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Blocks of cyclotomic Hecke algebras

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Abstract

This paper classifies the blocks of the cyclotomic Hecke algebras of type $G(r, 1, n)$ over an arbitrary field. Rather than working with the Hecke algebras directly we work instead with the cyclotomic Schur algebras. The advantage of these algebras is that the cyclotomic Jantzen sum formula gives an easy combinatorial characterization of the blocks of the cyclotomic Schur algebras. We obtain an explicit description of the blocks by analyzing the combinatorics of 'Jantzen equivalence.'

We remark that a proof of the classification of the blocks of the cyclotomic Hecke algebras was announced in 1999. Unfortunately, Cox has discovered that this previous proof is incomplete.

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1. Introduction

The Ariki–Koike algebras are the cyclotomic Hecke algebras of type $G(r, 1, n)$. These algebras first appeared in the work of Cherednik [9] and they were first systematically studied by Ariki and Koike [3]. Independently, and at about the same time, Broué and Malle [6] generalized the definition of Iwahori–Hecke algebras to attach a Hecke algebra to each complex reflection group. The cyclotomic Hecke algebras are central to the conjectures of Broué, Malle

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and Michel [5] which grew out of an attempt to understand Broué's abelian defect group conjecture for the finite groups of Lie type.

The Ariki–Koike algebras arise most naturally as 'cyclotomic quotients' of the (extended) affine Hecke algebras of type A. To make this explicit, let $\mathbb{F}$ be a field and let $\mathscr{H}_n^{\mathrm{aff}}$ be the affine Hecke algebra of type A_n. Using the Bernstein presentation, $\mathscr{H}_n^{\mathrm{aff}}$ can be written as a twisted tensor product $\mathscr{H}_q(\mathfrak{S}_n) \otimes \mathbb{F}[X_1^{\pm}, \dots, X_n^{\pm}]$ of the Iwahori–Hecke algebra $\mathscr{H}_q(\mathfrak{S}_n)$ of the symmetric group and the Laurent polynomial ring $\mathbb{F}[X_1^{\pm}, \dots, X_n^{\pm}]$. The Ariki–Koike algebra is then the quotient algebra $\mathscr{H}_{r,n}(q, \mathbf{Q}) = \mathscr{H}_n^{\mathrm{aff}} / \langle (X_1 - Q_1) \dots (X_1 - Q_r) \rangle$, where $\mathbf{Q} = (Q_1, \dots, Q_r) \in (\mathbb{F}^\times)^r$.

As $\mathscr{H}_{r,n}(q, \mathbf{Q})$ is a quotient of $\mathscr{H}_n^{\mathrm{aff}}$, every irreducible $\mathscr{H}_{r,n}(q, \mathbf{Q})$-module can be considered as an irreducible $\mathscr{H}_n^{\mathrm{aff}}$-module. Conversely, by quotienting out by the characteristic polynomial of X_1, every irreducible $\mathscr{H}_n^{\mathrm{aff}}$-module is an irreducible module for some Ariki–Koike algebra. The deep results of Ariki [2] and Grojnowski [17] show that the module categories of the affine Hecke algebras and the Ariki–Koike algebras are intimately intertwined. The main result of this paper shows that, combinatorially, the blocks of these algebras are the same.

If A is an algebra then two simple A-modules D and D' belong to the same *block* if there exist simple A-modules $D = D_1, D_2, \dots, D_k = D'$ such that either $\operatorname{Ext}_A^1(D_i, D_{i+1}) \neq 0$ or $\operatorname{Ext}_A^1(D_{i+1}, D_i) \neq 0$, for $1 \leqslant i < k$. More generally, two A-modules M and N belong to the same block if all of their composition factors belong to the same block.

The natural surjection $\mathscr{H}_n^{\mathrm{aff}} \to \mathscr{H}_{r,n}(q, \mathbf{Q})$ shows that if D and D' are in the same block as $\mathscr{H}_{r,n}(q, \mathbf{Q})$-modules then they are in the same block as $\mathscr{H}_n^{\mathrm{aff}}$-modules. The main result of this paper shows that the blocks of the Ariki–Koike algebras are determined by the affine Hecke algebra.

Theorem A. *Suppose that $\mathbb{F}$ is an algebraically closed field and that $q \neq 1$. Let D and D' be irreducible modules for the Ariki–Koike algebra $\mathscr{H}_{r,n}(q, \mathbf{Q})$. Then D and D' belong to the same block as $\mathscr{H}_{r,n}(q, \mathbf{Q})$-modules if and only if they belong to the same block as $\mathscr{H}_n^{\mathrm{aff}}$-modules.*

We also classify the blocks of the Ariki–Koike algebras when $q = 1$ and when some of the parameters $Q_1, \dots, Q_r$ are zero.

By a well-known theorem of Bernstein [22, Proposition 3.11], the centre of $\mathscr{H}_n^{\mathrm{aff}}$ is the set $\mathbb{F}[X_1^{\pm}, \dots, X_n^{\pm}]^{\mathfrak{S}_n}$ of symmetric Laurent polynomials in $X_1, \dots, X_n$. Consequently, the central characters of $\mathscr{H}_n^{\mathrm{aff}}$ are naturally indexed by $\mathfrak{S}_n$-orbits of $(F^\times)^n$. This observation gives a natural combinatorial criterion for two $\mathscr{H}_{r,n}(q, \mathbf{Q})$-modules to belong to the same block (see Theorem 2.11 for the precise statement), and it is this statement that we actually prove. We prove Theorem A by first showing that blocks of $\mathscr{H}_{r,n}(q, \mathbf{Q})$ are the 'same' as the blocks of the associated cyclotomic q-Schur algebra. This allows us to use a new characterization of the blocks in terms of 'Jantzen coefficients' (Proposition 2.9).

Observe that Theorem A is equivalent to the following property of the blocks of $\mathscr{H}_n^{\mathrm{aff}}$.

Corollary. *Suppose that $q \neq 1$ and let D and D' be two simple $\mathscr{H}_{r,n}(q, \mathbf{Q})$-modules. Then D and D' belong to the same block as $\mathscr{H}_n^{\mathrm{aff}}$-modules if and only if there exist simple $\mathscr{H}_{r,n}(q, \mathbf{Q})$-modules $D = D_1, D_2, \dots, D_k = D'$ such that either*

$$\operatorname{Ext}^1_{\mathscr{H}_n^{\mathrm{aff}}}(D_i, D_{i+1}) \neq 0 \quad \text{or} \quad \operatorname{Ext}^1_{\mathscr{H}_n^{\mathrm{aff}}}(D_{i+1}, D_i) \neq 0,$$

for $1 \leqslant i < k$.

In 1999 Grojnowski [18] announced a proof of Theorem A. Using an ingenious argument, what Grojnowski actually proves is that

$$\operatorname{Ext}^1_{\mathscr{H}_n^{\mathrm{aff}}}(D, D') = \operatorname{Ext}^1_{\mathscr{H}_{r,n}(q,\mathbf{Q})}(D, D')$$

whenever $D \neq D'$ are simple $\mathscr{H}_{r,n}(q,\mathbf{Q})$-modules. Unfortunately, as Anton Cox [10] has pointed out, this is not enough to classify the blocks of the Ariki–Koike algebras. For example, it could happen that there are no $\mathscr{H}_n^{\mathrm{aff}}$-module extensions between different $\mathscr{H}_{r,n}(q,\mathbf{Q})$-modules which belong to the same block as $\mathscr{H}_n^{\mathrm{aff}}$-modules. We note that Grojnowski's result does not follow from Theorem A.

Lusztig [22] introduced a graded, or degenerate, Hecke algebra for each affine Hecke algebra. Brundan [8] has shown that the centre of the degenerate affine Hecke algebra maps onto the centre of the degenerate cyclotomic Hecke algebras. This gives a classification of the blocks of the degenerate cyclotomic and affine Hecke algebras analogous to our Theorem A. It should be possible to use the arguments from this paper to classify the blocks of the degenerate cyclotomic Hecke algebras of type $G(r,1,n)$ and the associated degenerate cyclotomic Schur algebras. All of the combinatorics that we use goes through without change, however, it is necessary to check that arguments of [21] can be adapted to prove a sum formula for the Jantzen filtrations of the degenerate cyclotomic Schur algebras. This should be routine (cf. [4, §6]), however, we have not checked the details.

The outline of this paper is as follows. In the next section we introduce the Ariki–Koike algebras and the cyclotomic q-Schur algebras. Using the representation theory of these two algebras, we reduce the proof of Theorem A to a purely combinatorial problem of showing that two equivalence relations on the set of multipartitions coincide (Theorem 2.11). The first of these equivalence relations comes from the cyclotomic Jantzen sum formula [21]. The second equivalence relation is equivalent to the combinatorial criterion which classifies the central characters the affine Hecke algebras. In Section 3 we develop the combinatorial machinery needed to show that our two equivalence relations on the set of multipartitions coincide when $q \neq 1$ and when the parameters $Q_1, \dots, Q_r$ are non-zero. Here we are greatly aided by the recent work of Fayers [14,15] on 'core blocks' of Ariki–Koike algebras. Finally, in Section 4 we consider the blocks of the Ariki–Koike algebras with 'exceptional' parameters; that is, those algebras with $q = 1$ or with some of the parameters $Q_1, \dots, Q_r$ being zero. Quite surprisingly, the algebras with exceptional parameters have only a single block (unless $q = 1$ and $r = 1$).

2. Cyclotomic Hecke algebras and Schur algebras

This section begins by introducing the cyclotomic Hecke algebras and Schur algebras. We then reduce the proof of Theorem A to a purely combinatorial statement which amounts to showing that two equivalence relations on the set of multipartitions coincide.

2.1. Ariki–Koike algebras

Let $\mathbb{F}$ be a field of characteristic $p \in \{2, 3, \dots\} \cup \{\infty\}$ and fix positive integers n and r. Suppose that $q, Q_1, \dots Q_r$ are elements of $\mathbb{F}$ such that q is invertible and let $\mathbf{Q} = (Q_1, \dots, Q_r)$.

The *Ariki–Koike algebra* $\mathcal{H}_{r,n} = \mathcal{H}_{r,n}(q, \mathbf{Q})$ is the unital associative $\mathbb{F}$-algebra with generators $T_0, T_1, \dots, T_{n-1}$ and relations

$$\begin{aligned}
(T_i + q)(T_i - 1) &= 0, & 1 \leqslant i \leqslant n-1,\\
(T_0 - Q_1)\dots(T_0 - Q_r) &= 0, &\\
T_i T_j &= T_j T_i, & 0 \leqslant i < j-1 \leqslant n-2,\\
T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}, & 1 \leqslant i \leqslant n-2,\\
T_0 T_1 T_0 T_1 &= T_1 T_0 T_1 T_0. &
\end{aligned}$$

Define $e \geqslant 2$ to be minimal such that $1 + q + \dots + q^{e-1} = 0 \in \mathbb{F}$. Then $e \in \{2, 3, \dots\} \cup \{\infty\}$. Note that $e = p$ if and only if $q = 1$. If $e \neq p$ and p is finite then p does not divide e.

Recall that a partition $\lambda = (\lambda_1, \lambda_2, \dots)$ of n is a weakly decreasing sequence of non-negative integers which sum to $|\lambda| = n$. An r-multipartition of n, or more simply a multipartition, is an ordered r-tuple $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)})$ of partitions with $|\boldsymbol{\lambda}| = |\lambda^{(1)}| + \dots + |\lambda^{(r)}| = n$. Let $\Lambda^+_{r,n}$ be the set of multipartitions of n. We regard a partition as a multipartition with one component, so any subsequent definition concerning multipartitions specializes to a corresponding definition for partitions.

The set of multipartitions is naturally ordered by *dominance* where $\boldsymbol{\lambda} \trianglerighteq \boldsymbol{\mu}$ if

$$\sum_{t=1}^{s-1} \left|\lambda^{(t)}\right| + \sum_{j=1}^{i} \lambda_j^{(s)} \geqslant \sum_{t=1}^{s-1} \left|\mu^{(t)}\right| + \sum_{j=1}^{i} \mu_j^{(s)}$$

for $s = 1, 2, \dots, r$ and all $i \geqslant 1$. We write $\boldsymbol{\lambda} \vartriangleright \boldsymbol{\mu}$ if $\boldsymbol{\lambda} \trianglerighteq \boldsymbol{\mu}$ and $\boldsymbol{\lambda} \neq \boldsymbol{\mu}$.

The Ariki–Koike algebra $\mathcal{H}_{r,n}$ is a cellular algebra [12,16]. The cell modules of $\mathcal{H}_{r,n}$ are indexed by the multipartitions of n. The cell module indexed by the multipartition $\boldsymbol{\lambda}$ is the *Specht module* $S(\boldsymbol{\lambda})$. By the theory of cellular algebras [16,23], there is an $\mathcal{H}_{r,n}$-invariant bilinear form $\langle , \rangle_{\boldsymbol{\lambda}}$ on the Specht module $S(\boldsymbol{\lambda})$, so the radical $\operatorname{rad} S(\boldsymbol{\lambda}) = \{x \in S(\boldsymbol{\lambda}) \mid \langle x, y\rangle_{\boldsymbol{\lambda}} = 0 \text{ for all } y \in S(\boldsymbol{\lambda})\}$ is an $\mathcal{H}_{r,n}$-submodule of $S(\boldsymbol{\lambda})$. Set $D(\boldsymbol{\lambda}) = S(\boldsymbol{\lambda}) / \operatorname{rad} S(\boldsymbol{\lambda})$. Then the non-zero $D(\boldsymbol{\lambda})$ give a complete set of pairwise non-isomorphic simple $\mathcal{H}_{r,n}$-modules.

The theory of cellular algebras gives us the following fact which is vital for this paper because it allows us work with Specht modules rather than with the simple $\mathcal{H}_{r,n}$-modules.

2.1. Lemma. *(See Graham–Lehrer [16, 3.9.8], [23, Corollary 2.2].) Suppose that $\boldsymbol{\lambda}$ is a multipartition. Then all of the composition factors of $S(\boldsymbol{\lambda})$ belong to the same block.*

Thus we can talk of the block of $\mathcal{H}_{r,n}$ which contains the Specht module $S(\boldsymbol{\lambda})$.

2.2. Cyclotomic q-Schur algebras

Rather than working with Specht modules to classify the blocks we want to work with Weyl modules. To this end let $\{L_1^{a_1} \dots L_n^{a_n} T_w \mid 0 \leqslant a_i < r \text{ and } w \in \mathfrak{S}_n\}$ be the Ariki–Koike basis of $\mathcal{H}_{r,n}$ [3, Proposition 3.4]. That is, $L_1 = T_0$ and $L_{i+1} = q^{1-i} T_i L_i T_i$, for $1 \leqslant i < n$, and if $w \in \mathfrak{S}_n$

then $T_w = T_{i_1} \dots T_{i_k}$ whenever $w = (i_1, i_1+1)\dots(i_k, i_k+1)$ with k minimal (so this is a reduced expression of w). For each multipartition $\boldsymbol{\lambda}$ define

$$m_{\boldsymbol{\lambda}} = \prod_{s=1}^{r} \prod_{k=1}^{|\lambda^{(1)}|+\dots+|\lambda^{(s-1)}|} (L_k - Q_s) \cdot \sum_{w \in \mathfrak{S}_{\boldsymbol{\lambda}}} T_w,$$

where $\mathfrak{S}_{\boldsymbol{\lambda}} = \mathfrak{S}_{\lambda^{(1)}} \times \dots \times \mathfrak{S}_{\lambda^{(r)}}$ is the Young subgroup of $\mathfrak{S}_n$ associated to $\boldsymbol{\lambda}$. The *cyclotomic q-Schur algebra* is the endomorphism algebra

$$\mathcal{S}_{r,n} = \mathcal{S}_{r,n}(q, \mathbf{Q}) = \operatorname{End}_{\mathcal{H}_{r,n}}\Bigl(\bigoplus_{\boldsymbol{\lambda} \in \Lambda_{r,n}^+} m_{\boldsymbol{\lambda}} \mathcal{H}_{r,n}\Bigr).$$

We remark that this variant of the cyclotomic q-Schur algebra is Morita equivalent to one of the algebras introduced in [12]. The representation theory of $\mathcal{S}_{r,n}$ is discussed in [24].

