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Lipschitz-type functions on metric spaces [☆]

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Abstract

In order to find metric spaces X for which the algebra $Lip^*(X)$ of bounded Lipschitz functions on X determines the Lipschitz structure of X, we introduce the class of *small-determined* spaces. We show that this class includes precompact and quasi-convex metric spaces. We obtain several metric characterizations of this property, as well as some other characterizations given in terms of the uniform approximation and the extension of uniformly continuous functions. In particular we show that X is small-determined if and only if every uniformly continuous real function on X can be uniformly approximated by Lipschitz functions.

Keywords: Lipschitz functions; Banach–Stone theorem; Uniform approximation

1. Introduction

The classical Banach–Stone theorem asserts that, for a compact topological space K, the linear metric structure of $C(K)$, the space of continuous real functions on K (endowed with the sup-norm), determines the topology of K. More precisely, if K and L are compact spaces, $C(K)$ is linearly isometric to $C(L)$ if, and only if, K is homeomorphic to L. This result has found a large number of extensions, generalizations and variants in many different contexts. We refer for instance to [7] and references therein for further information on this topic.

We are interested here in results along this line for the case of metric spaces and families of Lipschitz real functions. In this sense, it was proved by Sherbert in [18] that the Lipschitz structure of a compact metric space X is determined by the algebra $Lip(X) = Lip^*(X)$, where $Lip(X)$ and $Lip^*(X)$ denote respectively the set of all Lipschitz real functions and the set of all bounded Lipschitz real functions on X. An analogous result was given by Li Pi Su [19] for the algebra $Lip_{loc}(X)$ of locally Lipschitz functions on certain metric spaces X. This was extended by Bustamante and Arrazola in [4] to the class of realcompact metric spaces, and later on in [8] we obtain it for general metric spaces. On the other hand, Weaver in [20] and [21] studies the normed lattice and algebra structures of spaces $Lip^*(X)$ and, more generally,

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of the so-called *Lip-spaces*. In particular, it follows from [20] that the Lipschitz structure of a complete metric space X with diameter $\leqslant 2$ is determined by $Lip(X) = Lip^*(X)$ as a normal-lattice.

More recently, we have proved in [8] that the Lipschitz structure of an arbitrary complete metric space X is determined by the unital vector lattice structure of $Lip(X)$. Simple examples show that this is not true in general for $Lip^*(X)$. Nevertheless, a positive result is given also in [8] for the class of complete quasiconvex metric spaces X considering $Lip^*(X)$ as a unital vector lattice or as an algebra.

Our main purpose in this paper is to find a larger class of metric spaces for which $Lip^*(X)$ determines X. To achieve this we introduce here the class of small-determined spaces. We will see that these spaces enjoy good properties and in particular can be characterized in terms of the approximation and extension of uniformly continuous functions. This leads us to consider different types of Lipschitz maps between metric spaces. Namely, a map $f:(X,d)\to(Y,d')$ is said to be

- *Lipschitz*, if there exists $K \geqslant 0$ such that $d'(f(x), f(y)) \leqslant K\cdot d(x,y)$, for all $x, y \in X$.
- *Lipschitz in the small*, if there exist $r > 0$ and $K \geqslant 0$ such that $d'(f(x), f(y)) \leqslant K\cdot d(x,y)$, whenever $d(x,y) < r$.
- *Locally Lipschitz*, if for every $x \in X$ there exist $r_x > 0$ and $K_x \geqslant 0$ such that $d'(f(y), f(z)) \leqslant K_x\cdot d(y,z)$, whenever $d(y,z) < r_x$.
- *Lipschitz for large distances*, if for all $\varepsilon > 0$ there exists $K_\varepsilon \geqslant 0$ such that $d'(f(x), f(y)) \leqslant K_\varepsilon\cdot d(x,y)$, whenever $d(x,y) \geqslant \varepsilon$.

According to the above definitions we have the following easy facts:

(i) Every bounded map $f:(X,d)\to(Y,d')$, continuous or not, is Lipschitz for large distances. Indeed, let $M > 0$ be a real number such that the diameter of $f(X)$ is less than M. Then $d'(f(x), f(y)) \leqslant (M/\varepsilon)\cdot d(x,y)$, whenever $d(x,y) \geqslant \varepsilon$.

(ii) Any map is Lipschitz if, and only if, it is Lipschitz in the small and Lipschitz for large distances. Then from (i), a bounded map is Lipschitz if, and only if, it is Lipschitz in the small. Nevertheless, the map $f:\mathbb{N}\to\mathbb{R}$ defined by $f(n) = n^2$ is Lipschitz in the small but not Lipschitz. Moreover, this map is uniformly continuous and it is not Lipschitz for large distances.

