar{S}_{t_0})n}.$$

Because of (5) the above implies that if n is large enough, then, except for a set of measures $< (1/L)(1/100)\xi$, we have that (a') if we know the $\bar{P}'$-name of x is under $\bar{S}_{t_0}$ (of length n) then there are less than $2^{(1/100)\delta n}$ possible $\bar{P}'$-names under $\bar{S}_{t_0/L}$ (of length $(n+1)L$) that x can have. The above implies (a).

[2] Take the name from $-L$ to nL.

(*b*) If $x \subset \bar{X}_1$, then its continuous $\bar{P}'$-name (of length nt_0) has the property that the distribution of continuous $\bar{P}'$-names of length $\bar{N}$ contained in it is within $(1/100\xi$ of the distribution of continuous $\bar{P}'$-names (of length $\bar{N}$) in $\bar{X}$. Furthermore, the distribution of $\bar{P}'$ in it is within $(1/100)\xi$ of the distribution of $\bar{P}'$ in $\bar{X}$. Also the parts not belonging to $\bigcup_1^K \bar{P}_i$ are less than $(1/100)\,\epsilon$ of its length.

After removing a set of measure $< (1/100)\xi$ *from* X (*call what remains* X_1) we have

(*c*) The $\hat{P}$-names of points in x_1 correspond to the $\bar{P}$-names of points in $\bar{X}_1$ (under S_{t_0} and $\bar{S}_{t_0}$ and of length n).

(*d*) If A is an atom in $\bigvee_0^n S_{t_0}\hat{P}'$ then

$$m(A \cap X_1) < \tfrac{1}{2}^{[E(\hat{P}', S_{t_0}) - (1/100)]\delta n};$$

if A is an atom in $\bigvee_0^{Ln} S^i_{t_0/L}(Q \vee P')$ then $m(A \cap X_1) = 0$, or

$$m(A \cap X_1) > \tfrac{1}{2}^{[E(S_{t_0}/L) + (1/L)\cdot(1/100)\delta]Ln} = \tfrac{1}{2}^{[E(S_{t_0}) + (1/100)\delta]n}.$$

(*e*) If x is in X_1, then the distribution of P in the continuous P-name of x (of length nt_0) satisfies (2). That is, the part of the name not belonging to $\bigcup_0^K P_i$ is less than $(1/100)\epsilon$ (recall assumption [3] in the statement of the lemma and the choice of δ).

After removing an additional set of measure $< (1/100)\epsilon$ *from* X (*call what remains* X_2, $X_2 \subset X_1$) we have

(*f*) The $\hat{P}'$-and P'-names (of length n) under S_{t_0} agree to within $(1/100)\epsilon$.

(*g*) (*d*) still holds on X_2.

(*h*) n is large enough to satisfy (7). Also $L \cdot 2^{(2/100)\delta n} < 2^{(3/100)\delta n}$

(9) We will next apply the marriage lemma as in [8]. We first divide each atom in $\bigvee_0^n S^i_{t_0}\hat{P}'$ into $2^{(1/100)\delta n}$ equal pieces. Each of these will have smaller measure than the part of any atom of $\bigvee_0^{Ln} S^i_{t_0/L}(Q \vee P')$ lying in X_1 or X_2 [see (5), (8*d*) and (8*g*)]. We can now assign to each atom in $\bigvee_0^{Ln} S^i_{t_0/L}(Q \vee P')$ that intersects X_2 one of the above pieces of an atom of $\bigvee_0^n S^i_{t_0}\hat{P}'$ that intersects it. (We can do this because of the marriage lemma.) Because of (8*d*) we can extend the above assignment and assign to each atom in $\bigvee_0^{Ln} S^i_{t_0/L}(Q \vee P')$ that intersects X_1 one of the pieces of the atoms of $\bigvee_0^n S^i_{t_0}\hat{P}'$ that intersects X_1. (8*d*) implies

that there are more pieces of atoms of $\bigvee_0^n S_{t_0}^i \hat{P}'$ intersecting X_1 than there are atoms of $\bigvee_0^{Ln} S_{t_0/L}^i(Q \vee P')$ intersecting X_1).