The cyclotomic q-Schur algebra $\mathcal{S}_{r,n}$ is a quasi-hereditary cellular algebra. The cell modules of $\mathcal{S}_{r,n}$ are the *Weyl modules* $\Delta(\boldsymbol{\lambda})$, for $\boldsymbol{\lambda} \in \Lambda_{r,n}^+$. For each $\boldsymbol{\lambda} \in \Lambda_{r,n}^+$, there is a non-zero simple module $L(\boldsymbol{\lambda}) = \Delta(\boldsymbol{\lambda})/\operatorname{rad}\Delta(\boldsymbol{\lambda})$. Just as with Lemma 2.1, the theory of cellular algebras tells us the following.

2.2. Lemma. *(See Graham–Lehrer [16, 3.9.8], [23, Corollary 2.2].) Suppose that $\boldsymbol{\lambda}$ is a multipartition. Then all of the composition factors of $\Delta(\boldsymbol{\lambda})$ belong to the same block.*

The next result shows that in order to classify the blocks of $\mathcal{H}_{r,n}$ it is enough to consider the blocks of $\mathcal{S}_{r,n}$. As we will see, this is an easy consequence of the double centralizer theory.

Let A be a finite dimensional algebra over a field. Then A decomposes in a unique way as a direct sum of indecomposable two-sided ideals $\mathcal{H} = B_1 \oplus \dots \oplus B_d$. Recall that two simple A-modules D and D' are in the same block if there exist simple modules $D_1 = D, D_2, \dots, D_k = D'$ such that $\operatorname{Ext}^1_A(D_i, D_{i+1}) \neq 0$ or $\operatorname{Ext}^1_A(D_{i+1}, D_i) \neq 0$, for $1 \leqslant i < k$. As Ext^1_A classifies non-trivial extensions, it follows that two simple modules D and D' belong to the same block if and only if D and D' are both composition factors of B_j or, equivalently, that $D = DB_j$ and $D' = D'B_j$, for some j. Abusing terminology, we call the indecomposable subalgebras $B_1, \dots, B_d$ the blocks of A and we say that an A-module M belongs to the block B_j if $MB_j = M$. Using an idempotent argument (cf. [11, Theorem 56.12]) it is now easy to show that two indecomposable A-modules P and Q belong to the same block if and only if they are in the same *linkage class*; that is, there exist indecomposable modules $P_1 = P, \dots, P_l = Q$ such that P_i and P_{i+1} have a common irreducible composition factor, for $i = 1, \dots, l-1$.

By Lemma 2.1 and the last paragraph, two Specht modules $S(\boldsymbol{\lambda})$ and $S(\boldsymbol{\mu})$ belong to the same block if and only if they belong to the same linkage class; that is, there exist multipartitions $\boldsymbol{\lambda}_1 = \boldsymbol{\lambda}, \dots, \boldsymbol{\lambda}_k = \boldsymbol{\mu}$ such that $S(\boldsymbol{\lambda}_i)$ and $S(\boldsymbol{\lambda}_{i+1})$ have a common composition factor, for $1 \leqslant i < k$. Similarly, two Weyl modules belong to the same block if and only if they are in the same linkage class. We will use this characterization of the blocks of $\mathcal{H}$ and $\mathcal{S}_{r,n}$ below without a mention.

2.3. Proposition. *Let $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ be multipartitions of n. Then $S(\boldsymbol{\lambda})$ and $S(\boldsymbol{\mu})$ are in the same block as $\mathcal{H}_{r,n}$-modules if and only if $\Delta(\boldsymbol{\lambda})$ and $\Delta(\boldsymbol{\mu})$ are in the same block as $\mathcal{S}_{r,n}$-modules.*

Proof. Suppose first that $S(\boldsymbol{\lambda})$ and $S(\boldsymbol{\mu})$ are in the same block. By Lemma 2.1 all of the composition factors of $S(\boldsymbol{\lambda})$ belong to the same block. Therefore, by the remarks above, it is enough to consider the case when $D(\boldsymbol{\mu}) \neq 0$ and $D(\boldsymbol{\mu})$ is a composition factor of $S(\boldsymbol{\lambda})$. By a standard Schur functor argument [21, Proposition 2.17], $[\Delta(\boldsymbol{\lambda}) : L(\boldsymbol{\mu})] = [S(\boldsymbol{\lambda}) : D(\boldsymbol{\mu})] \neq 0$. Therefore, $\Delta(\boldsymbol{\lambda})$ and $\Delta(\boldsymbol{\mu})$ are in the same block. Note that this implies that $\mathcal{S}_{r,n}$ cannot have more blocks (that is, indecomposable subalgebras) than $\mathcal{H}_{r,n}$.

To prove the converse let $M = \bigoplus_{\boldsymbol{\lambda} \in \Lambda^+_{r,n}} m_{\boldsymbol{\lambda}} \mathcal{H}_{r,n}$ and suppose that $\mathcal{H}_{r,n} = B_1 \oplus \cdots \oplus B_k$ is the unique decomposition of $\mathcal{H}_{r,n}$ into a direct sum of indecomposable subalgebras. Then

$$M = M\mathcal{H}_{r,n} = MB_1 + \cdots + MB_k.$$

In fact, this sum is direct because, by definition, $MB_i \cap MB_j = \emptyset$ if $i \neq j$, and $MB_i \neq 0$ since $\mathcal{H}_{r,n}$ is a submodule of M. Therefore,

$$\begin{aligned}\mathcal{S}_{r,n} &= \operatorname{End}_{\mathcal{H}_{r,n}}(M) = \operatorname{End}_{\mathcal{H}_{r,n}}(MB_1 \oplus \cdots \oplus MB_k)\\ &= \bigoplus_{1 \leqslant i,j \leqslant k} \operatorname{Hom}_{\mathcal{H}_{r,n}}(MB_i, MB_j) = \bigoplus_{i=1}^{k} \operatorname{End}_{\mathcal{H}_{r,n}}(MB_i),\end{aligned}$$

where the last equality follows because B_i and B_j have no common irreducible constituents if $i \neq j$. Consequently, $\mathcal{S}_{r,n}$ has at least as many blocks as $\mathcal{H}_{r,n}$.

Combining the last two paragraphs proves the proposition. □

Thus, to prove Theorem A it suffices to determine when two Weyl modules are in the same block. The advantage of working with Weyl modules is shown in Lemma 2.4 below. Before we can state this result we need some notation.

If A is an algebra let $K_0(A)$ be the Grothendieck group of finite dimensional A-modules and if M is an A-module let $[M]$ be its image in $K_0(A)$. In particular, the Grothendieck group $K_0(\mathcal{S}_{r,n})$ of $\mathcal{S}_{r,n}$ is the free $\mathbb{Z}$-module with basis $\{[L(\boldsymbol{\lambda})] \mid \boldsymbol{\lambda} \in \Lambda^+_{r,n}\}$. The images $\{[\Delta(\boldsymbol{\lambda})] \mid \boldsymbol{\lambda} \in \Lambda^+_{r,n}\}$ of the Weyl modules give a second basis of $K_0(\mathcal{S}_{r,n})$ since $[\Delta(\boldsymbol{\lambda}) : L(\boldsymbol{\lambda})] = 1$ and $[\Delta(\boldsymbol{\lambda}) : L(\boldsymbol{\mu})] > 0$ only if $\boldsymbol{\lambda} \trianglerighteq \boldsymbol{\mu}$, for all $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \Lambda^+_{r,n}$ (see [16, Cor. 4.17]). Hence, we have the following.

2.4. Lemma. *Suppose that $a_{\boldsymbol{\lambda}} \in \mathbb{Z}$. Then $\sum_{\boldsymbol{\lambda}} a_{\boldsymbol{\lambda}}[\Delta(\boldsymbol{\lambda})] = 0$ in $K_0(\mathcal{S}_{r,n})$ if and only if $a_{\boldsymbol{\lambda}} = 0$ for all $\boldsymbol{\lambda} \in \Lambda^+_{r,n}$.*

Note that, in general, there can exist non-zero integers $a_{\boldsymbol{\lambda}} \in \mathbb{Z}$ such that $\sum_{\boldsymbol{\lambda}} a_{\boldsymbol{\lambda}}[S(\boldsymbol{\lambda})] = 0$. This follows because $K_0(\mathcal{H}_{r,n})$ is a free $\mathbb{Z}$-module of rank $L = \#\{\boldsymbol{\lambda} \in \Lambda^+_{r,n} \mid D(\boldsymbol{\lambda}) \neq 0\}$ and $L = \#\Lambda^+_{r,n}$ (if and) only if $\mathcal{H}_{r,n}$ is semisimple.

2.3. The cyclotomic Jantzen sum formula

The next step is to recall (a special case of) the machinery of the cyclotomic Jantzen sum formula [21]. Let t be an indeterminate over $\mathbb{F}$ and let $\mathcal{O} = \mathbb{F}[t, t^{-1}]_\pi$ be the localization of $\mathbb{F}[t, t^{-1}]$ at the prime ideal $\pi = \langle t - 1 \rangle$. Let $\mathcal{S}_\mathcal{O} = \mathcal{S}_\mathcal{O}(qt, \mathbf{X})$ be the cyclotomic Schur algebra over $\mathcal{O}$ with parameters qt and $\mathbf{X} = (X_1, \ldots, X_r)$ where

$$X_a = \begin{cases} Q_a t^{na}, & \text{if } Q_a \neq 0,\\ (t-1)t^{na}, & \text{if } Q_a = 0.\end{cases}$$

Consider $\mathbb{F}$ as an $\mathcal{O}$-module by letting t act on $\mathbb{F}$ as multiplication by 1. Then $\mathcal{S}_{r,n} \cong \mathcal{S}_{\mathcal{O}} \otimes_{\mathcal{O}} \mathbb{F}$, since $\mathcal{S}_{\mathcal{O}}$ is free as an $\mathcal{O}$-module by [12, Theorem 6.6]. The algebra $\mathcal{S}_{\mathcal{O}} \otimes_{\mathcal{O}} \mathbb{F}(t)$ is split semisimple by Schur–Weyl duality [24, Theorem 5.3] and Ariki's criterion for the semisimplicity for $\mathcal{H}_{r,n}$ [1]. Thus we are in the general setting considered in [21, §4].

Let ν_π be the π-adic evaluation map on $\mathcal{O}^\times$; thus, $\nu_\pi(f(t)) = k$ if $k \geqslant 0$ is maximal such that $(t-1)^k$ divides $f(t) \in \mathbb{F}[t, t^{-1}]$. Let $\Delta_{\mathcal{O}}(\boldsymbol{\lambda})$ be the Weyl module of $\mathcal{S}_{\mathcal{O}}$ indexed by the multipartition $\boldsymbol{\lambda} \in \Lambda^+_{r,n}$. Recall that $\Delta_{\mathcal{O}}(\boldsymbol{\lambda})$ carries a bilinear form $\langle,\rangle_{\boldsymbol{\lambda}}$ by the general theory of cellular algebras. For each integer $i \geqslant 0$ define

$$\Delta_{\mathcal{O}}(\boldsymbol{\lambda})_i = \bigl\{ x \in \Delta_{\mathcal{O}}(\boldsymbol{\lambda}) \bigm| \nu_\pi\bigl(\langle x, y\rangle\bigr) \geqslant i \text{ for all } y \in \Delta_{\mathcal{O}}(\boldsymbol{\lambda}) \bigr\}.$$

Finally, let $\Delta(\boldsymbol{\lambda})_i = (\Delta_{\mathcal{O}}(\boldsymbol{\lambda})_i + \pi\Delta_{\mathcal{O}}(\boldsymbol{\lambda}))/\pi\Delta_{\mathcal{O}}(\boldsymbol{\lambda})$. Then

$$\Delta(\boldsymbol{\lambda}) = \Delta(\boldsymbol{\lambda})_0 \supset \Delta(\boldsymbol{\lambda})_1 \supseteq \Delta(\boldsymbol{\lambda})_2 \supseteq \cdots$$

is a *Jantzen filtration* of the $\mathcal{S}_{r,n}$-module $\Delta(\boldsymbol{\lambda})$. Then $\Delta(\boldsymbol{\lambda})_k = 0$ for $k \gg 0$ since $\Delta(\boldsymbol{\lambda})$ is finite dimensional.

To describe the Jantzen filtration of $\Delta(\boldsymbol{\lambda})$ we need some combinatorics. The *diagram* of a multipartition $\boldsymbol{\lambda}$ is the set $[\boldsymbol{\lambda}] = \{(i, j, a) \mid 1 \leqslant j \leqslant \lambda_i^{(a)} \text{ and } 1 \leqslant a \leqslant r\}$. A *node* is any ordered triple (i, j, a) in $\mathbb{N} \times \mathbb{N} \times \{1, \dots, r\}$. For example, all of the elements of $[\boldsymbol{\lambda}]$ are nodes.

Each node $x = (i, j, a) \in [\boldsymbol{\lambda}]$ determines a *rim hook*

$$r_x^{\boldsymbol{\lambda}} = \bigl\{ (k, l, a) \in [\boldsymbol{\lambda}] \bigm| k \geqslant i, l \geqslant j \text{ and } (k+1, l+1, a) \notin [\boldsymbol{\lambda}] \bigr\}.$$

We say that $r_x^{\boldsymbol{\lambda}}$ is an *h-rim hook* if $h = |r_x^{\boldsymbol{\lambda}}|$. Let i' be maximal such that $(i', j, a) \in [\boldsymbol{\lambda}]$; so i' is the length of column j of $\lambda^{(a)}$. Then $f_x^{\boldsymbol{\lambda}} = (i', j, a) \in [\boldsymbol{\lambda}]$ is the *foot* of $r_x^{\boldsymbol{\lambda}}$ and $r_x^{\boldsymbol{\lambda}}$ has *leg length* $\ell\ell(r_x^{\boldsymbol{\lambda}}) = i' - i$. Similarly, the node $(i, \lambda_j^{(a)}, a)$ is the *hand* of $r_x^{\boldsymbol{\lambda}}$. If $x \in [\boldsymbol{\lambda}]$ let $\boldsymbol{\lambda} \setminus r_x^{\boldsymbol{\lambda}}$ be the multipartition with diagram $[\boldsymbol{\lambda}] \setminus r_x^{\boldsymbol{\lambda}}$. We say that $\boldsymbol{\lambda} \setminus r_x^{\boldsymbol{\lambda}}$ is the multipartition obtained by *unwrapping* the rim hook $r_x^{\boldsymbol{\lambda}}$ from $\boldsymbol{\lambda}$, and that $\boldsymbol{\lambda}$ is the multipartition obtained from $\boldsymbol{\lambda} \setminus r_x^{\boldsymbol{\lambda}}$ by *wrapping* on the rim hook $r_x^{\boldsymbol{\lambda}}$.

Define the $\mathcal{O}$-residue of the node $x = (i, j, a)$ to be $\operatorname{res}_{\mathcal{O}}(x) = (qt)^{j-i} X_a$.

2.5. Definition. Suppose that $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)})$ and $\boldsymbol{\mu} = (\mu^{(1)}, \dots, \mu^{(r)})$ are multipartitions of n. The *Jantzen coefficient* $J_{\boldsymbol{\lambda}\boldsymbol{\mu}}$ is the integer

$$J_{\boldsymbol{\lambda}\boldsymbol{\mu}} = \begin{cases} \displaystyle\sum_{x \in [\boldsymbol{\lambda}]} \sum_{y \in [\boldsymbol{\mu}],\, [\boldsymbol{\mu}] \setminus r_y^{\boldsymbol{\mu}} = [\boldsymbol{\lambda}] \setminus r_x^{\boldsymbol{\lambda}}} (-1)^{\ell\ell(r_x^{\boldsymbol{\lambda}}) + \ell\ell(r_y^{\boldsymbol{\mu}})} \nu_\pi\bigl(\operatorname{res}_{\mathcal{O}}\bigl(f_x^{\boldsymbol{\lambda}}\bigr) - \operatorname{res}_{\mathcal{O}}\bigl(f_y^{\boldsymbol{\mu}}\bigr)\bigr), & \text{if } \boldsymbol{\lambda} \rhd \boldsymbol{\mu}, \\ 0, & \text{otherwise.} \end{cases}$$

The Jantzen coefficient $J_{\boldsymbol{\lambda}\boldsymbol{\mu}}$ depends on the choices of $\mathbb{F}$, q and $\mathbf{Q} = (Q_1, \dots, Q_r)$. In fact, we will see that $J_{\boldsymbol{\lambda}\boldsymbol{\mu}}$ depends only on p, e and $\mathbf{Q}$. By definition $J_{\boldsymbol{\lambda}\boldsymbol{\mu}}$ is an integer which is determined by the combinatorics of multipartitions. The definition of $J_{\boldsymbol{\lambda}\boldsymbol{\mu}}$ is reasonably involved, however, it turns out that these integers are computable. In Sections 3 and 4 we give simpler formulae for the Jantzen coefficients.