(iii) Since the composition of two Lipschitz in the small maps is Lipschitz in the small, then $f:(X,d)\to(Y,d')$ is Lipschitz in the small if, and only if, $f:(X,d)\to(Y,\inf\{1,d'\})$ is Lipschitz.

(iv) It is clear that Lipschitz in the small implies uniformly continuous. On the other hand, the map $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x) = \sqrt{|x|}$ is uniformly continuous but not Lipschitz in the small nor locally Lipschitz.

Now, if we denote by $Lip(X)$, $LS(X)$, $Lip_{\mathrm{loc}}(X)$ and $U(X)$, the sets of real-valued functions on X that are Lipschitz, Lipschitz in the small, locally Lipschitz and uniformly continuous, respectively, we have

$$Lip(X) \subset LS(X) \subset Lip_{\mathrm{loc}}(X) \cap U(X),$$

and there is no relation between $U(X)$ and $Lip_{\mathrm{loc}}(X)$. The next example shows that, in general, $LS(X) \neq Lip_{\mathrm{loc}}(X) \cap U(X)$.

Example 1. Let X be the subset of the real line $X = \bigcup_{n\in\mathbb{N}}[2n, 2n+1]$, and $f: X\to\mathbb{R}$ the function $f(x) = g_n(x-2n)$ for $x \in [2n, 2n+1]$, where $\{g_n\}$ is any sequence of Lipschitz functions on $[0,1]$ uniformly converging to $g(x) = \sqrt{x}$. Then, f is uniformly continuous and locally Lipschitz but not Lipschitz in the small.

If $\mathcal{F}$ is a family of real functions, we denote by $\mathcal{F}^*$ the family of all bounded functions in $\mathcal{F}$. Thus, in the case of bounded functions, we have

$$Lip^*(X) = LS^*(X) \subset Lip^*_{\mathrm{loc}}(X) \cap U^*(X),$$

and again Example 1 gives X for which $LS^*(X) \neq Lip^*_{\mathrm{loc}}(X) \cap U^*(X)$.

2. The space of Lipschitz in the small functions

In this section we will be interested in the space $LS(X)$, which will be very useful in the sequel. We refer to Luukainen [16] where these functions have also been considered. It is easy to check that $LS(X)$ is a unital vector lattice that is not, in general, an algebra. However, $LS^*(X)$ is a unital vector lattice which is also an algebra. Moreover, both spaces $LS(X)$ and $LS^*(X)$ are closed under bounded inversion. Recall that a family $\mathcal{F}$ of real functions is *closed under bounded inversion* if $f \in \mathcal{F}$ and $f \geqslant 1$ implies $1/f \in \mathcal{F}$.

Our first result concerning $LS(X)$ is its uniform density in $U(X)$. In order to obtain it we use some ideas from [9], where uniformly continuous partitions of unity subordinated to some special uniform covers are constructed.

Theorem 1 *(Uniform density). In any metric space, every uniformly continuous real function can be uniformly approximated by functions that are Lipschitz in the small.*

Proof. Let (X,d) be a metric space, $f \in U(X)$ and $\varepsilon > 0$. Now, from the uniform continuity of f, we take $0<\delta<1$ associated to this ε. For each $n \in \mathbb{Z}$, consider the set

$$C_n = \{x \in X\colon\ (n-1)\varepsilon < f(x) < (n+1)\varepsilon\}$$

and the 1-Lipschitz function $g_n(x) = \inf\{1, d(x, X \setminus C_n)\}$. Note that the family $\{C_n\}_{n\in\mathbb{Z}}$ is a uniform cover of X satisfying that, for every $x \in X$, the ball of radius δ around x is contained in some C_m. Moreover $C_n \cap C_m = \emptyset$ whenever $|n-m|>1$. So, the function $g(x) = \sum_{n\in\mathbb{Z}} g_n(x)$ is well defined and it satisfies $\delta \leqslant g(x) \leqslant 2$, for all $x \in X$. Moreover, if $x, y \in X$ with $d(x,y) < \delta$, there exists $m \in \mathbb{Z}$ such that

$$\begin{aligned}\big|g(x)-g(y)\big| &= \big|(g_{m-1}+g_m+g_{m+1})(x) - (g_{m-1}+g_m+g_{m+1})(y)\big| \\ &\leqslant \sum_{i=m-1}^{m+1} \big|g_i(x)-g_i(y)\big| \leqslant 3\cdot d(x,y).\end{aligned}$$

Now consider $h(x) = \frac{1}{g(x)}(\sum_{n\in\mathbb{Z}} n\cdot g_n(x))$. Note that, if $x \in C_m$, we have

$$\begin{aligned}h(x) &= \frac{(m-1)g_{m-1}(x)}{g(x)} + \frac{mg_m(x)}{g(x)} + \frac{(m+1)g_{m+1}(x)}{g(x)} \\ &= m - \frac{g_{m-1}(x)}{g(x)} + \frac{g_{m+1}(x)}{g(x)}.\end{aligned}$$