We will now rephrase the above, in terms of the names of points. First note that because of (8f), if an atom of $\bigvee_0^n S_{t_0}^i \hat{P}'$ intersects an atom of $\bigvee_0^{Ln} S_{t_0/L}^i(Q \vee P')$ on X_2, then the $\hat{P}'$-name (under S_{t_0} of length n) of any point in the first atom agrees to within $(1/100)\epsilon$ with the P'-name under S_{t_0} of length n of the second atom. We get the following: We can assign to each to each atom A, in $\bigvee_0^{Ln} S_{t_0/L}^i(Q \vee P')$ that intersects X_1 a $\hat{P}'$-name under S_{t_0} of length n and if A intersects X_2, then the above $\hat{P}'$-name will disagree with the P'-name (under S_{t_0} of length n) of all the points in A (they all have the same name) in less than $(1/100)\epsilon\, n$ places. Furthermore, each $\hat{P}'$-name (under S_{t_0} of length n) is used at most $2^{(1/100)\delta n}$ times.

(10) Form a "continuous gadget" of height nt_0. Let F be the part of the gadget of height $\leqslant t_0$ and F_L the part of the gadget of height $\leqslant t_0/L$ (if Y is the base, $F = \bigcup_{0 \leqslant t \leqslant t_0} S_t Y$). Let ${}_1F_L$ be $F_L \cap X_1$ and ${}_2F_L = F_L \cap X_2$. Because S_{t_0} is mixing, we can assume $m(F_L - {}_1F_L) < (1/100)\xi\, m(F_L)$ and $m(F_L - {}_2F_L) < (1/100)\epsilon\, m(F_L)$. We can also assume that the measure of the part of X not in the continuous gadget is $< (1/100)\xi$.

(11) We will now change $\bigvee_0^{Ln} S_{t_0/L}^i(Q \vee P')$ on F_L. We will call this new partition of F_L, R. R will have the following properties:

(a) There is a 1–1 correspondence between atoms of R and atoms of $\bigvee_0^{Ln} S_{t_0/L}^i(Q \vee P')$.

(b) If $x \in Y$, then $\bigcup_0^{t_0/L} S_t x$ lies entirely in one atom of R and the corresponding atom in $\bigvee_0^{Ln} S_{t_0/L}^i(Q \vee P')$ interesected $\bigcup_0^{t_1/L} S_t x$.

(c) The atoms of R that cover ${}_2F_L$ correspond to atoms in $\bigvee_0^{Ln} S_{t_0/L}^i(Q \vee P')$ which covered ${}_2F_L$.

(d) Because of (b) and (7b), we have: There is a partition $\tilde{Q}$ such that $|\tilde{Q} - Q| < (1/100)\xi$ and all points in one atom of R have the same $\tilde{Q}$-name under $S_{t_0/L}$ of length nL.

(12) To each atom of R corresponds a $\hat{P}'$-name (under S_{t_0}, of length n) of some point x in X_1 (because of (9) and (11a). The $\hat{P}'$-name of x is equal (because of (8c) to the $\bar{P}'$-name of some $\bar{x}$ in $\bar{X}_1$. We can now assign to each atom R the $\bar{P}'$-name of $\bar{x}$ under $S_{t_0/L}$ of length nL.

We can now define $\tilde{P}$ so that the $\tilde{P}$-name (under $S_{t_0/L}$ of length nL) of a point, x, in F_L is the same as the $\bar{P}'$-name (under $s_{t_0/L}$ of length nL) which was assigned to the atom of R containing x.

(13) We will now check that

(a) $|E(\tilde{P}, S_{t_0}) - E(\bar{P}, \bar{S}_{t_0})| < \delta$.

We first note that

(b) $E\left(\bigvee_0^L S^i_{t_0/L}R, S_{t_0}\right) > E(\bar{P}, \bar{S}_{t_0}) - \frac{2}{100}\delta$.

This follows from (6) (which says Q has good entropy), (11d) (which implies that $F_L \vee (\bigvee_0^{nL} S^i_{t_0/L} R) \supset \tilde{Q}$), and (6) (which says that if we change Q by less than γ, it still has good entropy).