2.6. Theorem. *(See James and Mathas [21, Theorem 4.3].) Suppose that $\boldsymbol{\lambda}$ is a multipartition of n. Then*

$$\sum_{i>0}\big[\Delta(\boldsymbol{\lambda})_i\big] = \sum_{\boldsymbol{\mu}\in\Lambda_{r,n}^+} J_{\boldsymbol{\lambda}\boldsymbol{\mu}}\big[\Delta(\boldsymbol{\mu})\big]$$

in $K_0(\mathcal{S}_{r,n})$.

For multipartitions $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ in $\Lambda_{r,n}^+$ let $d_{\boldsymbol{\lambda}\boldsymbol{\mu}} = [\Delta(\boldsymbol{\lambda}) : L(\boldsymbol{\mu})]$ be the number of composition factors of $\Delta(\boldsymbol{\lambda})$ which are isomorphic to $L(\boldsymbol{\mu})$. Define

$$J'_{\boldsymbol{\lambda}\boldsymbol{\mu}} = \sum_{\substack{\boldsymbol{\nu}\in\Lambda_{r,n}^+\\ \boldsymbol{\lambda}\rhd\boldsymbol{\nu}\trianglerighteq\boldsymbol{\mu}}} J_{\boldsymbol{\lambda}\boldsymbol{\nu}} d_{\boldsymbol{\nu}\boldsymbol{\mu}}.$$

By Theorem 2.6, $J'_{\boldsymbol{\lambda}\boldsymbol{\mu}}$ is the composition multiplicity of the simple module $L(\boldsymbol{\mu})$ in $\bigoplus_{i>0}\Delta(\boldsymbol{\lambda})_i$. Therefore, $J'_{\boldsymbol{\lambda}\boldsymbol{\mu}} \geqslant 0$, for all $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \Lambda_{r,n}^+$. As $\Delta(\boldsymbol{\lambda})_1 = \operatorname{rad}\Delta(\boldsymbol{\lambda})$ we obtain the following.

2.7. Corollary. *Suppose that $\boldsymbol{\lambda} \neq \boldsymbol{\mu}$ are multipartitions of n. Then $d_{\boldsymbol{\lambda}\boldsymbol{\mu}} \leqslant J'_{\boldsymbol{\lambda}\boldsymbol{\mu}}$ and, moreover, $d_{\boldsymbol{\lambda}\boldsymbol{\mu}} \neq 0$ if and only if $J'_{\boldsymbol{\lambda}\boldsymbol{\mu}} \neq 0$.*

We now use Theorem 2.6 to classify the blocks of $\mathcal{S}_{r,n}$.

2.8. Definition. Suppose that $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \Lambda_{r,n}^+$. Then $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are *Jantzen equivalent*, and we write $\boldsymbol{\lambda} \sim_J \boldsymbol{\mu}$, if there exists a sequence of multipartitions $\boldsymbol{\lambda}_0 = \boldsymbol{\lambda}, \boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_k = \boldsymbol{\mu}$ such that either

$$J_{\boldsymbol{\lambda}_i\boldsymbol{\lambda}_{i+1}} \neq 0 \quad \text{or} \quad J_{\boldsymbol{\lambda}_{i+1}\boldsymbol{\lambda}_i} \neq 0,$$

for $0 \leqslant i < k$.

Jantzen equivalence gives us our first combinatorial characterization of the blocks of $\mathcal{S}_{r,n}$.

2.9. Proposition. *Suppose that $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \Lambda_{r,n}^+$. Then $\Delta(\boldsymbol{\lambda})$ and $\Delta(\boldsymbol{\mu})$ belong to the same block as $\mathcal{S}_{r,n}$-modules if and only if $\boldsymbol{\lambda} \sim_J \boldsymbol{\mu}$.*

Proof. We first show that $\Delta(\boldsymbol{\lambda})$ and $\Delta(\boldsymbol{\mu})$ belong to the same block whenever $\boldsymbol{\lambda} \sim_J \boldsymbol{\mu}$. By definition $\Delta(\boldsymbol{\lambda})_i$ is a submodule of $\Delta(\boldsymbol{\lambda})$ for all i, so all of the composition factors of $\sum_{i>0}\Delta(\boldsymbol{\lambda})_i$ belong to the same block as $\Delta(\boldsymbol{\lambda})$ by Lemma 2.2. Consequently, all of the composition factors of the virtual module $\sum_{\boldsymbol{\nu}} J_{\boldsymbol{\lambda}\boldsymbol{\nu}}[\Delta(\boldsymbol{\nu})]$ belong to the same block. Let Λ' be the set of multipartitions $\boldsymbol{\nu}$ such that $\Delta(\boldsymbol{\nu})$ is not in the same block as $\Delta(\boldsymbol{\lambda})$. Then we have $\sum_{\boldsymbol{\nu}\in\Lambda'} J_{\boldsymbol{\lambda}\boldsymbol{\nu}}[\Delta(\boldsymbol{\nu})] = 0$. Hence, $J_{\boldsymbol{\lambda}\boldsymbol{\nu}} = 0$ whenever $\boldsymbol{\nu} \in \Lambda'$ by Lemma 2.4. It follows that $\Delta(\boldsymbol{\lambda})$ and $\Delta(\boldsymbol{\mu})$ belong to the same block whenever $\boldsymbol{\lambda} \sim_J \boldsymbol{\mu}$.

To prove the converse it is sufficient to show that $\boldsymbol{\lambda} \sim_J \boldsymbol{\mu}$ whenever $d_{\boldsymbol{\lambda}\boldsymbol{\mu}} \neq 0$. Hence, by Corollary 2.7 we must show that $\boldsymbol{\lambda} \sim_J \boldsymbol{\mu}$ whenever $J'_{\boldsymbol{\lambda}\boldsymbol{\mu}} \neq 0$. However, if $J'_{\boldsymbol{\lambda}\boldsymbol{\mu}} \neq 0$ then we can find a multipartition $\boldsymbol{\nu}_1$ such that $J_{\boldsymbol{\lambda}\boldsymbol{\nu}_1} \neq 0$, $d_{\boldsymbol{\nu}_1\boldsymbol{\mu}} \neq 0$ and $\boldsymbol{\lambda} \rhd \boldsymbol{\nu}_1 \trianglerighteq \boldsymbol{\mu}$. Consequently, $\boldsymbol{\lambda} \sim_J \boldsymbol{\nu}_1$. If $\boldsymbol{\nu}_1 \neq \boldsymbol{\mu}$ then $J'_{\boldsymbol{\nu}_1\boldsymbol{\mu}} \neq 0$ by Corollary 2.7 since $d_{\boldsymbol{\nu}_1\boldsymbol{\mu}} \neq 0$. Therefore, we can find a multipartition $\boldsymbol{\nu}_2$ such that $J_{\boldsymbol{\nu}_1\boldsymbol{\nu}_2} \neq 0$, $d_{\boldsymbol{\nu}_2\boldsymbol{\mu}} \neq 0$ and $\boldsymbol{\nu}_1 \rhd \boldsymbol{\nu}_2 \trianglerighteq \boldsymbol{\mu}$. Continuing in this way we can find multipartitions

$\boldsymbol{\nu}_0 = \boldsymbol{\lambda}, \boldsymbol{\nu}_1, \dots, \boldsymbol{\nu}_k = \boldsymbol{\mu}$ such that $J_{\boldsymbol{\nu}_{i-1}\boldsymbol{\nu}_i} \neq 0$, $d_{\boldsymbol{\nu}_i\boldsymbol{\mu}} \neq 0$, for $0 < i < k$, and $\boldsymbol{\lambda} \rhd \boldsymbol{\nu}_1 \rhd \dots \rhd \boldsymbol{\nu}_k = \boldsymbol{\mu}$. Note that we must have $\boldsymbol{\nu}_k = \boldsymbol{\mu}$ for some k since $\Lambda_{r,n}^+$ is finite. Therefore, $\boldsymbol{\lambda} \sim_J \boldsymbol{\nu}_1 \sim_J \dots \sim_J \boldsymbol{\nu}_k = \boldsymbol{\mu}$ as required. □

Remark. Under some very mild technical assumptions (see, for example, [25, §4.1]), Jantzen filtrations can be defined for the standard modules of an arbitrary quasi-hereditary algebra. The argument of Proposition 2.9 is completely generic: it shows that the blocks of a quasi-hereditary algebra are determined by the 'Jantzen coefficients.'

Remark. Without using the cyclotomic q-Schur algebras it is not clear that the Jantzen equivalence determines the blocks of $\mathscr{H}_{r,n}$. Applying the Schur functor to Theorem 2.6 gives an analogous description of the Jantzen filtration of the Specht modules: $\sum_{i>0}[S(\boldsymbol{\lambda})_i] = \sum_{\boldsymbol{\mu}} J_{\boldsymbol{\lambda}\boldsymbol{\mu}}[S(\boldsymbol{\mu})]$. The problem is that, *a priori*, the composition factors of $\bigoplus_{\boldsymbol{\mu}} J_{\boldsymbol{\lambda}\boldsymbol{\mu}} S(\boldsymbol{\mu})$ could belong to different blocks because the analogue of Lemma 2.4 fails for Specht modules.

2.4. A second combinatorial characterization of the blocks

Proposition 2.9 completely determines the blocks of $\mathscr{S}_{r,n}$, and hence the blocks of $\mathscr{H}_{r,n}$. Unfortunately, it is not obvious when two multipartitions are Jantzen equivalent.

The *residue* of the node $x = (i, j, a)$ is

$$\operatorname{res}(x) = \begin{cases} q^{j-i} Q_a, & \text{if } q \neq 1 \text{ and } Q_a \neq 0, \\ (\overline{j-i}, Q_a), & \text{if } q = 1 \text{ and } Q_a \neq Q_b \text{ for } b \neq a, \\ Q_a, & \text{otherwise,} \end{cases}$$

where $\bar{z} = z \pmod p$ for $z \in \mathbb{Z}$ (if $p = \infty$ we set $\bar{z} = z$). Let

$$\operatorname{Res}(\Lambda_{r,n}^+) = \{\operatorname{res}(x) \mid x \in [\boldsymbol{\lambda}] \text{ for some } \boldsymbol{\lambda} \in \Lambda_{r,n}^+\}$$

be the set of all possible residues. For any multipartition $\boldsymbol{\lambda} \in \Lambda_{r,n}^+$ and $f \in \operatorname{Res}(\Lambda_{r,n}^+)$ define

$$C_f(\boldsymbol{\lambda}) = \#\{x \in [\boldsymbol{\lambda}] \mid \operatorname{res}(x) = f\}.$$

We can now define our second combinatorial equivalence relation on $\Lambda_{r,n}^+$.

2.10. Definition. Suppose that $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are multipartitions. Then $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are *residue equivalent*, and we write $\boldsymbol{\lambda} \sim_C \boldsymbol{\mu}$, if $C_f(\boldsymbol{\lambda}) = C_f(\boldsymbol{\mu})$ for all $f \in \operatorname{Res}(\Lambda_{r,n}^+)$.

It is easy to determine if two multipartitions are residue equivalent, so the next result gives an effective characterization of the blocks of the algebras $\mathscr{H}_{r,n}$ and $\mathscr{S}_{r,n}$.

2.11. Theorem. *Suppose that $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are multipartitions of n. Then the following are equivalent.*

(a) *$S(\boldsymbol{\lambda})$ and $S(\boldsymbol{\mu})$ belong to the same block as $\mathscr{H}_n(\mathbf{Q})$-modules.*
(b) *$\Delta(\boldsymbol{\lambda})$ and $\Delta(\boldsymbol{\mu})$ belong to the same block as $\mathscr{S}_{r,n}(\mathbf{Q})$-modules.*
(c) *$\boldsymbol{\lambda} \sim_J \boldsymbol{\mu}$.*
(d) *$\boldsymbol{\lambda} \sim_C \boldsymbol{\mu}$.*

By Propositions 2.3 and 2.9, (a), (b) and (c) are equivalent. Therefore, to prove the theorem it is enough to prove that $\boldsymbol{\lambda} \sim_J \boldsymbol{\mu}$ if and only if $\boldsymbol{\lambda} \sim_C \boldsymbol{\mu}$. The proof of this fact is given in Sections 3 and 4. It turns out that, combinatorially, these equivalence relations depend very much on whether or not $q = 1$ and whether or not some of the parameters $Q_1, \ldots, Q_r$ are zero. The following result allows us to treat these cases separately.

2.12. Theorem. *(See Dipper and Mathas [13, Theorem 1.5 and Corollary 5.7].) Suppose that $\mathbf{Q} = \mathbf{Q}_1 \amalg \mathbf{Q}_2 \amalg \cdots \amalg \mathbf{Q}_\kappa$ is a partition of $\mathbf{Q}$ such that $q^c Q_a \in \mathbf{Q}_\alpha$ only if $Q_a \in \mathbf{Q}_\alpha$, for $c \in \mathbb{Z}$, $1 \leqslant a \leqslant r$ and $1 \leqslant \alpha \leqslant \kappa$. Set $r_\alpha = |\mathbf{Q}_\alpha|$, for $1 \leqslant \alpha \leqslant \kappa$. Then $\mathcal{S}_{r,n}(\mathbf{Q})$ is Morita equivalent to the algebra*

$$\bigoplus_{\substack{n_1,\ldots,n_\kappa \geqslant 0 \\ n_1+\cdots+n_\kappa=n}} \mathcal{S}_{r_1,n_1}(\mathbf{Q}_1) \boxtimes \mathcal{S}_{r_2,n_2}(\mathbf{Q}_2) \boxtimes \cdots \boxtimes \mathcal{S}_{r_\kappa,n_\kappa}(\mathbf{Q}_\kappa).$$

Moreover, if $\mathbf{Q}_\alpha = \{Q_{i_1^\alpha}, \ldots, Q_{i_{r_\alpha}^\alpha}\}$, for $1 \leqslant \alpha \leqslant \kappa$, then the Morita equivalence is induced by the map $\Delta(\boldsymbol{\lambda}) \mapsto \Delta(\boldsymbol{\lambda}_1) \boxtimes \cdots \boxtimes \Delta(\boldsymbol{\lambda}_\kappa)$, where $\boldsymbol{\lambda}_\alpha = (\lambda^{(i_1^\alpha)}, \ldots, \lambda^{(i_{r_\alpha}^\alpha)})$, for $1 \leqslant \alpha \leqslant \kappa$ and $\boldsymbol{\lambda} \in \Lambda_{r,n}^+$.

There is an analogous result for the Ariki–Koike algebra $\mathcal{H}_{r,n}$; see [13, Theorem 1.1].

Theorem 2.12 says that the blocks of $\mathcal{H}_{r,n}(\mathbf{Q})$ and $\mathcal{S}_{r,n}(\mathbf{Q})$ depend only on the orbits of the parameters under multiplication by q. Further, by Theorem 2.12 it is enough to consider the case where $\mathbf{Q}$ is contained in a single q-orbit to prove Theorem 2.11. Hence, by rescaling T_0, if necessary, we can assume that the parameters $Q_1, \ldots, Q_r$ are all zero or that they are all powers of q. More explicitly, we can assume that either $Q_a = 0$, or that there exist integers $c_1, \ldots, c_r$ such that $Q_a = q^{c_a}$, for $1 \leqslant a \leqslant r$. Consequently, to prove Theorem 2.11 we are reduced to considering the following mutually exclusive cases:

2.13. Case 1. $q \neq 1$ and $Q_a = q^{c_a}$, for $1 \leqslant a \leqslant r$.
Case 2. $r = 1$ and $q = 1$ (and Q_1 arbitrary).
Case 3. $r > 1$, $q = 1$ and $Q_1 = \cdots = Q_r = 1$.
Case 4. $r > 1$, $q = 1$ and $Q_1 = \cdots = Q_r = 0$.
Case 5. $r > 1$, $q \neq 1$ and $Q_1 = \cdots = Q_r = 0$.