We are going to check that h is Lipschitz in the small. Indeed, if $x, y \in X$ with $d(x,y) < \delta$, there exists $m \in \mathbb{Z}$ such that

$$\begin{aligned}\big|h(x)-h(y)\big| &= \left|m - \frac{g_{m-1}(x)}{g(x)} + \frac{g_{m+1}(x)}{g(x)} - m + \frac{g_{m-1}(y)}{g(y)} - \frac{g_{m+1}(y)}{g(y)}\right| \\ &\leqslant \left|\frac{g_{m-1}(x)g(y) - g_{m-1}(y)g(x)}{g(x)g(y)}\right| + \left|\frac{g_{m+1}(x)g(y) - g_{m+1}(y)g(x)}{g(x)g(y)}\right| \\ &\leqslant \big(10/\delta^2\big)\cdot d(x,y).\end{aligned}$$

Finally, if $x \in C_m$ we have that $|h(x)-m| \leqslant 1$. From this we obtain that $|\varepsilon h(x) - f(x)| \leqslant \varepsilon$, for every $x \in X$, and this completes the proof since $\varepsilon h \in LS(X)$. □

As a consequence of Theorem 1, and taking into account that $LS^*(X) = Lip^*(X)$, we can derive the following well-known result.

Corollary 1. *In any metric space, every bounded uniformly continuous real function can be uniformly approximated by Lipschitz functions.*

In [8] we proved that the family $Lip_{\mathrm{loc}}(X)$ is uniformly dense in $C(X)$, the set of all continuous real-valued functions on X. Now, since $LS(X) \subset Lip_{\mathrm{loc}}(X) \cap U(X)$, Theorem 1 gives an analogous result in the frame of uniformly continuous functions.

Corollary 2. *In any metric space, every uniformly continuous real function can be uniformly approximated by uniformly continuous functions that are locally Lipschitz.*

Next we are going to see that the unital vector lattice structure of $LS(X)$ determines the "Lipschitz in the small structure" of a complete metric space X. Thus we can say that there is a Banach–Stone type theorem for $LS(X)$. Note that completeness cannot be dropped since every metric space has the same Lipschitz in the small functions as its completion. We need the following lemma, which is analogous to a classical result by Efremovich given for uniformly continuous functions in metric spaces [5], and it is also analogous to Theorem 3.9 in [8] for Lipschitz functions.

Lemma 1. *If $h:(X,d)\to(Y,d')$ is a map such that $f\circ h\in LS(X)$ for each $f\in LS(Y)$, then h is Lipschitz in the small. The same is true when $f\circ h\in LS^*(X)$ for each $f\in LS^*(Y)$.*

Proof. Using result (iii) of the introduction, it is enough to check that the map $h:(X,d)\to(Y,\inf\{1,d'\})$ is Lipschitz. Now, applying Theorem 3.9 in [8], h is Lipschitz if so is the composition with every Lipschitz function $f:(Y,\inf\{1,d'\})\to\mathbb{R}$. But this is true since $f\circ h\in LS^*(X)=Lip^*(X)$. The same holds for the bounded case. □

Recall that two metric spaces are *LS-homeomorphic* if there exists a homeomorphism h between them, such that h and h^{-1} are Lipschitz in the small.

Theorem 2 *(Banach–Stone type). Let X and Y be complete metric spaces. The following are equivalent*:

(a) *X is LS-homeomorphic to Y.*
(b) *$LS(X)$ is isomorphic to $LS(Y)$ as unital vector lattices.*
(c) *$LS^*(X)$ is isomorphic to $LS^*(Y)$ as unital vector lattices.*
(c′) *$LS^*(X)$ is isomorphic to $LS^*(Y)$ as algebras.*
(d) *$Lip^*(X)$ is isomorphic to $Lip^*(Y)$ as algebras.*
(d′) *$Lip^*(X)$ is isomorphic to $Lip^*(Y)$ as unital vector lattices.*

Proof. Firstly, note that the equivalence between (c) and (c′), and between (d) and (d′) can be easily obtained from Corollary 2.4 in [8]. On the other hand, it is clear that (c′) is equivalent to (d), since $Lip^*(X)=LS^*(X)$.

That (a) implies (b) is clear because if $h:X\to Y$ is an LS-homeomorphism, then $T:LS(Y)\to LS(X)$, defined by $T(f)=f\circ h$, is an isomorphism of unital vector lattices.

On the other hand, it is easy to see that every unital lattice homomorphism between spaces of real functions should map bounded functions into bounded functions, and then (b) implies (c).