(c) There is a partition H of F_L with $< 2^{(1/100)\delta n}$ atoms such that $\bigvee_0^n S^{-i}_{t_0} \tilde{P} \vee H \supset R$ on F_L. This follows from (9), (which says that in assigning the atoms in $\bigvee_0^{nL} S^i_{t_0/L}(Q \vee P')$ $\hat{P}'$-names, we used each $\hat{P}'$-name more than $2^{(1/100)\delta n}$ times (11) and (12) imply, therefore, that in assigning the R atoms $\bar{P}'$-names, under S_{t_0} of length n, we used each $\bar{P}'$-name no more than $2^{(1/100)\delta n}$ times).

(d) If i is an integer $< L$, then there is a partition H_i of $S^i_{t_0/L}F_L$ such that H_i has $< 2^{(2/100)\delta n}$ atoms and $\bigvee_0^n S^i_{t_0}\tilde{P} \vee H_i \supset S^i_{t_0/L} R$ on $S^i_{t_0/L} F_L$. We get (*d*) as follows: Because of (*c*) we can find $\bar{H}_i$ such that (i) $\bigvee_0^{nL} S^{-i}_{t_0/L}\tilde{P} \vee \bar{H}_i \supset S^i_{t_0/L} R$ and $\bar{H}_i$ has $< 2^{(1/100)\delta n}$ atoms. Because of (8a) (and because the $\tilde{P}$-name under $S_{t_0/L}$ is the same as the $\bar{P}'$-name under $S_{t_0/L}$ of some $\bar{x}$ in $\bar{X}_1$), we have an $\hat{H}_i$ having $< 2^{(1/100)\delta n}$ atoms, and such that (ii) $\bigvee_0^n S^{-i}_{t_0} \tilde{P} \vee \hat{H}_i \supset \bigvee_{-L}^{nL} S^{-i}_{t_0/L} \tilde{P}$ on $S^i_{t_0/L} F_L$ (i) and (ii) give (d).

(e) There is a partition H' of F having $< 2^{(3/100)\delta n}$ atoms such that $\bigvee_0^n S^i_{t_0} \tilde{P} \vee H' \supset \bigvee_0^L S^i_{t_0/L} R$. This follows immediately from (d) and (8h). Because

$$m(F) < \frac{1}{n}, \qquad E(H') < \frac{1}{n} \cdot \frac{3}{100}\,\delta n = \frac{3}{100}\,\delta,$$

(a) follows immediately from (e) and (b).

(*f*) Because of (a), we get conclusion [$\bar{4}$] of our lemma which says that $|E(\tilde{P}, S_{\bar{t}_0}) - E(\bar{P}, \bar{S}_{\bar{t}_0}) < \delta$. We can see this as follows: Because $\bar{P}$ generates under S_{t_0}, we have $E(\bar{P}, \bar{S}_{\bar{t}_0}) = \bar{t}_{0/t_0} E(\bar{P}, \bar{S}_{t_0})$. We also have $E(\tilde{P}, \bar{S}_{\bar{t}_0}) \geq \bar{t}_{0/t_0} E(\tilde{P}\, \bar{S}_{t_0})$. Since $E(\tilde{P}, \bar{S}_{\bar{t}_0}) \leq E(\bar{P}, \bar{S}_{\bar{t}_0})$, we get [$\bar{4}$].

(14) We will now check conclusion [$\bar{3}$]. Because of (5), it is enough to show that (a) $\bar{d}(\{S_t\tilde{P}\}_0^{\bar{N}}, \{\bar{S}_t\bar{P}'\}_0^{\bar{N}}) < (1/10)\delta$. Because of (11) and (12),