Note that $\mathcal{H} = \mathcal{H}_{1,n}$ is independent of Q_1 when $r = 1$.

The proof of Theorem 2.11 for Case 1 is given in Section 3. Cases 2–5 are considered in Section 4 using similar, but easier, arguments. Given a node $x = (i, j, a)$ note that $\operatorname{res}(x) = q^{j-i} Q_a$ in Case 1, $\operatorname{res}(x) = (\overline{j-i}, Q_1)$ in Case 2 and $\operatorname{res}(x) = Q_a$ in the other three cases.

The basic strategy for proving Theorem 2.11 for each of these five cases is the same, however, the proof breaks up into three cases because the combinatorics of residue equivalence is different for Case 1, Case 2 and Cases 3–5. Fayers has pointed out that the Ariki–Koike algebras in Cases 3 and 4 are isomorphic via the algebra homomorphism determined by $T_0 \mapsto (T_0 - 1)$ and $T_i \mapsto T_i$, for $1 \leqslant i < n$, so we do not actually need to consider Case 4.

2.5. The blocks of the affine Hecke algebra

Assuming Theorem 2.11 we now prove Theorem A from the introduction.

As the centre $Z(\mathcal{H}_n^{\mathrm{aff}})$ of $\mathcal{H}_n^{\mathrm{aff}}$ is the set of symmetric Laurent polynomials in $X_1, \dots, X_n$, the central characters of $\mathcal{H}_n^{\mathrm{aff}}$ are indexed by $\mathfrak{S}_n$-orbits of $(\mathbb{F}^\times)^n$. More precisely, if $\gamma \in (\mathbb{F}^\times)^n/\mathfrak{S}_n$ then the central character χ_γ is given by evaluation at γ.

By Lemma 2.1, all of the composition factors of the Specht module $S(\boldsymbol{\lambda})$ belong to the same block as $\mathcal{H}_{r,n}$-modules. Therefore, all of the composition factors of $S(\boldsymbol{\lambda})$ belong to the same block as $\mathcal{H}_n^{\mathrm{aff}}$-modules. We need to know the central characters of the Specht modules.

2.14. Lemma. *Suppose that $q \neq 1$ and that $D(\boldsymbol{\lambda}) \neq 0$, for some multipartition $\boldsymbol{\lambda} \in \Lambda_{r,n}^+$. Then $f(X) \in \mathbb{Z}(\mathcal{H}_n^{\mathrm{aff}})$ acts on $D(\boldsymbol{\lambda})$ as multiplication by $f(\gamma)$, where $\gamma = (\mathrm{res}(x_1), \mathrm{res}(x_2), \dots, \mathrm{res}(x_n))$ and $[\boldsymbol{\lambda}] = \{x_1, \dots, x_n\}$ (in any order).*

Proof. As all of the composition factors of $S(\boldsymbol{\lambda})$ belong to the same block as $D(\boldsymbol{\lambda})$, $f(X)$ acts on $S(\boldsymbol{\lambda})$ and on $D(\boldsymbol{\lambda})$ as multiplication by the same scalar. By [21, Proposition 3.7] this scalar is given by evaluating the polynomial $f(X)$ at $(\mathrm{res}(x_1), \mathrm{res}(x_2), \dots, \mathrm{res}(x_n))$. □

2.15. Theorem. *Suppose that $q \neq 1$ and that $\mathbb{F}$ is algebraically closed. Then two simple $\mathcal{H}_n^{\mathrm{aff}}$-modules D and D' belong to the same block if and only if they have the same central character.*

Proof. Any two simple modules in the same block have the same central character. Conversely, suppose that D and D' are simple $\mathcal{H}_n^{\mathrm{aff}}$-modules which have the same central character. Let $(X_1 - Q_1)\dots(X_1 - Q_s)$ and $(X_1 - Q_{s+1})\dots(X_1 - Q_r)$, respectively, be the minimal polynomials for X_1 acting on D and D'. (Note that $Q_1, \dots, Q_r$ are non-zero since $X_1, \dots, X_n$ are invertible.) Then D and D' are both simple modules for the Ariki–Koike algebra $\mathcal{H}_{r,n}$ with parameters $Q_1, \dots, Q_r$. Therefore, $D \cong D(\boldsymbol{\lambda})$ and $D' \cong D(\boldsymbol{\mu})$ for some multipartitions $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \Lambda_{r,n}^+$. By assumption, D and D' have the same central characters. The central character of $D(\boldsymbol{\lambda})$ is uniquely determined by the multiset of residues $\{\mathrm{res}(x) \mid x \in [\boldsymbol{\lambda}]\}$ by Lemma 2.14. Similarly, the central character of $D(\boldsymbol{\mu})$ is determined by the multiset $\{\mathrm{res}(x) \mid x \in [\boldsymbol{\mu}]\}$. Hence, $C_f(\boldsymbol{\lambda}) = C_f(\boldsymbol{\mu})$, for all $f \in \mathrm{Res}(\Lambda_{r,n}^+)$. Therefore, $\boldsymbol{\lambda} \sim_C \boldsymbol{\mu}$, so $D \cong D(\boldsymbol{\lambda})$ and $D' = D(\boldsymbol{\mu})$ are in the same block as $\mathcal{H}_{r,n}$-modules by Theorem 2.11. Hence, D and D' are in the same block as $\mathcal{H}_n^{\mathrm{aff}}$-modules. □

Theorem 2.15 is not new. We are grateful to Iain Gordon for pointing out that the classification of the blocks of $\mathcal{H}_n^{\mathrm{aff}}$ by central characters is an immediate corollary of a general result of Müller [26, Theorem 7] since $\mathcal{H}_n^{\mathrm{aff}}$ is finite dimensional over its centre. See also [7, III.9].

Combining Theorems 2.11 and 2.15 we obtain a more descriptive version of Theorem A.

2.16. Corollary *(Theorem A). Suppose that $\mathbb{F}$ is an algebraically closed field, $q \neq 1$ and that the parameters $Q_1, \dots, Q_r$ are non-zero. Let $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ be multipartitions in $\Lambda_{r,n}^+$ with $D(\boldsymbol{\lambda}) \neq 0$ and $D(\boldsymbol{\mu}) \neq 0$. Then the following are equivalent:*

(a) *$D(\boldsymbol{\lambda})$ and $D(\boldsymbol{\mu})$ belong to the same block as $\mathcal{H}_{r,n}$-modules.*
(b) *$D(\boldsymbol{\lambda})$ and $D(\boldsymbol{\mu})$ belong to the same block as $\mathcal{H}_n^{\mathrm{aff}}$-modules.*
(c) *$D(\boldsymbol{\lambda})$ and $D(\boldsymbol{\mu})$ have the same central character as $\mathcal{H}_n^{\mathrm{aff}}$-modules.*
(d) *$\boldsymbol{\lambda} \sim_C \boldsymbol{\mu}$.*

3. Combinatorics

In this section, we prove $\boldsymbol{\lambda} \sim_J \boldsymbol{\mu}$ if and only if $\boldsymbol{\lambda} \sim_C \boldsymbol{\mu}$, for $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \Lambda_{r,n}^+$ in the cases when $q \neq 1$ and all of the parameters $Q_1, \dots, Q_r$ are powers of q. This is Case 1 of 2.13. The basic idea is to reduce the comparison of the Jantzen and residue equivalence relations to the case where the multipartitions $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are both 'cores.' The complication is that, unlike for partitions (the case $r = 1$), we do not have a good notion of 'core' for multipartitions when $r > 1$. We circumvent this difficulty using ideas of Fayers [14,15].

As we assume that the parameters $Q_1, \ldots, Q_r$ are all powers of q, there exist integers $c_1, \ldots, c_r$ such that $Q_a = q^{c_a}$, for $1 \leqslant a \leqslant r$. The sequence $\mathbf{c} = (c_1, \ldots, c_r)$ is called the *multi-charge* of $\mathbf{Q}$.

Now that $\mathbf{Q}$ is contained in a single q-orbit, we redefine the *residue* of a node $x = (i, j, a)$ to be

$$\operatorname{res}(x) = (j - i + c_a) \pmod e.$$

Therefore, $\{\operatorname{res}(x) \mid x \in [\boldsymbol{\lambda}] \text{ for some } \boldsymbol{\lambda} \in \Lambda^+_{r,n}\} \subseteq \mathbb{Z}/e\mathbb{Z}$.

For $\boldsymbol{\lambda} \in \Lambda^+_{r,n}$ and $f \in \mathbb{Z}/e\mathbb{Z}$ put $C_f(\boldsymbol{\lambda}) = \#\{x \in [\boldsymbol{\lambda}] \mid \operatorname{res}(x) = f\}$. It is straightforward to check that with these new conventions $\boldsymbol{\lambda} \sim_C \boldsymbol{\mu}$ if and only if $C_f(\boldsymbol{\lambda}) = C_f(\boldsymbol{\mu})$, for all $f \in \mathbb{Z}/e\mathbb{Z}$.

3.1. Abacuses

Abacuses first appeared in the work of Gordon James [19] and have since been used extensively in the modular representation theory of the symmetric groups and related algebras. An *e-abacus* is an abacus with e vertical runners, which are infinite in both directions. If e is finite then we label the runners $0, 1, \ldots, e - 1$ from left to right and position $z \in \mathbb{Z}$ on the abacus is the bead position in row x on runner y, where $z = xe + y$ and $0 \leqslant y < e$. If $e = \infty$ then we label the runners $\ldots, -1, 0, 1, \ldots$ and position z on the abacus is the bead position in row 0 on runner z.

Let $\boldsymbol{\lambda} \in \Lambda^+_{r,n}$ be a multipartition and recall that we have fixed a sequence of integers $\mathbf{c} = (c_1, \ldots, c_r)$. Fix a with $1 \leqslant a \leqslant r$ and, for $i \geqslant 0$, define

$$\beta^a_i = \lambda^{(a)}_i - i + c_a.$$

Then the *β-numbers* (with charge c_a) for the partition $\lambda^{(a)}$ are the integers $\beta^a_1, \beta^a_2, \ldots$ and we define $B_a = \{\beta^a_1, \beta^a_2, \ldots\}$. The *e-abacus display* of $\lambda^{(a)}$ (with multi-charge $(c_1, \ldots, c_r)$) is the e-abacus with a bead at position β^a_i, for $i \geqslant 1$. The e-abacus display of the multipartition $\boldsymbol{\lambda}$ is the ordered r-tuple of abacuses for the partitions $\lambda^{(1)}, \ldots, \lambda^{(r)}$.

It is easy to check that a multipartition is uniquely determined by its abacus display and that every abacus display corresponds to some multipartition.

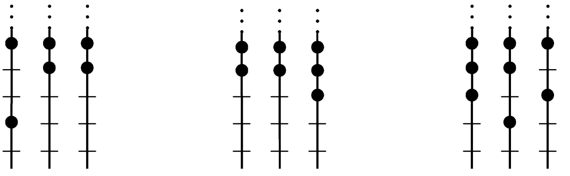
3.1. Example. Suppose that $e = 3$, $r = 3$ and $\mathbf{c} = (0, 1, 2)$. Let

$$\boldsymbol{\lambda} = \big((4, 1, 1), (2), (3, 2, 1)\big).$$

Then

$$B_1 = \{3, -1, -2, -4, -5, \ldots\}, \qquad B_2 = \{2, -1, -2, \ldots\},$$
$$B_3 = \{4, 2, 0, -2, -3, \ldots\}$$

and the abacus display for $\boldsymbol{\lambda}$ is given by



Let λ be a partition and suppose that $B = \{\beta_1, \beta_2, \ldots\}$ is the set of β-numbers for λ. Then the e-abacus for λ has beads at positions β_i, for $i \geqslant 0$. If $\beta_i + h \notin B$ then *moving* the bead at position β_i to the *right* h positions gives a new abacus display with beads at positions $\{\beta_1, \beta_2, \ldots, \beta_{i-1}, \beta_i + h, \beta_{i+1}, \ldots\}$. Similarly, if $\beta_i - h \notin B$ then *moving* this bead h positions to the *left* creates a new abacus display with beads at positions $\{\beta_1, \beta_2, \ldots, \beta_{i-1}, \beta_i - h, \beta_{i+1}, \ldots\}$. The conditions $\beta_i \pm h \notin B$ are needed to ensure that the abacus display for λ does not already have a bead at the new position. Note that with these conventions moving a bead on runner 0 one position to the left moves the bead to a position on runner $e - 1$ in the preceding row. Similarly, moving a bead on runner $e - 1$ to the right moves a bead to a position on runner 0 in the next row. We also talk of moving beads in the abacus displays of multipartitions.

Recasting the above discussion in terms of the abacus we have the following well-known result which goes back to Littlewood and James. (Recall that we defined rim hooks in Section 2.)

3.2. Lemma. *Suppose that λ is a partition. Then moving a bead to the right h positions from runner f to runner f' corresponds to wrapping an h-rim hook with foot residue $f + 1$, and hand residue f', onto λ. Similarly, moving a bead h positions to the left, from runner f to runner f' corresponds to unwrapping an h-rim hook from λ with foot residue $f' + 1$ and hand residue f.*

That increasing a β-number by h corresponds to wrapping on an h-rim hook is proved in [23, Lemma 5.26]. The remaining claim about residues follows easily from our definitions. As a consequence we obtain the following.

3.3. Corollary. *Suppose that λ is a partition and $f \in \mathbb{Z}/e\mathbb{Z}$, where $e < \infty$. Then*

(a) *Moving a bead down one row on a runner corresponds to wrapping an e-rim hook onto $[\lambda]$. If this bead is on runner f then the rim hook has foot residue $f + 1$.*
(b) *Moving a bead up one row on a runner corresponds to unwrapping an e-rim hook from $[\lambda]$. If this bead is on runner f the rim hook has foot residue $f + 1$.*
(c) *Moving the lowest bead on runner $f - 1$ down one row corresponds to wrapping on an e-hook with foot residue f. Consequently, we can add an e-hook with foot residue f to any partition.*

Suppose that λ is a partition. The *e-core* of λ is the partition $\bar{\lambda}$ whose e-abacus display is obtained from the e-abacus display for λ by moving all beads as high as possible on their runners, that is, successively removing all e-hooks from the diagram of λ. If $e = \infty$ then the e-core of λ is λ itself. Define the *e-weight* of the partition, $\mathrm{w}_e(\lambda)$, to be the number of e-hooks that we remove in order to construct $\bar{\lambda}$.

3.2. Jantzen equivalence

In order to prove Theorem 2.11 we first simplify the formula for $J_{\boldsymbol{\lambda\mu}}$. Let $\boldsymbol{\lambda}$ be a multipartition and recall that if $x \in [\boldsymbol{\lambda}]$ then $r_x^{\boldsymbol{\lambda}} \subseteq [\boldsymbol{\lambda}]$ is the associated rim hook. To ease notation we let $h_x^{\boldsymbol{\lambda}} = |r_x^{\boldsymbol{\lambda}}|$ be the *hook length* of $r_x^{\boldsymbol{\lambda}}$.

Recall that $\mathbb{F}$ is a field of characteristic p. Define $\nu_p' : \mathbb{Z}^\times \to \mathbb{N}$ to be the map

$$\nu_p'(h) = \begin{cases} p^k, & \text{if } p \text{ is finite,} \\ 1, & \text{if } p = \infty, \end{cases}$$

where $k \geqslant 0$ is maximal such that p^k divides h. We caution the reader that ν_p' is not the standard p-adic valuation map.