Finally, in order to see that (c) implies (a), we need to make use of the *structure space* $H(\mathcal{L})$ associated to a unital vector lattice $\mathcal{L}$, which is defined as the space of all real-valued unital vector lattice homomorphisms on $\mathcal{L}$ endowed with the pointwise topology, that is, the topology that inherits as a subspace of the product $\mathbb{R}^{\mathcal{L}}$ (see [6] or [8]). Thus, we have that $H(LS^*(X))=H(Lip^*(X))$ is a compact topological space containing X densely, and we know that a point in $H(LS^*(X))$ has a countable neighborhood basis if, and only if, it belongs to X (see [8]). Now if $T:LS^*(X)\to LS^*(Y)$ is an isomorphism of unital vector lattices, then $h:H(LS^*(Y))\to H(LS^*(X))$ defined by $h(\psi)=\psi\circ T$, is an homeomorphism that takes Y onto X. In addition, for each $f\in LS^*(X)$, we have that $f\circ h_{|Y}=T(f)\in LS^*(Y)$, and then, from Lemma 1, $h_{|Y}$ is Lipschitz in the small. The same holds for $(h_{|Y})^{-1}$. □

We note that the equivalence between (a) and (d) was obtained by Luukainen in [16].

3. Small-determined metric spaces

Note that, in the class of complete metric spaces X, the following function spaces determine the corresponding structure on X, namely, $C(X)$ and $C^*(X)$ the topological structure (see Gillman and Jerison [11]), $U(X)$ and $U^*(X)$ the uniform structure (see [6]), and, as we have seen in above Theorem 2, $LS(X)$ and $LS^*(X)$ the LS-structure. In addition, according to [8], we can say that $Lip(X)$ determines the Lispchitz structure on X, but the same is not true for $Lip^*(X)$ (a simple example is obtained by considering (X,d) and $(X,\inf\{1,d\})$, where d is unbounded).

In fact, Theorem 2 says that *Lip** only determines the *LS*-structure of complete metric spaces, but not necessarily their *Lip*-structure. Motivated for this fact we give next definition.

Definition 1. A metric space X is said to be *small-determined* whenever $LS(X) = Lip(X)$.

With this terminology and Theorem 3.10 in [8], we have the following immediate consequence of Theorem 2.

Theorem 3 *(Banach–Stone type). In the class of complete small-determined metric spaces X, the Lip-structure of X is determined by $Lip^*(X)$ as an algebra as well as an unital vector lattice.*

It is clear that every metric space which is Lipschitz homeomorphic to a small-determined space is itself small-determined. Then we can say that to be small-determined is a Lipschitzian property. Of course, not every metric space enjoys this property; for instance, if we consider on $\mathbb{N}$ the usual metric then, every function on $\mathbb{N}$ is Lipschitz in the small but not necessarily Lipschitz. On the other hand, it is easy to see that every precompact metric space, or more generally every weakly precompact metric space is small-determined. Recall that a metric (or uniform) space X is said to be *precompact* if every net in X has a Cauchy subnet. In the same way, X is said to be *weakly precompact* if it is precompact with the weak uniformity given by the family $U(X)$, that is, the uniformity that X inherits as a uniform subspace of the product $\mathbb{R}^{U(X)}$. It is not difficult to check that a metric (or uniform) space is weakly precompact if and only if $U(X) = U^*(X)$. Then every precompact space is weakly precompact, but the converse is not true. This can be seen, for instance, considering the unit ball of any infinite dimensional normed space.

Our next result gives a characterization of those bounded metric spaces that are small-determined.

Theorem 4. *A bounded metric space X is small-determined if and only if it is weakly precompact.*

Proof. It is clear that, if $U(X) = U^*(X)$ then $Lip(X) = Lip^*(X) = LS^*(X) = LS(X)$, and so X is small-determined. Conversely, if X is bounded and small-determined then $Lip(X) = Lip^*(X)$ and $Lip(X) = LS(X)$. Hence, by Theorem 1 we obtain that $U(X) = U^*(X)$, which is equivalent to the weak precompactness of X. □

Now we are going to see that the remarkable class of length spaces (or more generally quasi-convex spaces) are also small-determined. Recall that the length of a path $\sigma : [a, b] \to X$ in a metric space (X, d) is defined as:

$$\ell(\sigma) = \sup \sum_{i=1}^{n} d\big(\sigma(t_i), \sigma(t_{i-1})\big),$$

where the supremum is taken over all partitions $a = t_0 < t_1 < \cdots < t_n = b$. A metric space (X, d) is said to be a *length space* if for every $x, y \in X$, the distance $d(x, y)$ coincides with the infimum of all lengths of continuous paths in X from x to y. Note that in particular normed spaces are length spaces. Other typical examples are Riemannian manifolds endowed with its geodesic distance, but the class of length spaces also includes many other "singular" spaces. We refer to the book by Burago, Burago and Ivanov [3] for an account on this topic.