the $\tilde{P}$-name under $S_{t_0/L}$ of length nL of all the points $S_t(x)$ where $0 \leqslant t \leqslant t_{0/L}$ and $x \in Y$ is the $\bar{P}'$-name under $\bar{S}_{t_0/L}$ of length nL of some point $\bar{x}$ in $\bar{X}_1$. Because of (7a), the continuous $\tilde{P}$-name of length nt_0 of x differs from the continuous $\bar{P}'$-name of length nt_0 of $\bar{x}$ by less than $(1/100)\xi$ which is less than $(1/100)\bar{\delta}$. Because of (8b), the distribution of continuous $\bar{P}'$-names of length $\bar{N}$ in the continuous $\bar{P}'$-name of $\bar{x}$ (of length nt_0) is within $(1/100)\xi$ of the distribution of continuous $\bar{P}'$-names of length $\bar{N}$ in $\bar{X}$. We therefore have that the distribution of continuous $\tilde{P}$-name of length $\bar{N}$, in the $\tilde{P}$-name of x of length nt_0, is within $(1/10)\bar{\delta}$ of the distribution of continuous $\bar{P}'$-name of length $\bar{N}$ in $\bar{X}$. This gives (a).

(15) We will now check [$\bar{2}$]. Because of (5), it is enough to check that (a) $\bar{e}(\tilde{P}, \bar{P}') < (1/10)\bar{\delta}$. As in (14) we get that the continuous $\tilde{P}$-name (of length nt_0) of a point x in Y differs by less than $(1/100)\xi$ from the continuous $\bar{P}'$-name (of length nt_0) of some point x in $\bar{X}_1$. Because of (8b), we get that the distribution $\tilde{P}$ in the continuous $\tilde{P}$ of x differs by less than ξ from the distribution of $\bar{P}'$ in $\bar{X}$. Because of our choice of ξ in (6), we get (*a*).

(16) We will now check that $| \tilde{P} - P | < \epsilon$. (This is [$\bar{0}$]). (a) In (9) we assigned to each atom A of $\bigvee_0^{Ln} S^i_{t_0/L}(Q \vee P')$ which intersects X_2 a $\hat{P}'$-name, $\{\hat{\alpha}_i'\}_1^n$ under S_{t_0} that disagrees with the P'-name, $\{\hat{\alpha}_i'\}_1^n$, (under S_{t_0}) of the points, in A in less than $(1/100)n\epsilon$ places. Let A_1 be the atom of R corresponding to A (see (11a). Let $\bar{x}$ be a point in $\bar{X}_1$ whose $\bar{P}'$-name under $\bar{S}_{t_0}$ of length n is $\{\hat{\alpha}_i'\}_1^n$. Then (b) the continuous $\tilde{P}$-name of any point in A_1 will disagree with the continuous $\bar{P}'$-name of $\bar{x}$ by less than $(1/10)\xi$ because of (7a). But (c), the continuous $\bar{P}'$-name of $\bar{x}$ will differ from the continuous P'-name of any point in A by $<(1/10)\epsilon$ because $\{\hat{\alpha}_i'\}_1^n$ and $\{\hat{\alpha},'\}_1^n$ differ in $<(1/100)\ \epsilon n$ places [see (a)]; and because we can apply (2), since any point in A satisfies (8*e*) and $\bar{x}$ satisfies (8*b*). Because of (11b) and (7d), we have (d) the continuous P'-name (of length nt_0) of any point in A_1 differs by $<(1/100)\epsilon$ from the continuous P'-name of some point in A. Because of (*b*), (c), and (d), and because the atom of R corresponding to the A_1 covers ${}_2F_L$ (see (11c)), we get that $\tilde{P} - P' | < (1/5)\epsilon$. Because of (5), we get that that $| \tilde{P} - P | < \epsilon$.

(17) We will now check [1]. Because of (7c), all of the $\tilde{P}_i$ except the one (call it $\tilde{P}_m$) that correspond to the atoms of $\bar{P}'$ that were lumped together satisfy [1]. Because of (5) and 8b $m(\tilde{P}_m) < \delta$. We must also

check that $\tilde{P}_m$ is good. We could assume that the part of X not in the continuous gadget defined in (10) is in $\tilde{P}_m$. We could also change $\tilde{P}$ slightly by putting a small part around the top and bottom of the continuous gadget in $\tilde{P}_m$. This will insure that $\tilde{P}_m$ is good, and will not disturb any of the other conclusions of the lemma.