If $\sigma = (\sigma_1, \sigma_2, \ldots)$ is a partition let $\sigma' = (\sigma_1', \sigma_2', \ldots)$ be its conjugate. Then $\sigma_i' = c$ if c is maximal such that $(c, i) \in [\sigma]$. (So σ_i' is the length of column i of $[\sigma]$.) For any integer $h \in \mathbb{Z}$ let $[h]_t = (t^h - 1)/(t - 1) \in \mathbb{F}[t, t^{-1}]$.

3.4. Lemma. *Suppose that $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are multipartitions of n and that $[\boldsymbol{\lambda}] \setminus r_x^{\boldsymbol{\lambda}} = [\boldsymbol{\mu}] \setminus r_y^{\boldsymbol{\mu}}$, for some nodes $x = (i, j, a) \in [\boldsymbol{\lambda}]$ and $y = (k, l, b) \in [\boldsymbol{\mu}]$. Then $\nu_\pi(\mathrm{res}_\mathcal{O}(f_x^{\boldsymbol{\lambda}}) - \mathrm{res}_\mathcal{O}(f_y^{\boldsymbol{\mu}})) \neq 0$ if and only if $\mathrm{res}(f_x^{\boldsymbol{\lambda}}) = \mathrm{res}(f_y^{\boldsymbol{\mu}})$, in which case*

$$\nu_\pi\big(\mathrm{res}_\mathcal{O}\big(f_x^{\boldsymbol{\lambda}}\big) - \mathrm{res}_\mathcal{O}\big(f_y^{\boldsymbol{\mu}}\big)\big) = \nu_p'\big(n(a-b) + j - \lambda_i^{(a)\prime} - l + \mu_k^{(b)\prime}\big).$$

Proof. Let $i' = \lambda_i^{(a)\prime}$ and $k' = \mu_k^{(b)\prime}$ so that $f_x^{\boldsymbol{\lambda}} = (i', j, a)$ and $f_y^{\boldsymbol{\mu}} = (k', l, b)$. Then

$$\begin{aligned} \mathrm{res}_\mathcal{O}\big(f_x^{\boldsymbol{\lambda}}\big) - \mathrm{res}_\mathcal{O}\big(f_y^{\boldsymbol{\mu}}\big) &= q^{j-i'+c_a} t^{na+j-i'} - q^{l-k'+c_b} t^{nb+l-k'} \\ &= q^{l-k'+c_b} t^{nb+l-k'} \big(q^{j-i'-l+k'+c_a-c_b} t^{n(a-b)+j-i'-l+k'} - 1\big). \end{aligned}$$

Therefore, $\nu_\pi(\mathrm{res}_\mathcal{O}(x) - \mathrm{res}_\mathcal{O}(y)) \neq 0$ if and only if $q^{j-i'-l+k'+c_a-c_b} = 1$, which is if and only if $\mathrm{res}(f_x^{\boldsymbol{\lambda}}) = q^{j-i'+c_a} = q^{l-k'+c_b} = \mathrm{res}(f_y^{\boldsymbol{\mu}})$.

Now suppose that $\mathrm{res}(f_x^{\boldsymbol{\lambda}}) = \mathrm{res}(f_y^{\boldsymbol{\mu}})$ and let $h = n(a-b) + j - i' - l + k'$. Note that h is non-zero because if $a = b$ then h is the axial distance from x to y. Then

$$\nu_\pi\big(\mathrm{res}_\mathcal{O}\big(f_x^{\boldsymbol{\lambda}}\big) - \mathrm{res}_\mathcal{O}\big(f_y^{\boldsymbol{\mu}}\big)\big) = \nu_\pi\big(t^{n(a-b)+j-i'-l+k'} - 1\big) = 1 + \nu_\pi\big([h]_t\big).$$

If $p = \infty$ then $(t-1)$ does not divide $[h]_t$, so that $\nu_\pi(\mathrm{res}_\mathcal{O}(x) - \mathrm{res}_\mathcal{O}(y)) = 1 = \nu_p'(h)$. If p is finite then write $h = p^k h'$, where $p \nmid h'$. Then

$$[h]_t = \big[p^k h'\big]_t = \big[p^k\big]_t [h']_{t^{p^k}} = (t-1)^{p^k - 1} [h']_t^{p^k}.$$

Now, $t - 1$ does not divide $[h']_t$ since $p \nmid h'$. Therefore, $\nu_\pi([h]_t) = \nu_p'(h) - 1$ and the result follows. □

We can now prove that (c) ⇒ (d) in Theorem 2.11.

3.5. Corollary. *Suppose that $\boldsymbol{\lambda} \sim_J \boldsymbol{\mu}$, where $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \Lambda_{r,n}^+$. Then $\boldsymbol{\lambda} \sim_C \boldsymbol{\mu}$.*

Proof. Without loss of generality we may assume that $J_{\boldsymbol{\lambda\mu}} \neq 0$. By Lemma 3.4 and Definition 2.5, $J_{\boldsymbol{\lambda\mu}}$ is non-zero only if there exist nodes $x \in [\boldsymbol{\lambda}]$ and $y \in [\boldsymbol{\mu}]$ such that $[\boldsymbol{\lambda}] \setminus r_x^{\boldsymbol{\lambda}} = [\boldsymbol{\mu}] \setminus r_y^{\boldsymbol{\mu}}$

and $\operatorname{res}(f_x^{\boldsymbol\lambda}) = \operatorname{res}(f_y^{\boldsymbol\mu})$. These two conditions imply that $C_f(\boldsymbol\lambda) = C_f(\boldsymbol\mu)$, for all $f \in \mathbb{Z}/e\mathbb{Z}$, so that $\boldsymbol\lambda \sim_C \boldsymbol\mu$. $\square$

Establishing the reverse implication in Theorem 2.11 takes considerably more effort. We start by explicitly describing the Jantzen coefficients.

3.6. Proposition. *Let $\boldsymbol\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)})$ and $\boldsymbol\mu = (\mu^{(1)}, \dots, \mu^{(r)})$ be multipartitions in $\Lambda_{r,n}^+$.*

(a) *Suppose that there exist integers $a < b$ such that $\lambda^{(c)} = \mu^{(c)}$, for $c \neq a, b$, and that $\lambda^{(a)} \neq \mu^{(a)}$ and $\lambda^{(b)} \neq \mu^{(b)}$. Then $J_{\boldsymbol\lambda\boldsymbol\mu} \neq 0$ only if there exist nodes $x = (i, j, a) \in [\boldsymbol\lambda]$ and $y = (k, l, b) \in [\boldsymbol\mu]$ such that* $\operatorname{res}(f_x^{\boldsymbol\lambda}) = \operatorname{res}(f_y^{\boldsymbol\mu})$ *and $[\boldsymbol\lambda] \setminus r_x^{\boldsymbol\lambda} = [\boldsymbol\mu] \setminus r_y^{\boldsymbol\mu}$. In this case*

$$J_{\boldsymbol\lambda\boldsymbol\mu} = (-1)^{\ell\ell(r_x^{\boldsymbol\lambda}) + \ell\ell(r_y^{\boldsymbol\mu})} \nu_p'\big(n(a-b) + j - \lambda_i^{(a)'} - l + \mu_k^{(b)'}\big).$$

(b) *Suppose that e is finite and for some integer a we have $\lambda^{(c)} = \mu^{(c)}$, for $c \neq a$. Then $J_{\boldsymbol\lambda\boldsymbol\mu} \neq 0$ only if there exist nodes $x = (i, j, a), (i, m, a) \in [\boldsymbol\lambda]$ such that $m < j$, $e \mid h_{(i,m,a)}^{\boldsymbol\lambda}$ and $\boldsymbol\mu$ is obtained by wrapping a rim hook of length $h_x^{\boldsymbol\lambda}$ onto $\boldsymbol\lambda \setminus r_x^{\boldsymbol\lambda}$ with its hand node in column m. In this case*

$$J_{\boldsymbol\lambda\boldsymbol\mu} = \begin{cases} (-1)^{\ell\ell(r_x^{\boldsymbol\lambda}) + \ell\ell(r_y^{\boldsymbol\mu})} \nu_p'(h_{(i,m,a)}^{\boldsymbol\lambda}), & \text{if } e \nmid h_{(i,j,a)}^{\boldsymbol\lambda}, \\ (-1)^{\ell\ell(r_x^{\boldsymbol\lambda}) + \ell\ell(r_y^{\boldsymbol\mu})} (\nu_p'(h_{(i,m,a)}^{\boldsymbol\lambda}) - \nu_p'(h_{(i,j,a)}^{\boldsymbol\lambda})), & \text{if } e \mid h_{(i,j,a)}^{\boldsymbol\lambda}, \end{cases}$$

where the node $y \in [\boldsymbol\mu]$ is determined by $[\boldsymbol\mu] \setminus r_y^{\boldsymbol\mu} = [\boldsymbol\lambda] \setminus r_x^{\boldsymbol\lambda}$.

(c) *In all other cases, $J_{\boldsymbol\lambda\boldsymbol\mu} = 0$.*

Proof. Suppose that $J_{\boldsymbol\lambda\boldsymbol\mu} \neq 0$. Then $\boldsymbol\lambda \rhd \boldsymbol\mu$ by Definition 2.5 and $\operatorname{res}(f_x^{\boldsymbol\lambda}) = \operatorname{res}(f_y^{\boldsymbol\mu})$ by Lemma 3.4. Furthermore, there exist nodes $x = (i, j, a) \in [\boldsymbol\lambda]$ and $y = (k, l, b) \in [\boldsymbol\mu]$ such that $[\boldsymbol\lambda] \setminus r_x^{\boldsymbol\lambda} = [\boldsymbol\mu] \setminus r_y^{\boldsymbol\mu}$. Consequently, $\lambda^{(c)} \neq \mu^{(c)}$ for at most two values of c. Therefore, since $\boldsymbol\lambda \rhd \boldsymbol\mu$, we may assume that we have integers $1 \leqslant a \leqslant b \leqslant r$ such that $\lambda^{(c)} = \mu^{(c)}$, for $c \neq a, b$.

If $a \neq b$ then the nodes x and y are uniquely determined because $r_x^{\boldsymbol\lambda} = [\lambda^{(a)}] \setminus [\mu^{(a)}]$ and $r_y^{\boldsymbol\mu} = [\mu^{(b)}] \setminus [\lambda^{(b)}]$. Therefore, $\lambda^{(a)} \neq \mu^{(a)}$, $\lambda^{(b)} \neq \mu^{(b)}$ and we are in the situation considered in part (a). The formula for $J_{\boldsymbol\lambda\boldsymbol\mu}$ now follows directly from Definition 2.5 and Lemma 3.4.

Now assume that $a = b$. If $e = \infty$ then $\operatorname{res}(f_x^{\boldsymbol\lambda}) = \operatorname{res}(f_y^{\boldsymbol\mu})$ if and only if $x = y$ since $h_x^{\boldsymbol\lambda} = h_y^{\boldsymbol\mu}$. This forces $\boldsymbol\lambda = \boldsymbol\mu$, which is not possible since $\boldsymbol\lambda \rhd \boldsymbol\mu$. Hence, e must be finite. By Lemma 3.2 the abacus display for $\mu^{(a)}$ is obtained from the abacus display for $\lambda^{(a)}$ by moving one bead $h_x^{\boldsymbol\lambda}$ positions to the left *from* runner r, and the other bead h_x^{λ} positions to the right to runner r, where $r = f_x^{\lambda} + h_x^{\lambda} - 1$.

Case 1: $e \nmid h_{(i,j,a)}^{\boldsymbol\lambda}$. By Lemma 3.2 and the remarks above, the beads on the abacus displays of $\lambda^{(a)}$ and $\mu^{(a)}$ are being moved between different runners. Therefore, the nodes $x = (i, j, a) \in [\boldsymbol\lambda]$ and $y = (k, l, a) \in [\boldsymbol\mu]$ are uniquely determined by the conditions $\operatorname{res}(f_x^{\boldsymbol\lambda}) = \operatorname{res}(f_y^{\boldsymbol\mu})$ and $[\boldsymbol\lambda] \setminus r_x^{\boldsymbol\lambda} = [\boldsymbol\mu] \setminus r_y^{\boldsymbol\mu}$. Let $m = \mu_k^{(a)}$. Then $h_{(i,m,a)}^{\boldsymbol\lambda} = (j - \lambda_i^{(a)'}) - (l - \mu_k^{(a)'})$ is the 'axial distance' from $f_x^{\boldsymbol\lambda}$ to $f_y^{\boldsymbol\mu}$, so that $e \mid h_{(i,m,a)}^{\boldsymbol\lambda}$. (In fact, $h_{(i,m,a)}^{\boldsymbol\lambda}$ is the axial distance between the corresponding hand nodes, but this distance is, of course, the same. Note also that, since $\operatorname{res}(f_x^{\boldsymbol\lambda}) = \operatorname{res}(f_y^{\boldsymbol\mu})$, we have that $e \mid h_{(i,m,a)}^{\boldsymbol\lambda}$.) Hence, $J_{\boldsymbol\lambda\boldsymbol\mu} = (-1)^{\ell\ell(r_x^{\boldsymbol\lambda}) + \ell\ell(r_y^{\boldsymbol\mu})} \nu_p'(h_{(i,m,a)}^{\boldsymbol\lambda})$ by Definition 2.5 and Lemma 3.4.

Case 2: $e \mid h^{\boldsymbol{\lambda}}_{(i,j,a)}$. Since $h^{\boldsymbol{\lambda}}_x \equiv 0 \pmod e$ unwrapping $r^{\boldsymbol{\lambda}}_x$ from $\boldsymbol{\lambda}$ and wrapping $r^{\boldsymbol{\mu}}_y$ back onto $\boldsymbol{\lambda} \setminus r^{\boldsymbol{\lambda}}_x$ correspond to moving one bead on runner $\operatorname{res}(f^{\boldsymbol{\lambda}}_x) - 1$ up $\frac{1}{e}h^{\boldsymbol{\lambda}}_x$ rows and another bead on runner $\operatorname{res}(f^{\boldsymbol{\lambda}}_x) - 1$ down $\frac{1}{e}h^{\boldsymbol{\lambda}}_x$ rows. If in the abacus display for $\boldsymbol{\lambda}$ these beads were moved from rows $r_1 > r_2$ to rows r'_1 and r'_2, respectively, then the abacus display for $\boldsymbol{\mu}$ can also be obtained from abacus display for $\boldsymbol{\lambda}$ by moving the bead in row r_1 to row r'_2 and moving the bead in row r_2 to row r'_1. That is, there exist nodes $x' \neq x$ and $y' \neq y$ such that we can obtain $\boldsymbol{\mu}$ by unwrapping $r^{\boldsymbol{\lambda}}_{x'}$ from $\boldsymbol{\lambda}$ and wrapping $r^{\boldsymbol{\mu}}_{y'}$ back onto $\boldsymbol{\lambda} \setminus r^{\boldsymbol{\lambda}}_{x'}$. By Lemma 3.2 there are no other ways of obtaining $\boldsymbol{\mu}$ by unwrapping a rim hook from $\boldsymbol{\lambda}$ and wrapping it back on again. Since $\boldsymbol{\lambda} \rhd \boldsymbol{\mu}$ we can choose the nodes $x = (i, j, a)$ and $y = (k, l, a)$ above so that $r_1 > r'_1 > r'_2 > r_2$. Then $x' = (i, m, a)$, where $m = \mu^{(a)}_k$, and $y' = (\lambda^{(a)'}_j, l, a)$. Further,

$$\ell\ell(r^{\boldsymbol{\lambda}}_x) + \ell\ell\big(r^{\boldsymbol{\mu}}_y\big) = \lambda^{(a)'}_j - i + \mu^{(a)'}_l - k \quad \text{and} \quad \ell\ell\big(r^{\boldsymbol{\lambda}}_{x'}\big) + \ell\ell\big(r^{\boldsymbol{\mu}}_{y'}\big) = \lambda^{(a)'}_m - i + \mu^{(a)'}_l - \lambda^{(a)'}_j.$$

By construction, $k = \lambda^{(a)'}_m + 1$, so $\ell\ell(r^{\boldsymbol{\lambda}}_x) + \ell\ell(r^{\boldsymbol{\mu}}_y)$ and $\ell\ell(r^{\boldsymbol{\lambda}}_{x'}) + \ell\ell(r^{\boldsymbol{\mu}}_{y'})$ have opposite parities. The axial distance from $f^{\boldsymbol{\lambda}}_x$ to $f^{\boldsymbol{\mu}}_y$ is $h^{\boldsymbol{\lambda}}_{(i,m,a)}$ (where $e \mid h^{\boldsymbol{\lambda}}_{(i,m,a)}$ since $\operatorname{res}(f^{\boldsymbol{\lambda}}_x) = \operatorname{res}(f^{\boldsymbol{\mu}}_y)$) and the axial distance from $f^{\boldsymbol{\lambda}}_{x'}$ to $f^{\boldsymbol{\mu}}_{y'}$ is $h^{\boldsymbol{\lambda}}_{(i,j,a)}$. Therefore,

$$J_{\boldsymbol{\lambda}\boldsymbol{\mu}} = (-1)^{\ell\ell(r^{\boldsymbol{\lambda}}_x)+\ell\ell(r^{\boldsymbol{\mu}}_y)}\big(\nu'_p\big(h^{\boldsymbol{\lambda}}_{(i,m,a)}\big) - \nu'_p\big(h^{\boldsymbol{\lambda}}_{(i,j,a)}\big)\big)$$

as required.