A metric space (X, d) is said to be *quasi-convex* if there is a constant $C > 0$ so that every pair of points $x, y \in X$ can be joined by a continuous path σ whose length satisfies $\ell(\sigma) \leqslant C \cdot d(x, y)$. It is easily seen that a metric space is quasi-convex if and only if it is Lipschitz homeomorphic to some length space.

Proposition 1. *Every quasi-convex metric space is small-determined.*

Proof. Since a quasi-convex space is Lispchitz equivalent to a length space, we can suppose that (X, d) is a length space. Then, let $f \in LS(X)$, $r > 0$ and $K \geqslant 0$ such that $|f(x) - f(y)| \leqslant K \cdot d(x, y)$ whenever $d(x, y) < r$. Let two arbitrary points $z_1, z_2 \in X$, $\varepsilon > 0$, and a continuous path $\sigma : [a, b] \to X$ from z_1 to z_2 with $\ell(\sigma) \leqslant d(z_1, z_2) + \varepsilon$. Using the uniform continuity of σ we can choose a partition $a = t_0 < t_1 < \cdots < t_n = b$ such that $d(\sigma(t_i), \sigma(t_{i-1})) < r$, for $i = 1, \ldots, n$. Then

$$\begin{aligned}\left|f(z_1)-f(z_2)\right| &\leqslant \sum_{i=1}^{n}\left|f\big(\sigma(t_i)\big)-f\big(\sigma(t_{i-1})\big)\right| \leqslant \sum_{i=1}^{n} K\cdot d\big(\sigma(t_i),\sigma(t_{i-1})\big)\\ &\leqslant K\cdot \ell(\sigma)\leqslant K\cdot\big(d(z_1,z_2)+\varepsilon\big).\end{aligned}$$

Therefore, $|f(z_1)-f(z_2)|\leqslant K\cdot d(z_1,z_2)$, for every $z_1,z_2\in X$. □

Note that in particular every normed space is small-determined. On the other hand, there are simple examples of small-determined spaces which are neither weakly precompact nor quasi-convex. Consider, for instance, the subset of the real line, $X=\{1+\frac{1}{2}+\cdots+\frac{1}{n}\colon n\in\mathbb{N}\}$. Another interesting example is the subset of $\mathbb{R}^2$, $X=(\mathbb{R}\times\{0\})\cup([0,1]\times\{1\})$. The fact that this space is small-determined is a consequence of the following general result.

Proposition 2. *If a metric space X can be written as the union of two small-determined spaces, where one of these is also bounded, then it is small-determined.*

Proof. Suppose $X=Y\cup Z$, where Y and Z are small-determined and Y is bounded. Let $f\in LS(X)$ and $r>0$, such that $|f(x)-f(y)|\leqslant K\cdot d(x,y)$ when $d(x,y)<r$. Now, since the restriction maps $f_{|Y}$ and $f_{|Z}$ are Lipschitz on Y and Z, respectively, then to show $f\in Lip(X)$, it will be enough to find $\widetilde{K}>0$ such that

$$\left|f(y)-f(z)\right|\leqslant \widetilde{K}\cdot d(y,z)$$

whenever $y\in Y$, $z\in Z$ and $d(y,z)\geqslant r$. To this end, choose a point $z_0\in Z$, and let $C>0$ a real number greater than the respective Lipschitz constants of $f_{|Y}$ and $f_{|Z}$. Let $A=\sup_{y\in Y}|f(y)-f(z_0)|$ and $B=\sup_{y\in Y}d(y,z_0)$. Note that both suprema A and B exist since Y is weakly precompact (Theorem 4) and hence every uniformly continuous function on Y is bounded. Then we have,

$$\begin{aligned}\left|f(y)-f(z)\right| &\leqslant \left|f(y)-f(z_0)\right|+\left|f(z_0)-f(z)\right|\leqslant A+C\cdot d(z_0,z)\\ &\leqslant A+C\cdot\big(d(z_0,y)+d(y,z)\big)\leqslant A+C\cdot B+C\cdot d(y,z)\\ &\leqslant \frac{A+C\cdot B}{r}\cdot d(y,z)+C\cdot d(y,z)\leqslant \widetilde{K}\cdot d(y,z). \qquad \square\end{aligned}$$

Note that, in general, the union of two small-determined spaces needs not to be small-determined. A simple example for this is the subset of $\mathbb{R}^2$, $X=(\mathbb{R}\times\{0\})\cup(\mathbb{R}\times\{1\})$, since the function $f(x,0)=x$ and $f(x,1)=-x$ belongs to $LS(X)$ but is not Lipschitz.

According to the above results we can say that the class of (complete) small-determined metric spaces is strictly bigger than the class of (complete) quasi-convex metric spaces, and hence Theorem 3 is an extension of the Banach–Stone type result Theorem 3.3 in [8].