COROLLARY. *Under the same hypothesis on the main lemma we can conclude that there is a $\tilde{P}$ such that*

$$|\tilde{P} - P| < \epsilon \quad \text{and} \quad \bar{d}(\{S_t\tilde{P}\}_0^N, \{\bar{S}_t\bar{P}\}_0^N) = 0 \quad \text{for all } N.$$

Furthermore, $\tilde{P}$ will be good.

LEMMA. *Let S_t be a Bernoulli flow. Let P and Q be good partitions such that P generates under S_1 and that if we let X_Q be the σ-algebra generated by Q under S_t, then Q generates X_Q under S_1 (S_t acting on X_Q is automatically a Bernoulli flow by* [4]). *We also assume $E(P, S_1) = E(Q, S_1)$.*

Given ϵ, there is a Q_1 such that

(1) $(Q_1, S_t) \sim (Q, S_t)$ i.e., $\bar{d}(\{S_tQ_1\}_0^N, \{S_tQ\}_0^N) = 0$ for all N;

(2) $\bigvee_{-\infty}^{\infty} S_1^i Q_1 \overset{\epsilon}{\supset} P$;

(3) $|Q_1 - Q| < \epsilon$.

Proof. (1) Since P generates under S_1, we can choose K_1 so that

(a) $\bigvee_{-K_1}^{K_1} S_{1i} P \overset{\epsilon/10}{\supset} Q$.

Apply the corollary to the main lemma to S_t acting on X_Q to get t_0, N, and δ such that $t_0 = (1/M)$, M an integer, and if P' is a partition in X_Q satisfying

(b) dist P' = dist P;

(c) $\bar{d}(\{S_tP\}_0^N, \{S_tP'\}_0^N) < \delta$;

(d) $|E(P, S_{t_0}) - E(P', S_{t_0})| < \delta$;

then there exists a P_1 in X_Q such that

(e) $|P_1 - P| < \epsilon/30K_1$, and

(f) $(P_1, S_t) \sim (P, S_t)$.[3]

Choose $\bar{\epsilon}$ such that if $\bar{\epsilon} < \delta$, $\bar{\epsilon} < \epsilon$, and if Q' is in X_Q and $|Q' - Q| < \bar{\epsilon}$, then $|E(Q', S_{t_0}) - E(Q, S_{t_0})| < (\delta/100)$. Choose $K_2 > K_1$ so that

(g) $$\bigvee_{-K_2}^{K_2} S_{t_0} P \overset{\bar{\epsilon}/10}{\supset} Q.$$

Choose $\bar{N}$ so that $t_0 K_2 < (\bar{\epsilon}/100)\bar{N}$ and $N < (\bar{\epsilon}/100)\bar{N}$.

(2) Apply Lemma 1 to form a continuous gadget on X_1 of height $\bar{N}$ where $m(X - X_1) < \bar{\epsilon}/100$. Let Y_1 be its base. Change Q to $\hat{Q}$ in X_Q so that points in Y_1 have only a finite number of continuous $\hat{Q}$-names of length $\bar{N}$. Do this so that

(a) $|Q - \hat{Q}| < (\epsilon \cdot \bar{\epsilon}/10)$ and $|E(Q, S_{t_0}) - E(\hat{Q}, S_{t_0})| < (1/100)\bar{\delta}$ (note that $E(Q, S_{t_0}) = E(P, S_{t_0}) = t_0$). Let $\mathscr{J}_1$ be the continuous gadget obtained by partitioning X_1 by $\hat{Q}$ and P. Choose P' in X_Q so that if $\mathscr{J}_2$ is the gadget obtained by partitioning X_1 by $\hat{Q}$ and P', then

(b) $\mathscr{J}_1$ and $\mathscr{J}_2$ are isomorphic

Extend the definition of P' to $X - X_1$ so that dist P = dist P'.

(3) (1a) implies that there is an operator L (see the definition on p. 0) such that

(a) $$|L_P - Q| < \epsilon/10.$$

Therefore, (b) $|L_P - \hat{Q}| < (2\epsilon/10)$. Because $\mathscr{J}_1$ and $\mathscr{J}_2$ are isomorphic,

(c) $|L_{P'} - \hat{Q}| < (3\epsilon/10)$. (1) ($g$) implies

$$\bigvee_{-K_2}^{K_2} S_{t_0} P \overset{2\bar{\epsilon}/10}{\supset} \hat{Q}.$$

[3] We can apply the lemma to X_Q because we never used that the σ-algebra of X separated points. We only need that $L1$ holds (to get F and F_L measurable), and this follows from the existence of a good partition in X_Q.