We have now exhausted all of the cases where $J_{\boldsymbol{\lambda}\boldsymbol{\mu}}$ is non-zero, so the proposition is proved. $\square$

3.3. *Residue equivalence*

We are now ready to start proving that $\boldsymbol{\lambda} \sim_J \boldsymbol{\mu}$ whenever $\boldsymbol{\lambda} \sim_C \boldsymbol{\mu}$.

A rim hook of $\boldsymbol{\lambda}$ is *vertical* if it is contained within a single column of $[\boldsymbol{\lambda}]$.

A partition λ is an (e, p)-*Carter partition* if it has the property that

$$\nu'_p\big(h^{\lambda}_{(i,m,1)}\big) = \nu'_p\big(h^{\lambda}_{(i,j,1)}\big), \quad \text{for all } (i, m, 1), (i, j, 1) \in [\lambda].$$

These partitions arise because $\Delta(\lambda)$ is irreducible if and only if λ is (e, p)-irreducible. The (e, p)-Carter partitions are described explicitly in [23, Theorem 5.45]. For us the most important properties of these partitions are that if λ is an (e, p)-Carter partition then:

- all of the e-hooks which can be unwrapped from λ when constructing its e-core $\bar{\lambda}$ are vertical;
- ν'_p is constant on the rows of $[\lambda]$; and
- $\bar{\lambda}'_i \equiv \bar{\lambda}'_{i-1} - 1 \pmod e$ whenever $\lambda'_i \neq \bar{\lambda}'_i$.

3.7. Proposition. *Suppose that* $\boldsymbol{\lambda} \in \Lambda^+_{r,n}$ *and* $1 \leqslant a \leqslant r$. *Define*

$$\Lambda_a(\boldsymbol{\lambda}) = \big\{\boldsymbol{\mu} \in \Lambda^+_{r,n} \bigm| \overline{\mu^{(a)}} = \overline{\lambda^{(a)}} \text{ and } \mu^{(c)} = \lambda^{(c)} \text{ when } c \neq a\big\}.$$

Then $\boldsymbol{\lambda} \sim_J \boldsymbol{\mu}$ *for all* $\boldsymbol{\mu} \in \Lambda_a(\boldsymbol{\lambda})$.

Proof. Suppose that $\boldsymbol{\mu} \in \Lambda_a(\boldsymbol{\lambda})$. If $e = \infty$ then $\overline{\lambda^{(a)}} = \overline{\mu^{(a)}}$ if and only if $\lambda^{(a)} = \mu^{(a)}$ so there is nothing to prove. Assume then that e is finite and let $w_a = \mathrm{w}_e(\lambda^{(a)})$. If $w_a = 0$ then $\overline{\lambda^{(a)}} = \lambda^{(a)}$ so that $\boldsymbol{\lambda} = \boldsymbol{\mu}$ and there is nothing to prove. So we can assume that $w_a > 0$.

Let $\boldsymbol{\rho}$ be the multipartition in $\Lambda_a(\boldsymbol{\lambda})$ where $\rho^{(a)}$ is the partition obtained by wrapping w_a vertical e-hooks onto the first column of the e-core of $\lambda^{(a)}$. Then $\boldsymbol{\mu} \trianglerighteq \boldsymbol{\rho}$ for all $\boldsymbol{\mu} \in \Lambda_a(\boldsymbol{\lambda})$. To prove the lemma it is enough to show that $\boldsymbol{\mu} \sim_J \boldsymbol{\rho}$, for all $\boldsymbol{\mu} \in \Lambda_a(\boldsymbol{\lambda})$. By induction on dominance we may assume that $\boldsymbol{\nu} \sim_J \boldsymbol{\rho}$ whenever $\boldsymbol{\nu} \in \Lambda_a(\boldsymbol{\lambda})$ and $\boldsymbol{\mu} \rhd \boldsymbol{\nu}$. If $J_{\boldsymbol{\mu}\boldsymbol{\nu}} \neq 0$ for some $\boldsymbol{\nu} \in \Lambda_a(\boldsymbol{\lambda})$ then $\boldsymbol{\mu} \sim_J \boldsymbol{\nu}$. As $\boldsymbol{\mu} \rhd \boldsymbol{\nu}$, we have that $\boldsymbol{\nu} \sim_J \boldsymbol{\rho}$ by induction, so that $\boldsymbol{\mu} \sim_J \boldsymbol{\nu} \sim_J \boldsymbol{\rho}$.

It remains to consider the case when $\boldsymbol{\mu} \rhd \boldsymbol{\rho}$ and $J_{\boldsymbol{\mu}\boldsymbol{\nu}} = 0$ for all $\boldsymbol{\nu} \in \Lambda_a(\boldsymbol{\lambda})$. By Lemma 3.6(b),

$$\nu_p'\big(h^{\boldsymbol{\mu}}_{(i,m,a)}\big) = \nu_p'\big(h^{\boldsymbol{\mu}}_{(i,j,a)}\big), \quad \text{for all } (i,m,a), (i,j,a) \in [\boldsymbol{\mu}],$$

so that $\mu^{(a)}$ is an (e,p)-Carter partition. Since $w_a > 0$ we can find a (unique) node $(i,j,a) \in [\boldsymbol{\mu}]$ such that

$$h^{\mu}_{(i,j,a)} \equiv 0 \pmod e \qquad \text{and} \qquad h^{\mu}_{(i',j',a)} \not\equiv 0 \pmod e,$$

for all $(i',j',a) \in [\boldsymbol{\mu}]$ with $(i',j') \neq (i,j)$, $i' \leqslant i$ and $j' \geqslant j$. Let $\boldsymbol{\nu}$ be the multipartition obtained by unwrapping $r^{\boldsymbol{\mu}}_{(i,j,a)}$ from $[\boldsymbol{\mu}]$ and wrapping it back on to the end of the first row of $[\boldsymbol{\mu}] \setminus r^{\boldsymbol{\mu}}_{(i,j,a)}$. Similarly, let $\boldsymbol{\eta}$ be the multipartition obtained by unwrapping this same hook from $\boldsymbol{\mu}$ and wrapping it back on to the end of the first column of $[\boldsymbol{\mu}] \setminus r^{\boldsymbol{\mu}}_{(i,j,a)}$. Therefore, $J_{\boldsymbol{\nu}\boldsymbol{\mu}} \neq 0$ and $J_{\boldsymbol{\nu}\boldsymbol{\eta}} \neq 0$, by Lemma 3.6(b), so that $\boldsymbol{\mu} \sim_J \boldsymbol{\nu} \sim_J \boldsymbol{\eta}$. Note that $\boldsymbol{\mu} \rhd \boldsymbol{\rho}$ implies that $j > 1$, so that $\boldsymbol{\mu} \rhd \boldsymbol{\eta}$. Consequently, $\boldsymbol{\mu} \sim_J \boldsymbol{\rho}$ by induction. □

Recall that the e-cores of the partitions of n completely determine the blocks when $r = 1$. We have the following imperfect generalization when $r > 1$.

3.8. Definition. Suppose that $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)})$ is a multipartition. Then the *e-multicore* of $\boldsymbol{\lambda}$ is the multipartition $\bar{\boldsymbol{\lambda}} = (\bar{\lambda}^{(1)}, \dots, \bar{\lambda}^{(r)})$. We abuse notation and say that $\boldsymbol{\lambda}$ is a multicore if $\boldsymbol{\lambda} = \bar{\boldsymbol{\lambda}}$.

By Corollary 3.3(a), the e-multicore $\bar{\boldsymbol{\lambda}}$ of $\boldsymbol{\lambda}$ is obtained from $\boldsymbol{\lambda}$ by sequentially unwrapping all e-rim hooks from the diagram of $\boldsymbol{\lambda}$, in any order. Note that if $e = \infty$ then every multipartition is an e-multicore.

Mimicking the representation theory of the symmetric groups, we extend the definition of w_e to multipartitions by defining $\mathrm{w}_e(\boldsymbol{\lambda})$ to be the number of e-hooks that have to be unwrapped from $\boldsymbol{\lambda}$ to construct $\bar{\boldsymbol{\lambda}}$. If e is finite then $\mathrm{w}_e(\boldsymbol{\lambda}) = \frac{1}{e}(|\boldsymbol{\lambda}| - |\bar{\boldsymbol{\lambda}}|)$, whereas $\mathrm{w}_\infty(\boldsymbol{\lambda}) = 0$. Now define

$$\mathrm{W}_e(\boldsymbol{\lambda}) = \max\big\{\mathrm{w}_e(\boldsymbol{\mu}) \bigm| \boldsymbol{\mu} \sim_C \boldsymbol{\lambda}\big\}.$$

Note that while $\mathrm{W}_e(\boldsymbol{\lambda})$ is well defined, it is not immediately clear how to compute it.

3.9. Lemma. *Suppose that $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \Lambda^+_{r,n}$ and that $\bar{\boldsymbol{\lambda}} = \bar{\boldsymbol{\mu}}$. Then $\boldsymbol{\lambda} \sim_J \boldsymbol{\mu}$.*

Proof. We argue by induction on $d(\boldsymbol{\lambda}, \boldsymbol{\mu})$, where

$$d(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \frac{1}{e^2} \sum_{a=1}^{r} \big(|\lambda^{(a)}| - |\mu^{(a)}|\big)^2.$$

Note that $d(\boldsymbol{\lambda},\boldsymbol{\mu})$ is a non-negative integer because our assumption $\bar{\boldsymbol{\lambda}}=\bar{\boldsymbol{\mu}}$ implies that $|\lambda^{(a)}|\equiv|\mu^{(a)}| \pmod e$, for $1\leqslant a\leqslant r$.

Suppose first that $d(\boldsymbol{\lambda},\boldsymbol{\mu})=0$. Then $|\lambda^{(a)}|=|\mu^{(a)}|$, for $1\leqslant a\leqslant r$. Define a sequence of multipartitions $\boldsymbol{\nu}_0=\boldsymbol{\lambda},\boldsymbol{\nu}_1,\dots,\boldsymbol{\nu}_r=\boldsymbol{\mu}$ by setting

$$\boldsymbol{\nu}_i^{(j)}=\begin{cases}\lambda^{(j)}, & i<j,\\ \mu^{(j)}, & i\geqslant j.\end{cases}$$

Then $\boldsymbol{\nu}_i\sim_J\boldsymbol{\nu}_{i+1}$ for $0\leqslant i<r$, by Proposition 3.7, so that $\boldsymbol{\lambda}\sim_J\boldsymbol{\mu}$ by transitivity.

Now suppose that $d(\boldsymbol{\lambda},\boldsymbol{\mu})>0$. Since $\bar{\boldsymbol{\lambda}}=\bar{\boldsymbol{\mu}}$ and $|\boldsymbol{\lambda}|=|\boldsymbol{\mu}|$, there exist integers b and c such that $|\lambda^{(b)}|<|\mu^{(b)}|$ and $|\lambda^{(c)}|>|\mu^{(c)}|$. By Corollary 3.3 it is possible to construct a new multipartition $\boldsymbol{\nu}$ by unwrapping an e-hook from $\lambda^{(c)}$ and wrapping it back on to $\lambda^{(b)}$ without changing the residue of the foot node. Then $\boldsymbol{\lambda}\sim_J\boldsymbol{\nu}$ by Proposition 3.6 (and Lemma 3.2). Moreover, $\bar{\boldsymbol{\nu}}=\bar{\boldsymbol{\lambda}}=\bar{\boldsymbol{\mu}}$ and $d(\boldsymbol{\nu},\boldsymbol{\mu})<d(\boldsymbol{\lambda},\boldsymbol{\mu})$. Therefore, $\boldsymbol{\nu}\sim_J\boldsymbol{\mu}$ by induction, so that $\boldsymbol{\lambda}\sim_J\boldsymbol{\mu}$ as required. □

We now need several results and definitions of Fayers from the papers [14,15]. It should be noted that there is a certain symbiosis between these two papers and the present paper because Fayers wrote his papers believing that the classification of the blocks of the Ariki–Koike algebras had already been established. Fortunately, Fayers' results do not depend on the block classification so when he discovered that there was a gap in the previous proof of the classification he changed his papers so that they now refer to 'combinatorial blocks,' or residue classes of multipartitions. Thanks to the main result of this paper, the 'combinatorial blocks' studied by Fayers are indeed blocks.

3.10. Definition.

(a) (Fayers [15]) Suppose that $\boldsymbol{\lambda}$ is a multicore and, if $e=\infty$, suppose further that the abacus display for $\lambda^{(a)}$ contains a bead in position i but not in position j, while the abacus display for $\lambda^{(b)}$ contains a bead in position j but not in position i. Define $s_{ij}^{ab}(\boldsymbol{\lambda})$ to be the multicore whose abacus display is obtained by moving a bead from runner i to runner j on the abacus for $\lambda^{(a)}$ and moving a bead from runner j to runner i on the abacus for $\lambda^{(b)}$.

(b) Suppose that e is finite and let $\boldsymbol{\lambda}$ be a multipartition. Define $t_{iw}^{a}(\boldsymbol{\lambda})$ to be the multipartition whose abacus display is obtained by moving the lowest bead on runner i of the abacus for $\lambda^{(a)}$ down w rows.

3.11. Lemma. *Suppose that $\boldsymbol{\lambda}\sim_C\boldsymbol{\mu}$ and that $\bar{\boldsymbol{\mu}}=s_{ij}^{ab}(\bar{\boldsymbol{\lambda}})$. Then $\boldsymbol{\lambda}\sim_J\boldsymbol{\mu}$.*

Proof. Let $\boldsymbol{\nu}=t^a_{i\mathrm{w}_e(\boldsymbol{\lambda})}(\bar{\boldsymbol{\lambda}})$ and $\boldsymbol{\rho}=t^a_{j\mathrm{w}_e(\boldsymbol{\mu})}(\bar{\boldsymbol{\mu}})$. Then $\boldsymbol{\lambda}\sim_J\boldsymbol{\nu}$ and $\boldsymbol{\rho}\sim_J\boldsymbol{\mu}$ by Lemma 3.9. Furthermore, the multipartitions $\boldsymbol{\nu}$ and $\boldsymbol{\rho}$ satisfy the conditions of Proposition 3.6(a), so $\boldsymbol{\lambda}\sim_J\boldsymbol{\nu}\sim_J\boldsymbol{\rho}\sim_J\boldsymbol{\mu}$ as required. □

3.12. Definition. (See Fayers [14, §2.1].) Suppose that $\boldsymbol{\lambda}$ is a multipartition. Then the *e-weight* of $\boldsymbol{\lambda}$ is the integer

$$\mathrm{wt}(\boldsymbol{\lambda})=\sum_{j=1}^{r}C_{c_j}(\boldsymbol{\lambda})-\frac{1}{2}\sum_{f\in\mathbb{Z}/e\mathbb{Z}}\big(C_f(\boldsymbol{\lambda})-C_{f+1}(\boldsymbol{\lambda})\big)^2.$$

Fayers [14] shows that $\mathrm{wt}(\boldsymbol{\lambda}) \geqslant 0$ for all multipartitions $\boldsymbol{\lambda}$, and that $\mathrm{wt}(\lambda) = \mathrm{w}_e(\lambda)$ when $r = 1$; that is, Fayers' definition of weight coincides with the usual definition of weight on the set of partitions. Further, if $\boldsymbol{\lambda} \sim_C \boldsymbol{\mu}$ then $\mathrm{wt}(\boldsymbol{\lambda}) = \mathrm{wt}(\boldsymbol{\mu})$, so the function $\mathrm{wt}(\cdot)$ is constant on the residue classes of $\Lambda^+_{r,n}$. The results of [14, Proposition 3.8] show how to use the abacus display of $\boldsymbol{\lambda}$ to calculate $\mathrm{wt}(\boldsymbol{\lambda})$. Combining this method with Lemma 3.16 below gives a way of computing $\mathrm{W}_e(\boldsymbol{\lambda})$ using the abacus display of $\boldsymbol{\lambda}$. We leave the details to the reader.