Another interesting class of metric spaces containing length spaces and quasi-convex spaces are the so called well-chained spaces. Recall that a metric space (X,d) is said to be *well-chained* or *chainable* if for every $x,y\in X$ and every $\varepsilon>0$ there exists an ε-chain joining x and y, i.e., a finite sequence $x=x_0,x_1,\ldots,x_n=y$ such that $d(x_i,x_{i-1})<\varepsilon$, for $i=1,\ldots,n$. In a well-chained metric space (X,d) we can define a family of metrics $\{d_\varepsilon\}_{\varepsilon>0}$ as follows:

$$d_\varepsilon(x,y)=\inf\sum_{i=1}^{n}d(x_i,x_{i-1}),$$

where the infimum is taken over all the ε-chains, $x=x_0,x_1,\ldots,x_n=y$. Note that for X well-chained, every metric d_ε is *LS*-equivalent to d. Indeed, from the triangle inequality of d it follows that $d\leqslant d_\varepsilon$. Moreover, $d_\varepsilon(x,y)=d(x,y)$ whenever $d(x,y)<\varepsilon$. It is clear that, if (X,d) is a length space, then $d=d_\varepsilon$, for every $\varepsilon>0$. Recall that the converse is also true for complete metric spaces (Corollary 2.4.17 in [3]).

Note that a metric space is well-chained if and only if it is *uniformly connected*, that is, it is not the union of two non-empty subsets which are a positive distance apart. So every connected metric space, and in particular every length space and every quasi-convex space, is uniformly connected. But clearly the converse is not true. For instance, the (complete) subspace of $\mathbb{R}^2$ defined by $X=\{(x,y)\colon x\cdot y=\pm 1\}$ is uniformly connected but not connected. Our next example shows that not every well-chained (even connected) metric space is small-determined.

Example 2. Let $X = ([0,\infty) \times \{0,1\}) \cup (\{0\} \times [0,1])$ with the usual metric in $\mathbb{R}^2$. Then X is uniformly connected but not small-determined since the function defined by $f(x,0) = x$, $f(x,1) = -x$, for $x \geqslant 0$; and $f(0,y) = 0$, for $y \in [0,1]$ is Lipschitz in the small but not Lipschitz.

In order to characterize those uniformly connected metric spaces that are small-determined, we need the following lemma.

Lemma 2. *A metric space (X,d) is small-determined if and only if for any metric space (Y,d'), every Lipschitz in the small map $h:(X,d) \to (Y,d')$ is Lipschitz.*

Proof. Suppose that X is small-determined and let $h:(X,d) \to (Y,d')$ a Lipschitz in the small map. From Theorem 3.9 in [8], h is Lipschitz if the composition with every Lipschitz function $f:(Y,d') \to \mathbb{R}$ is Lipschitz. But, this is clear since $f \circ h \in LS(X) = Lip(X)$. The converse is obvious. □

Theorem 5. *Let (X,d) be a uniformly connected metric space. Then X is small-determined if and only if the metrics d and d_ε are Lipschitz equivalent for every $\varepsilon > 0$.*

Proof. When (X,d) is small-determined then, for every $\varepsilon > 0$, the metrics d and d_ε are Lipschitz equivalent. Indeed, from the inequality $d \leqslant d_\varepsilon$, it follows that identity map $i:(X,d_\varepsilon) \to (X,d)$ is Lipschitz. And, by Lemma 2, we have that the map $i:(X,d) \to (X,d_\varepsilon)$, which is always Lipschitz in the small, must be Lipschitz too.

For the converse we will use the same techniques as in the proof of Proposition 1 above. Let $f \in LS(X)$, and $r > 0$ such that $|f(x) - f(y)| \leqslant K \cdot d(x,y)$ when $d(x,y) < r$. Since d and $d_{r/2}$ are Lipschitz equivalent then $d_{r/2} \leqslant M \cdot d$, for some $M > 0$. Let two arbitrary points $x, y \in X$, let $\delta > 0$ and, $x = x_0, x_1, \dots, x_n = y$, some $r/2$-chain joining x and y, and such that $\sum_{i=1}^n d(x_i, x_{i-1}) \leqslant d_{r/2}(x,y) + \delta$. Therefore,

$$\begin{aligned}\bigl|f(x) - f(y)\bigr| &\leqslant \sum_{i=1}^n \bigl|f(x_i) - f(x_{i-1})\bigr| \leqslant \sum_{i=1}^n K \cdot d(x_i, x_{i-1}) \\ &\leqslant K \cdot \bigl(d_{r/2}(x,y) + \delta\bigr).\end{aligned}$$

Hence, $|f(x) - f(y)| \leqslant K \cdot d_{r/2}(x,y) \leqslant K \cdot M \cdot d(x,y)$, and $f \in Lip(X)$. □