Because $\mathscr{J}_1$ and $\mathscr{J}_2$ are isomorphic we have (d)

$$\bigvee_{-K_2}^{K_2} S_{t_0} P' \overset{3\bar{\epsilon}/10}{\supset} \hat{Q}.$$

Because of (d) and our choice of $\bar{\epsilon}$ we get that P' satisfies (1) (d). Because $\mathscr{J}_1 \sim \mathscr{J}_2$ and $\bar{N}$ was large compared to N, we see that P' satisfies (1c). Therefore, we can find P_1 such that

(e) P_1 is in X_Q ;

(B) $(P_1, S_t) \sim (P, S_t)$;

(g) $| P_1 - P < \epsilon/30\text{K}_1$;

(h) $| L_{P_1} - Q | < \frac{1}{5}\epsilon$.

(4) Now choose $K_3 > K_2$ so that

(a) $$\bigvee_{-K_4}^{K_3} S_1{}^i Q \overset{\epsilon/10}{\supset} P_1 .$$

Apply the corollary to the main lemma to get t_1, N_1, and δ_1 where $t_1 = (1/M_1)$, M_1 an integer, and if Q^* satisfies

(b) $\text{dist}\, Q^* = \text{dist}\, Q$,

(c) $\bar{d}(\{S_t Q^*\}_0^{N_1}, \{S_t Q\}_0^{N_1}) < \delta_1$, and

(d) $| E(Q^*, S_{t_1}) - E(Q, S_{t_1})| < \delta_1$, then there is a Q_1 such that

(e) $(Q_1 S_t) \sim (Q, S_t)$, $| Q_1 - Q^* | < \epsilon/30\text{K}_3$. Now choose $\tilde{\epsilon}$, $\tilde{\epsilon} < \delta_1$, $\tilde{\epsilon} < \epsilon$ such that if $|{}_1P - P| < \tilde{\epsilon}$, then $| E({}_1P, S_{t_1}) - E(P, S_{t_1}) | < \delta_1$. Now choose $K_4 > K_3$ so that

(f) $$\bigvee_{-K_4}^{K_4} S_{t_1}^i Q \overset{\tilde{\epsilon}/10}{\supset} P_1 .$$

Choose $\bar{N}_1$ so that $t_1 K_4 < (\tilde{\epsilon}/100)\bar{N}_1$ and $N_1 < (\bar{\epsilon}/100)\bar{N}_1$.

(5) We will now apply Lemma 1 and get a continuous gadget of height $\bar{N}_1$ consisting of X_2 partitioned by P. We can assume that

(a) $m(X - X_2) < \tilde{\epsilon}/100$. We can assume that the distribution of continuous P-names of length $\bar{N}_1$ in the base Y_2 of the gadget is as close as we want to the distribution of continuous P-names of length $\bar{N}_1$ in X. (To see this, let ${}_\beta Y_2 = \bigcup_{|t|<\beta} S_t$, Y_2. Now by the mixing we can assume the distribution of names in ${}_\beta Y_2$ is very good. If β is small enough, this is close to the distribution of P-names in Y_2 since P is good).

We also have that the distribution of P_1-names of length $\bar{N}_1$ in Y_2 is as close as we want to the distribution of P_1-names in X. We can then find partitions, $\tilde{P}$ and $\tilde{P}_1$, of X_2 so that the points in Y_2 have only a finite number of different $\tilde{P}$ — (or $\tilde{P}_1$ —) names of length $\bar{N}_1$ and

$$\text{(b)} \quad |\tilde{P} - P| < (\tilde{\epsilon}/100) \cdot (1/K_1) \quad \text{and} \quad |\tilde{P}_1 - P_1| < (\tilde{\epsilon}/100) \cdot (1/K_1)$$

and the gadgets formed by partitioning X_2 by $\tilde{P}$ and $\tilde{P}_1$ are isomorphic. Now pick $\underset{\sim}{Q}^*$ so that the gadget $\tilde{\mathscr{J}}$ formed by partitioning X_2 by $\tilde{P}$ and $\underset{\sim}{Q}^*$ is isomorphic to the gadget $\tilde{\mathscr{J}}_1$ formed by partitioning X_2 by $\tilde{P}_1$ and $\underset{\sim}{Q}$. Define $\underset{\sim}{Q}^*$ on $X - Y_2$ so that dist $\underset{\sim}{Q}^* =$ dist $\underset{\sim}{Q}$.