Recall that a node $(i, j, a) \in [\boldsymbol{\lambda}]$ is *removable* if $[\boldsymbol{\lambda}] \setminus \{(i, j, a)\}$ is the diagram of some multipartition in $\Lambda^+_{r,n-1}$. Similarly, a node $(i, j, a) \notin [\boldsymbol{\lambda}]$ is *addable* if $[\boldsymbol{\lambda}] \cup \{(i, j, a)\}$ is the diagram of some multipartition in $\Lambda^+_{r,n+1}$. The node $x = (i, j, a)$ is an f-node if $\mathrm{res}(x) = f$.

Let $\boldsymbol{\lambda}$ be a multipartition. For $f \in \mathbb{Z}/e\mathbb{Z}$ and $a \in \{1, \dots, r\}$, define

$$\delta^a_f(\boldsymbol{\lambda}) = \#\{\text{removable } f\text{-nodes of } [\lambda^{(a)}]\} - \#\{\text{addable } f\text{-nodes of } [\lambda^{(a)}]\}$$

and set

$$\delta_f(\boldsymbol{\lambda}) = \sum_{j=1}^{r} \delta^j_f(\boldsymbol{\lambda}).$$

The sequence $(\delta_f(\boldsymbol{\lambda}) \mid f \in \mathbb{Z}/e\mathbb{Z})$ is the *hub* of $\boldsymbol{\lambda}$. The hub of $\boldsymbol{\lambda}$ can be read off the abacus display of $\boldsymbol{\lambda}$ using Lemma 3.2.

Observe that Corollary 3.3 implies that if e is finite then the hub is unchanged by wrapping he-hooks onto $[\boldsymbol{\lambda}]$, for $h \geqslant 1$. Furthermore, $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ have the same hub if $\boldsymbol{\mu} = s^{ab}_{ij}(\boldsymbol{\lambda})$, for some a, b, i, j.

3.13. Proposition. *(See Fayers [14, Proposition 3.2 and Lemma 3.3].) Suppose that $\boldsymbol{\lambda}$ is a multipartition of n and $\boldsymbol{\mu}$ is a multipartition of m. Then*

(a) *If $e < \infty$ and $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ have the same hub then $m \equiv n \bmod e$ and*

$$\mathrm{wt}(\boldsymbol{\lambda}) - \mathrm{wt}(\boldsymbol{\mu}) = \frac{r(n-m)}{e};$$

(b) *If $n = m$ then $\boldsymbol{\lambda} \sim_C \boldsymbol{\mu}$ if and only if they have the same hub.*

Consequently, if $\boldsymbol{\mu}$ is obtained from $\boldsymbol{\lambda}$ by wrapping on an e-hook, then $\mathrm{wt}(\boldsymbol{\mu}) = \mathrm{wt}(\boldsymbol{\lambda}) + r$.

The next result will let us determine when $\mathrm{W}_e(\boldsymbol{\lambda}) = \mathrm{w}_e(\boldsymbol{\lambda})$.

3.14. Proposition. *(See Fayers [15, Theorem 3.1].) Suppose that $\boldsymbol{\lambda} \in \Lambda^+_{r,n}$ is a multipartition. Then the following are equivalent.*

(a) *$\boldsymbol{\mu}$ is a multicore whenever $\boldsymbol{\mu} \sim_C \boldsymbol{\lambda}$.*
(b) $\mathrm{wt}(\boldsymbol{\mu}) \geqslant \mathrm{wt}(\boldsymbol{\lambda})$ *whenever $\boldsymbol{\mu}$ and $\boldsymbol{\lambda}$ have the same hub.*

3.15. Definition. A multipartition $\boldsymbol{\lambda}$ is a *reduced multicore* if it satisfies the conditions of Proposition 3.14.

Not every multicore is reduced. If $\boldsymbol{\lambda}$ is a reduced multicore then the block which contains $\Delta(\boldsymbol{\lambda})$ is, in general, not simple. In contrast, when $r = 1$ every core is a reduced multicore and the block containing a core is always simple. If $\boldsymbol{\lambda}$ is an reduced multicore then Fayers [15] calls the set of multipartitions $\{\boldsymbol{\mu} \mid \boldsymbol{\mu} \sim_C \boldsymbol{\lambda}\}$ a 'core block.'

3.16. Lemma. *Suppose that $\boldsymbol{\lambda} \in \Lambda^+_{n,r}$. Then $\bar{\boldsymbol{\lambda}}$ is a reduced multicore if and only if* $\mathrm{w}_e(\boldsymbol{\lambda}) = \mathrm{W}_e(\boldsymbol{\lambda})$.

Proof. Suppose $\mathrm{w}_e(\boldsymbol{\lambda}) \neq \mathrm{W}_e(\boldsymbol{\lambda})$. By definition, there exists a multipartition $\boldsymbol{\mu}$ such that $\boldsymbol{\mu} \sim_C \boldsymbol{\lambda}$ and $\mathrm{w}_e(\boldsymbol{\mu}) > \mathrm{w}_e(\boldsymbol{\lambda})$. Now $\bar{\boldsymbol{\mu}}$ and $\bar{\boldsymbol{\lambda}}$ have the same hub, and by Proposition 3.13, $\mathrm{wt}(\bar{\boldsymbol{\mu}}) < \mathrm{wt}(\bar{\boldsymbol{\lambda}})$, contradicting condition (b) of Proposition 3.14. Therefore, $\bar{\boldsymbol{\lambda}}$ is not a reduced multicore.

Now suppose that $\bar{\boldsymbol{\lambda}}$ is not a reduced multicore. Then there exists a multipartition $\boldsymbol{\mu}$, which is not a multicore, such that $\boldsymbol{\mu} \sim_C \bar{\boldsymbol{\lambda}}$. Let $\boldsymbol{\nu} = t^1_{0\mathrm{w}_e(\boldsymbol{\lambda})}(\boldsymbol{\mu})$. Then $\boldsymbol{\nu} \sim_C \boldsymbol{\lambda}$ and $\mathrm{w}_e(\boldsymbol{\nu}) > \mathrm{w}_e(\boldsymbol{\lambda})$. Hence, $\mathrm{W}_e(\boldsymbol{\lambda}) > \mathrm{w}_e(\boldsymbol{\lambda})$. □

3.17. Lemma. *(See Fayers [15, Proof of Proposition 3.7(1)].) Suppose that $\boldsymbol{\lambda}$ is a multicore which is not reduced. Then there exists a sequence of multicores $\boldsymbol{\lambda}_0 = \boldsymbol{\lambda}, \boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_k = \boldsymbol{\mu}$ such that* $\mathrm{wt}(\boldsymbol{\mu}) < \mathrm{wt}(\boldsymbol{\lambda})$, *and $\boldsymbol{\lambda}_{m+1} = s^{a_m b_m}_{i_m j_m}(\boldsymbol{\lambda}_m)$ and* $\mathrm{wt}(\boldsymbol{\lambda}_m) \leqslant \mathrm{wt}(\boldsymbol{\lambda})$, *for $0 \leqslant m < k$.*

3.18. Lemma. *(See Fayers [15, Proof of Proposition 3.7(2)].) Suppose that $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are reduced multicores and that $\boldsymbol{\lambda} \sim_C \boldsymbol{\mu}$. Then there exists a sequence of multicores $\boldsymbol{\lambda}_0 = \boldsymbol{\lambda}, \boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_k = \boldsymbol{\mu}$ such that $\boldsymbol{\lambda}_{m+1} = s^{a_m b_m}_{i_m j_m}(\boldsymbol{\lambda}_m)$ and $\boldsymbol{\lambda}_{m+1} \sim_C \boldsymbol{\lambda}_m$, for $0 \leqslant m < k$.*

We can now complete the proof of Theorem 2.11 when $q \neq 1$ and the parameters $Q_1, \dots, Q_r$ are non-zero. Consequently, this completes the proofs of Theorem A from the introduction.

3.19. Theorem. *Suppose that $q \neq 1$ and that the parameters $Q_1, \dots, Q_r$ are non-zero. Let $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ be multipartitions in $\Lambda^+_{n,r}$. Then $\boldsymbol{\lambda} \sim_C \boldsymbol{\mu}$ if and only if $\boldsymbol{\lambda} \sim_J \boldsymbol{\mu}$.*

Proof. By Corollary 3.5 if $\boldsymbol{\lambda} \sim_J \boldsymbol{\mu}$ then $\boldsymbol{\lambda} \sim_C \boldsymbol{\mu}$. Suppose then that $\boldsymbol{\lambda} \sim_C \boldsymbol{\mu}$. To show that $\boldsymbol{\lambda} \sim_J \boldsymbol{\mu}$ it is sufficient to prove the following two statements. Let $\boldsymbol{\nu} \in \Lambda^+_{r,n}$.

(a) Suppose that $\mathrm{w}_e(\boldsymbol{\nu}) < \mathrm{W}_e(\boldsymbol{\nu})$. Then there exists $\boldsymbol{\eta} \in \Lambda^+_{r,n}$ such that $\boldsymbol{\eta} \sim_J \boldsymbol{\nu}$ and $\mathrm{w}_e(\boldsymbol{\eta}) > \mathrm{w}_e(\boldsymbol{\nu})$.
(b) Suppose that $\boldsymbol{\nu} \sim_C \boldsymbol{\eta}$ and that $\mathrm{w}_e(\boldsymbol{\nu}) = \mathrm{W}_e(\boldsymbol{\nu}) = \mathrm{w}_e(\boldsymbol{\eta})$. Then $\boldsymbol{\eta} \sim_J \boldsymbol{\nu}$.

Suppose, as in (a), that $\mathrm{w}_e(\boldsymbol{\nu}) < \mathrm{W}_e(\boldsymbol{\nu})$. Then e is finite and by Lemma 3.16, $\bar{\boldsymbol{\nu}}$ is not a reduced multicore. By Lemma 3.17, there exists a sequence of multicores $\boldsymbol{\nu}_0 = \bar{\boldsymbol{\nu}}, \boldsymbol{\nu}_1, \dots, \boldsymbol{\nu}_k$ such that $\mathrm{wt}(\boldsymbol{\nu}_k) < \mathrm{wt}(\bar{\boldsymbol{\nu}})$ and for $0 \leqslant m < k$ we have

$$\boldsymbol{\nu}_{m+1} = s^{a_m b_m}_{i_m j_m}(\boldsymbol{\nu}_m) \quad \text{and} \quad \mathrm{wt}(\boldsymbol{\nu}_m) \leqslant \mathrm{wt}(\bar{\boldsymbol{\nu}}).$$

For all m with $0 \leqslant m \leqslant k$, we have that $\boldsymbol{\nu}_m$ and $\bar{\boldsymbol{\nu}}$ have the same hub, so Proposition 3.13 says that $|\boldsymbol{\nu}_m| \leqslant |\bar{\boldsymbol{\nu}}|$, that $|\bar{\boldsymbol{\nu}}| \equiv |\boldsymbol{\nu}_m| \pmod e$ and that $|\boldsymbol{\nu}_k| < |\bar{\boldsymbol{\nu}}|$. Define

$$w_m = \mathrm{w}_e(\boldsymbol{\nu}) + \frac{1}{e}\big(|\bar{\boldsymbol{\nu}}| - |\boldsymbol{\nu}_m|\big)$$

and set

$$\boldsymbol{\eta}_m = t^1_{0w_m}(\boldsymbol{\nu}_m) \quad \text{and} \quad \boldsymbol{\eta} = \boldsymbol{\eta}_k.$$

Then $\boldsymbol{\eta}_m \sim_J \boldsymbol{\eta}_{m+1}$, for $0 \leqslant m < k$, by Lemma 3.11, so that by Lemma 3.7 and transitivity, $\boldsymbol{\nu} \sim_J \boldsymbol{\eta}_0 \sim_J \boldsymbol{\eta}_m = \boldsymbol{\eta}$. Moreover, $\mathrm{w}_e(\boldsymbol{\eta}) = \mathrm{w}_e(\boldsymbol{\nu}) + \frac{1}{e}(|\bar{\boldsymbol{\nu}}| - |\boldsymbol{\nu}_k|) > \mathrm{w}_e(\boldsymbol{\nu})$ as required.

Now consider (b), that is, suppose that $\boldsymbol{\nu} \sim_C \boldsymbol{\eta}$ and $\mathrm{w}_e(\boldsymbol{\nu}) = \mathrm{W}_e(\boldsymbol{\nu}) = \mathrm{w}_e(\boldsymbol{\eta})$. By Lemma 3.16, $\bar{\boldsymbol{\nu}}$ and $\bar{\boldsymbol{\eta}}$ are reduced multicores. Then, by Lemma 3.18, there exist multicores $\boldsymbol{\nu}_0 = \bar{\boldsymbol{\nu}}, \boldsymbol{\nu}_1, \dots, \boldsymbol{\nu}_k = \bar{\boldsymbol{\eta}}$ such that

$$\boldsymbol{\nu}_{m+1} = s^{a_m b_m}_{i_m j_m}(\boldsymbol{\nu}_m) \quad \text{and} \quad \boldsymbol{\nu}_{m+1} \sim_C \boldsymbol{\nu}_m \quad \text{for } 0 \leqslant m < k.$$

For $0 \leqslant m \leqslant k$, define $\boldsymbol{\xi}_m = t^1_{0\mathrm{w}_e(\boldsymbol{\nu})}(\boldsymbol{\nu}_m)$. Then by Lemma 3.11, $\boldsymbol{\xi}_m \sim_J \boldsymbol{\xi}_{m+1}$ and by Lemma 3.7 and transitivity, $\boldsymbol{\nu} \sim_J \boldsymbol{\xi}_0 \sim_J \boldsymbol{\xi}_k \sim_J \boldsymbol{\eta}$ as required. □

4. The blocks for algebras with exceptional parameters

In this section we classify the blocks of the Ariki–Koike algebras for the remaining cases from 2.13. That is, we assume that the parameters satisfy one of the following four cases:

Case 2. $r = 1$ and $q = 1$ (and Q_1 arbitrary).
Case 3. $r > 1$, $q = 1$ and $Q_1 = \cdots = Q_r = 1$.
Case 4. $r > 1$, $q = 1$ and $Q_1 = \cdots = Q_r = 0$.
Case 5. $r > 1$, $q \neq 1$ and $Q_1 = \cdots = Q_r = 0$.

As in the previous section the basic strategy is to use the Jantzen sum formula to analyze the combinatorics of the Jantzen coefficients.

We distinguish between Cases 2 and 3 because the blocks differ dramatically in these two cases. In fact, the blocks in Case 2 behave like the blocks when $q \neq 1$ and the parameters $Q_1, \dots, Q_r$ are non-zero. Quite surprisingly, the algebras $\mathscr{H}_{r,n}$ and $\mathscr{S}_{r,n}$ have only one block in Cases 3–5.

In all cases the blocks of the algebras $\mathscr{H}_{r,n}$ and $\mathscr{S}_{r,n}$ are determined by the Jantzen equivalence by Proposition 2.9. This section gives an explicit description of when two multipartitions are Jantzen equivalent in Cases 2–5 above.