Next, we are going to obtain a characterization of general small-determined metric spaces. To this end we will use some ideas of Levy and Rice [15]. We also need to recall a well-known result by Arens and Eells [1], saying that every metric space (X,d) is always isometrically contained in some normed space. An easy way to do this is the following: fix a base-point $x_0 \in X$, and consider $U^*(X)$ endowed with the sup-norm. Then the map $x \rightsquigarrow f_x$, where $f_x(z) = d(z,x) - d(x,x_0)$, is an isometry from X into the Banach space $(U^*(X), \|\cdot\|_\infty)$. Nevertheless, the original construction given by Arens and Eells is different. They define a Banach space $B(X)$ canonically associated to X, containing it isometrically, and with the following nice universal property: for every Banach space E, every Lipschitz map $\varphi: X \to E$ vanishing at x_0, can be extended to a unique continuous linear map $T_\varphi: B(X) \to E$ (see e.g. Weaver [21, Theorem 2.2.4]). The space $B(X)$ is said to be the *Arens–Eells space* associated to X in [21], and it is also called the *free Banach space* over X in [12].

Theorem 6. *A metric space (X,d) is small-determined if and only if for each $\varepsilon > 0$ there is some metric space $(M,\hat{d})$ (depending upon ε) isometrically containing X such that $M \setminus X$ is finite, and where the metrics $\hat{d}$ and $\hat{d}_\varepsilon$ are Lipschitz equivalent.*

Proof. Given $\varepsilon > 0$, we define the equivalence relation $\simeq_\varepsilon$ on X as follows: $x \simeq_\varepsilon y$ if and only if there exists an ε-chain joining x and y. Note that two different classes of equivalence are ε apart. So, if X is small-determined, there must be a finite number of these classes. Otherwise, X could be written as an infinite union of subsets that are ε-apart, $X = \bigcup_{n=1}^\infty A_n$. In this case, if we choose $x_n \in A_n$, and define the function $f(x) = n \cdot d(x_1, x_n)$ when $x \in A_n$, then f Lipschitz in the small but not Lipschitz, which is a contradiction.

Then, suppose $X = A_1 \cup \cdots \cup A_n$, where $A_1, \dots, A_n$ are the different equivalence classes, and take $x_i \in A_i$, for $i = 1, \dots, n$. Now, let $(E, \|\cdot\|)$ be any normed space isometrically containing X. Let L_i be the line segment in E from x_i to x_{i+1}, for $i = 1, \dots, n-1$. As $L_1 \cup \cdots \cup L_{n-1}$ is compact, we can select a finite ε-net, F_ε. Now, if we take $M = X \cup F_\varepsilon$, then this metric space is ε-chained. In addition, from Proposition 2, M is small-determined, and hence, both metrics $\hat{d} = \|\cdot\|_{|M}$ and $\hat{d}_\varepsilon$, that are always *LS*-equivalent, must be in fact Lipschitz equivalent on M.

Conversely, let $f \in LS(X)$ and $r > 0$ such that $|f(x) - f(y)| \leqslant K \cdot d(x, y)$ when $d(x, y) < r$. From the hypothesis, let $(M, \hat{d})$ be the metric space associated to $\varepsilon = r/2$, such that $M = X \cup F$ where F is finite. Let

$$H = \{t \in F\colon \exists x \in X,\ \hat{d}(x, t) < r/2\}.$$

Now, if $t \in H$ the set $B_t = \{x \in X,\ \hat{d}(x, t) < r/2\}$ is non-empty, and we can define

$$\hat{f}(t) = \inf\{f(x) + K \cdot \hat{d}(x, t)\colon x \in B_t\}.$$

Note that, since the restriction of f to B_t is in fact K-Lipschitz, we obtain in this way a K-Lipschitz extension of $f_{|B_t}$ to $B_t \cup \{t\}$. Repeating this for all $t \in H$, we have a function $\hat{f}$ defined on the finite subset H, which can be obviously extended to a $\hat{K}$-Lipschitz function $\hat{f}$ on F, for some $\hat{K} \geqslant K$. Consider now the function g on M given by $g = f$ on X and $g = \hat{f}$ on F. Then g satisfies

$$|g(u) - g(v)| \leqslant \hat{K} \cdot \hat{d}(u, v) \quad \text{if } \hat{d}(u, v) < r/2.$$

Indeed, if $u, v \in X$, then $|g(u) - g(v)| = |f(u) - f(v)| \leqslant K \cdot d(u, v)$; if $u, v \in F$, then $|g(u) - g(v)| = |\hat{f}(u) - \hat{f}(v)| \leqslant \hat{K} \cdot \hat{d}(u, v)$; finally when $u \in X$ and $v \in F$, then $u \in B_v$ and $|g(u) - g(v)| = |f(u) - \hat{f}(v)| \leqslant K \cdot \hat{d}(u, v)$.