$$\text{(6)} \quad \text{(4f) and (5b) imply (a)} \quad \bigvee_{-K_4}^{K_4} S_{t_1}^i \underset{\sim}{Q} \overset{2\tilde{\epsilon}/10}{\supset} \tilde{P}_1 .$$

Because $\tilde{\mathscr{J}} \sim \tilde{\mathscr{J}}_1$ we get

$$\text{(b)} \quad \bigvee_{-K_4}^{K_4} S_{t_1}^i \underset{\sim}{Q}^* \overset{3\tilde{\epsilon}/10}{\supset} \tilde{P}$$

and hence

$$\text{(c)} \quad \bigvee_{-K_4}^{K_4} S_{t_1}^i \underset{\sim}{Q}^* \overset{\tilde{\epsilon}}{\supset} P.$$

Because of (c) and our choice of $\tilde{\epsilon}$, $\underset{\sim}{Q}^*$ satisfies (4d). Because of our choice of $\bar{N}_1$ and the measure of $X - X_2$, $\underset{\sim}{Q}^*$ satisfies (4c). Therefore, we can find $\underset{\sim}{Q}_1$, such that

$$\text{(d)} \qquad (\underset{\sim}{Q}_1, S_t) \sim (\underset{\sim}{Q}, S_t)$$

and (e) $|\underset{\sim}{Q}_1 - \underset{\sim}{Q}^*| < \epsilon/30\mathrm{K}_3$. Because of (4a) and (5b) we get

$$\text{(f)} \quad \bigvee_{-K_3}^{K_3} S_1{}^i \underset{\sim}{Q} \overset{\epsilon/10}{\supset} \tilde{P}_1 .$$ Since $\mathscr{G} \sim \mathscr{G}_1$ we get

$$\text{(g)} \quad \bigvee_{-K_3}^{K_3} S_1{}^i \underset{\sim}{Q}^* \overset{2\epsilon/10}{\supset} \tilde{P}$$ and hence (h) $$\bigvee_{-K_3}^{K_3} S_1{}^i \underset{\sim}{Q}^* \overset{\epsilon}{\supset} P,$$ and by (e) we get

$$\text{(i)} \quad \bigvee_{-K_3}^{K_3} S_1{}^i \underset{\sim}{Q}_1 \overset{\epsilon}{\supset} P.$$

Because of (3h) and (5b,j) $|L_{\bar{P}_1} - Q| < (5\epsilon/10)$. Because $\tilde{\mathscr{J}} \sim \tilde{\mathscr{J}}_1$,

(k) $|L_{\bar{P}} - Q^*| < (6\epsilon/10)$ and, by (6) (e), (l) $|L_{\bar{P}} - Q_1| < (7\epsilon/10)$. Thus

(m) $|L_P - Q_1| < (8\epsilon/10)$. But (3a) says $|L_P - Q| < (\epsilon/10)$. Therefore, by (m),

$$\text{(n)} \qquad |Q - Q_1| < \epsilon.$$

Our lemma follows from (n), (i), and (d).

Repeated application of this lemma (see the argument at the end of [1]) yields:

THEOREM. *Let S_t and $\bar{S}_t$ be Bernoulli flows and $E(S_1) = E(\bar{S}_1)$. Let P and R be good partitions such that P generates under S_1 and R generates under $\bar{S}_1$. Then we can find a Q such that Q generates under S_1 and $(Q, S_t) \sim (R, \bar{S}_t)$.*

The above theorem easily implies that S_t and $\bar{S}_t$ are isomorphic by Lemma 0.

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