4.1. The blocks when $r = 1$ and $q = 1$

Assume that we are in Case 2 above and let $\mathscr{H}_n = \mathscr{H}_{1,n}$ and $\mathscr{S}_n = \mathscr{S}_{1,n}$. In this case the Specht modules and Weyl modules are indexed by partitions, rather than multipartitions, so we write λ in place of $\boldsymbol{\lambda}$, and so on. The nodes in the diagrams of partitions are all of the form $(i, j, 1)$, for $i, j \geqslant 1$, so we drop the trailing 1 from this notation and consider a node to be an ordered pair (i, j), so that $[\lambda] = \{(i, j) \mid 1 \leqslant j \leqslant \lambda_i\}$.

As $q = 1$ we have that $e = p$. Following Section 3 define the *residue* of a node $x = (i, j)$ to be

$$\mathrm{res}(x) = (j - i) \pmod{p}.$$

Once again, $\{\operatorname{res}(x) \mid x \in [\boldsymbol{\lambda}] \text{ for some } \boldsymbol{\lambda} \in \Lambda_{r,n}^+\} \subseteq \mathbb{Z}/p\mathbb{Z}$. For a partition λ and $f \in \mathbb{Z}/p\mathbb{Z}$ put $C_f(\boldsymbol{\lambda}) = \#\{x \in [\boldsymbol{\lambda}] \mid \operatorname{res}(x) = f\}$ and define $\lambda \sim_C \mu$ if $C_f(\lambda) = C_f(\mu)$, for all $f \in \mathbb{Z}/p\mathbb{Z}$. Then it is well known (and easy to prove using Corollary 3.3(a)) that $\boldsymbol{\lambda} \sim_C \boldsymbol{\mu}$ if and only if λ and μ have the same p-core.

We can now prove Theorem 2.11 when $q = 1$ and $r = 1$. To prove this result we need to show that the Jantzen and residue equivalence relations on the set of partitions coincide. We follow the argument of the previous section.

The analogue of Lemma 3.4 in Case 2 is as follows.

4.1. Lemma. *Suppose that λ and μ are partitions of n and that $[\lambda] \setminus r_x^\lambda = [\mu] \setminus r_y^\mu$, for some nodes $x = (i, j) \in [\lambda]$ and $y = (k, l) \in [\mu]$. Then*

$$\nu_\pi\big(\operatorname{res}_{\mathcal{O}}\big(f_x^\lambda\big) - \operatorname{res}_{\mathcal{O}}\big(f_y^\mu\big)\big) = \nu'_p\big(j - \lambda'_j - l + \mu'_l\big).$$

Proof. Let $i' = \lambda'_i$ and $k' = \mu'_k$ so that $f_x^\lambda = (i', j)$ and $f_y^\mu = (k', l)$. Then

$$\operatorname{res}_{\mathcal{O}}\big(f_x^\lambda\big) - \operatorname{res}_{\mathcal{O}}\big(f_y^\mu\big) = t^{na+j-i'} - t^{na+l-k'} = t^{na+l-k'}\big(t^{j-i'-l+k'} - 1\big).$$

Mimicking the proof of Lemma 3.4, let $h = j - i' - l + k'$. Then

$$\nu_\pi\big(\operatorname{res}_{\mathcal{O}}\big(f_x^\lambda\big) - \operatorname{res}_{\mathcal{O}}\big(f_y^\mu\big)\big) = \nu_\pi\big(t^{j-i'-l+k'} - 1\big) = 1 + \nu_\pi\big([h]_t\big).$$

Repeating the second half of the proof of Lemma 3.4 completes the proof. □

The only difference between Lemmas 3.4 and 4.1 is that now $\nu_\pi(\operatorname{res}_{\mathcal{O}}(f_x^\lambda) - \operatorname{res}_{\mathcal{O}}(f_y^\mu))$ is non-zero whenever $[\lambda] \setminus r_x^\lambda = [\mu] \setminus r_y^\mu$; that is, we no longer require that $\operatorname{res}(f_x^\lambda) = \operatorname{res}(f_y^\mu)$.

4.2. Proposition. *Let λ and μ are partitions of n. Then $J_{\lambda\mu}$ is non-zero only if p is finite and there exist nodes $x = (i, j), (i, m) \in [\lambda]$ such that $m < j$, $p \mid h_{(i,m)}^\lambda$ and μ is obtained by wrapping a rim hook of length h_x^λ onto $\lambda \setminus r_x^\lambda$ with its highest node in column m. In this case*

$$J_{\lambda\mu} = \begin{cases} (-1)^{\ell\ell(r_x^\lambda)+\ell\ell(r_y^\mu)}\nu'_p(h_{(i,m)}^\lambda), & \text{if } p \nmid h_{(i,j)}^\lambda, \\ (-1)^{\ell\ell(r_x^\lambda)+\ell\ell(r_y^\mu)}(\nu'_p(h_{(i,m)}^\lambda) - \nu'_p(h_{(i,j)}^\lambda)), & \text{if } p \mid h_{(i,j)}^\lambda, \end{cases}$$

where the node $y \in [\mu]$ is determined by $[\mu] \setminus r_y^\mu = [\lambda] \setminus r_x^\lambda$.

Proof. Suppose that $J_{\lambda\mu} \neq 0$. Then $\lambda \rhd \mu$ by Definition 2.5 and there exist nodes $x = (i, j) \in [\lambda]$ and $y = (k, l, b) \in [\mu]$ such that $[\lambda] \setminus r_x^\lambda = [\mu] \setminus r_y^\mu$.

Case 1: $\operatorname{res}(f_x^\lambda) \neq \operatorname{res}(f_y^\mu)$. Unwrapping the rim hook r_x^λ from λ moves a bead on the abacus for λ from runner r_1, say, to runner $\operatorname{res}(f_x^\lambda) - 1$, and wrapping the rim hook r_y^μ back on to $\lambda \setminus r_x^\lambda$ moves a bead from runner $\operatorname{res}(f_y^\mu) - 1$ to runner r_2, say. Since $\operatorname{res}(f_x^\lambda) \neq \operatorname{res}(f_y^\mu)$ we can also construct the partition μ from λ by moving a bead from runner r_1 to runner r_2 and then moving a bead from runner $\operatorname{res}(f_x^\lambda) - 1$ to runner $\operatorname{res}(f_y^\mu) - 1$. Comparing the abacus displays of λ and μ, there are no other ways of obtaining μ from λ by moving a single rim hook. As in the proof of Proposition 3.6, the sums of the leg lengths for the two different ways of changing λ into μ

by moving a rim hook have different parities, so their contributions to $J_{\lambda\mu}$ cancel out. Hence, $J_{\lambda\mu} = 0$ when $\mathrm{res}(f_x^\lambda) \neq \mathrm{res}(f_y^\mu)$.

Case 2: $\mathrm{res}(f_x^\lambda) = \mathrm{res}(f_y^\mu)$. The proof of Proposition 3.6 in the case when $a = b$ can now be repeated without a change to complete the proof of the proposition. □

4.3. Corollary. *Suppose that λ and μ are partitions of n. Then $\lambda \sim_J \mu$ if and only if $\lambda \sim_C \mu$.*

Proof. By Proposition 4.2, $\lambda \sim_C \mu$ whenever $\lambda \sim_J \mu$. The reverse implication follows by the argument of Proposition 3.7 since this proof only uses part (b) of Proposition 3.6, which is the same as the statement of Proposition 4.2. □

Remark. Corollary 4.3 completes the classification of the blocks of the q-Schur algebras and the Hecke algebras of type A; that is when $r = 1$. Unfortunately, the classification of the blocks of the q-Schur algebras given in [20, Theorem 4.24] (and reproduced in [23, Theorem 5.47]), contains a small error. Fortunately, the classification of the blocks of the Hecke algebras of type A given in [20, Theorem 4.29] is correct—indeed, when $r = 1$ our proof is a streamlined version of this argument.

4.2. The blocks when $r > 1$ and $q = 1$ or $Q_1 = \cdots = Q_r = 0$

We now consider the blocks in the remaining cases, that is, when $r > 1$ and either $q = 1$ or $Q_1 = \cdots = Q_r = 0$. In this case all simple modules belong to the same block. We use the same strategy to prove Theorem 2.11 in these cases as in the previous sections.

Note that, in Cases 3–5, $\mathrm{res}(x) = Q_a = Q_1$ for any node $x = (i, j, a)$. Therefore, in these cases, $\Lambda_{r,n}^+$ forms a single residue class. Hence, in order to prove Theorem 2.11, we need to show that any two multipartitions in $\Lambda_{r,n}^+$ are Jantzen equivalent. Consequently, in Cases 3–5, Theorem 2.11 asserts that the algebras $\mathscr{H}_{r,n}$ and $\mathscr{S}_{r,n}$ have only one block. That is, in Cases 3–5, $\mathscr{H}_{r,n}$ and $\mathscr{S}_{r,n}$ are indecomposable algebras.

We adopt the same strategy that we used to prove Theorem 3.19. To state the analogue of Lemma 3.4 set

$$\epsilon = \begin{cases} 1, & \text{if } Q_1 = \cdots = Q_r = 0 \text{ (Cases 4 and 5)}, \\ 0, & \text{otherwise.} \end{cases}$$

4.4. Lemma. *Suppose that $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are multipartitions of n and that $[\boldsymbol{\lambda}] \setminus r_x^{\boldsymbol{\lambda}} = [\boldsymbol{\mu}] \setminus r_y^{\boldsymbol{\mu}}$, for some nodes $x = (i, j, a) \in [\boldsymbol{\lambda}]$ and $y = (k, l, b) \in [\boldsymbol{\mu}]$. Then*

$$\nu_\pi\big(\mathrm{res}_{\mathcal{O}}\big(f_x^{\boldsymbol{\lambda}}\big) - \mathrm{res}_{\mathcal{O}}\big(f_y^{\boldsymbol{\mu}}\big)\big) = \nu'_p\big(n(a-b) + j - \boldsymbol{\lambda}_i^{(a)'} - l + \boldsymbol{\mu}_k^{(b)'}\big) + \epsilon.$$

The proof of Lemma 4.4 is similar to proofs of Lemmas 3.4 and 4.1, so we leave the details to the reader. Note, in particular, that if $a \neq b$ then $\nu_\pi(\mathrm{res}_{\mathcal{O}}(f_x^{\boldsymbol{\lambda}}) - \mathrm{res}_{\mathcal{O}}(f_y^{\boldsymbol{\mu}}))$ is always non-zero since $\nu'_p(h) \geqslant 0$, for all $h \in \mathbb{Z} \setminus \{0\}$. This crucial difference leads to $J_{\boldsymbol{\lambda}\boldsymbol{\mu}}$ being non-zero whenever there exist nodes $x = (i, j, a) \in [\boldsymbol{\lambda}]$ and $y = (k, l, b) \in [\boldsymbol{\mu}]$ with $a < b$ and $[\boldsymbol{\lambda}] \setminus r_x^{\boldsymbol{\lambda}} = [\boldsymbol{\mu}] \setminus r_y^{\boldsymbol{\mu}}$. More explicitly, we have the following analogue of Propositions 3.6 and 4.2. Again, we leave details to the reader.

4.5. Proposition. *Let* $\boldsymbol{\lambda} = (\boldsymbol{\lambda}^{(1)}, \dots, \boldsymbol{\lambda}^{(r)})$ *and* $\boldsymbol{\mu} = (\boldsymbol{\mu}^{(1)}, \dots, \boldsymbol{\mu}^{(r)})$ *be multipartitions in* $\Lambda^+_{r,n}$.

(a) *Suppose that there exist integers* $a \neq b$ *such that* $\boldsymbol{\lambda}^{(c)} = \boldsymbol{\mu}^{(c)}$, *for* $c \neq a, b$. *Then* $J_{\boldsymbol{\lambda\mu}} \neq 0$ *only if* $a < b$ *and there exist nodes* $x = (i, j, a) \in [\boldsymbol{\lambda}]$ *and* $y = (k, l, b) \in [\boldsymbol{\mu}]$ *such that* $[\boldsymbol{\lambda}] \setminus r^{\boldsymbol{\lambda}}_x = [\boldsymbol{\mu}] \setminus r^{\boldsymbol{\mu}}_y$. *In this case*

$$J_{\boldsymbol{\lambda\mu}} = (-1)^{\ell\ell(r^{\boldsymbol{\lambda}}_x)+\ell\ell(r^{\boldsymbol{\mu}}_y)}\big(\nu'_p\big(n(a-b) + j - \boldsymbol{\lambda}^{(a)'}_i - l + \boldsymbol{\mu}^{(b)'}_k\big) + \epsilon\big).$$

(b) *Suppose that e is finite and for some integer a we have* $\boldsymbol{\lambda}^{(c)} = \boldsymbol{\mu}^{(c)}$, *for* $c \neq a$. *Then* $J_{\boldsymbol{\lambda\mu}} \neq 0$ *only if there exist nodes* $x = (i, j, a), (i, m, a) \in [\boldsymbol{\lambda}]$ *such that* $m < j$, $e \mid h^{\lambda}_{(i,m,a)}$ *and* $\boldsymbol{\mu}$ *is obtained by wrapping a rim hook of length* $h^{\boldsymbol{\lambda}}_x$ *onto* $\boldsymbol{\lambda} \setminus r^{\boldsymbol{\lambda}}_x$ *with its highest node in column m. In this case*

$$J_{\boldsymbol{\lambda\mu}} = \begin{cases} (-1)^{\ell\ell(r^{\boldsymbol{\lambda}}_x)+\ell\ell(r^{\boldsymbol{\mu}}_y)}(\nu'_p(h^{\boldsymbol{\lambda}}_{(i,m,a)}) + \epsilon), & \text{if } e \nmid h^{\boldsymbol{\lambda}}_{(i,j,a)}, \\ (-1)^{\ell\ell(r^{\boldsymbol{\lambda}}_x)+\ell\ell(r^{\boldsymbol{\mu}}_y)}(\nu'_p(h^{\boldsymbol{\lambda}}_{(i,m,a)}) - \nu'_p(h^{\boldsymbol{\lambda}}_{(i,j,a)})), & \text{if } e \mid h^{\boldsymbol{\lambda}}_{(i,j,a)}, \end{cases}$$

where $y \in [\boldsymbol{\mu}]$ *is determined by* $[\boldsymbol{\mu}] \setminus r^{\boldsymbol{\mu}}_y = [\boldsymbol{\lambda}] \setminus r^{\boldsymbol{\lambda}}_x$.

(c) *In all other cases,* $J_{\boldsymbol{\lambda\mu}} = 0$.

We can now complete the proof of Theorem 2.11.

Proof of Theorem 2.11 for Cases 3–5. Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)})$ be a multipartition of n and fix integers $a \neq b$ with $\lambda^{(a)} \neq (0)$ and $1 \leqslant a, b \leqslant r$. Let $\boldsymbol{\mu}$ be any multipartition that can be obtained by unwrapping a rim hook from $[\lambda^{(a)}]$ and wrapping it back on to component b of $\boldsymbol{\lambda}$. Then $\boldsymbol{\lambda} \sim_J \boldsymbol{\mu}$ by Proposition 4.5(a). In particular, note that $\boldsymbol{\lambda} \sim_J \boldsymbol{\mu}$ if $\boldsymbol{\mu}$ is obtained from $\boldsymbol{\lambda}$ by moving a removable node from $\lambda^{(a)}$ to $\lambda^{(b)}$. Consequently, by moving the nodes in $[\boldsymbol{\lambda}]$ to the right, one by one, we see that $\boldsymbol{\lambda}$ is Jantzen equivalent to a multipartition $\boldsymbol{\mu}$, where $\boldsymbol{\mu} = ((0), \dots, (0), \mu^{(r)})$ for some partition $\mu^{(r)}$. Similarly, moving nodes in $\boldsymbol{\mu}$ to the left, one-by-one, now shows that $\boldsymbol{\lambda} \sim_J \boldsymbol{\mu} \sim_J ((n), (0), \dots, (0))$. Hence, every multipartition in $\Lambda^+_{r,n}$ is Jantzen equivalent to $((n), (0), \dots, (0))$. This shows that there is only one block in Cases 3, 4 and 5, so the theorem follows. □

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