Since $\hat{d}$ and $\hat{d}_{r/2}$ are Lipschitz equivalent, reasoning as in the precedent Theorem 5 it follows that in fact f is Lipschitz on X. $\square$

We finish this section with some further remarks about small-determined spaces. It is easy to check that a metric space is small-determined if and only if so is its completion. Moreover, the Lipschitzian property of being small-determined is not a uniform property. In fact, every small-determined non-weakly precompact metric space (X, d) is uniformly homeomorphic to the non-small-determined bounded space $(X, \inf\{1, d\})$ (see Theorem 4). As well, this property is not hereditary. Indeed, every metric space (small-determined or not) is a subspace of a small-determined space, namely as a subspace of a normed space. In addition, this property is not stable for products. A simple example for this is $X = \mathbb{R} \times \{0, 1\}$, since the function defined by $f(x, 0) = x$ and $f(x, 1) = -x$ belongs to $LS(X)$ but is not Lipschitz. Nevertheless we note that this property is stable for products in the frame of uniformly connected metric spaces, which is an easy consequence of Theorem 5. Finally, it is not difficult to check that the class of small-determined spaces is closed under *Lipschitz quotients*. Recall that a map between metric spaces $\varphi : X \to Y$ is said to be a Lipschitz quotient if it is surjective, Lipschitz, and there is a constant $C > 0$ such that for every $x \in X$ and every $r > 0$ we have that $\varphi(B(x, r)) \supset B(\varphi(x), r/C)$ (see e.g. Benyamini and Lindenstrauss [2]).

4. Small-determined spaces and U-embedding

In this section we are going to obtain a characterization of small-determined spaces in terms of the extension of uniformly continuous real functions, as well as in connection with the uniform approximation of uniformly continuous functions. Recall that a subset X of a metric space M is said to be *U-embedded* (respectively *U^*-embedded*) in M if every member of $U(X)$ (respectively $U^*(X)$) can be extended to a member of $U(M)$ (respectively $U^*(M)$). We refer to Levy and Rice [14] and [15] for an account of this topic. A classical result due to Katetov [13] gives that, in fact, every subset of a metric space is always U^*-embedded. It is easy to see that the analogous result for U-embedding is not true, even for closed subsets. For instance it is clear that $\mathbb{N}$ is not U-embedded in $\mathbb{R}$. On the other hand, according to McShane [17], every Lipschitz real function defined on a subset of a metric space can be extended to a Lipschitz function on the whole space (with the same Lipschitz constant).

Theorem 7. *Let (X, d) be a metric space. The following are equivalent*:

(a) *X is small-determined.*
(b) *Every uniformly continuous real function can be uniformly approximated by Lipschitz functions.*

(c) *Every uniformly continuous real function is Lipschitz for large distances.*
(d) *X is U-embedded in every metric space bi-Lipschitz containing it.*
(e) *X is U-embedded in its Arens–Eells space $B(X)$.*
(f) *X is U-embedded in some normed space bi-Lipschitz containing it.*

Proof. That (a) implies (b) follows at once from Theorem 1. Now, suppose (b) and let $f \in U(X)$. Take $g \in Lip(X)$ such that $|f - g| < 1$. Since $f - g$ is bounded, then it is Lipschitz for large distances (see (i) in the introduction). Therefore, if we write $f = g + (f - g)$, then it is clear that f is also Lipschitz for large distances, and (c) holds.

On the other hand, it is easy to see that (c) implies (a). Indeed, if $f \in LS(X)$ and (c) is true, we have that f is in particular Lipschitz for large distances, and then Lipschitz. Hence, X is small-determined.

Now suppose (b), and in order to prove (d), consider X bi-Lipschitz contained in some metric space M, i.e., there exists a bi-Lipschitz map $i : X \to i(X) \subset M$. Let $f \in U(X)$ and $g \in Lip(X)$ with $|f - g| < 1$. Identifying X with $i(X)$ and using the result of McShane, g can be extended to a Lipschitz function on M. Now, from the result of Katetov, the function $f - g \in U^*(X)$ can be also extended to a uniformly continuous function on M. Hence $f = g + (f - g)$ can be clearly extended, and therefore X is U-embedded in M.

That (d) implies (e) and (e) implies (f) are obvious. Finally, that (f) implies (b) follows at once since any normed space is small-determined (Proposition 1). Therefore every uniformly continuous real function on a normed space can be uniformly approximated by Lipschitz functions, and the same is true for those subspaces that are U-embedded. □

The equivalence between (b) and (c) was proved by Géher in [10]. On the other hand, Levy and Rice in [14] showed that, when X is a subset of a normed space M, both conditions are also equivalent to the fact that X is U-embedded in M. They also noted that these conditions imply (d).

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