In this paper, we shall employ the method of cone-valued Lyapunov functions and comparison principle to investigate the $\phi_0$-stability of impulsive hybrid systems on time scales.

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1. Introduction

Since there is a striking similarity or even duality between the theories of continuous and discrete dynamic systems, many results in the theory of difference equations have been obtained as more or less natural discrete analogs of corresponding results of differential equations. From a modelling point of view, it is more realistic to model a phenomena by a dynamic system that incorporates both continuous and discrete times, namely, time as an arbitrary closed set of reals known as time scales or measure chains. Recently, the theory of dynamic systems on time scales has gained impetus because it provides a framework which permits us to handle both continu-
ous and discrete dynamic systems simultaneously so that one can get some insight and a better understanding of the subtle differences of these two different systems [1].


Recently, Lakshmikantham and Liu [5] gave the concept of hybrid systems, Wang and Liu [6,7] obtained the stability criteria for impulsive hybrid systems on time scales. The identifying characteristic of hybrid systems in general is that they incorporate both continuous components, usually called plants, which are governed by differential equations, and also digital components such as digital computers, sensors and actuators controlled by programs.

In this paper, we shall employ the method of cone-valued Lyapunov functions to investigate the $\phi_0$-stability of impulsive hybrid systems on time scales, and give some stability results via comparison principle. At the same time, we give an example to illustrate our result.

2. Preliminaries

Let $\mathbb{T}$ be a time scale with $t_0 \geq 0$ as minimal element and no maximal element.

**Definition 2.1.** (See [1].) The mappings $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ defined as

$$\sigma(t) = \inf\{s \in \mathbb{T}: s > t\}$$

and

$$\rho(t) = \sup\{s \in \mathbb{T}: s < t\}$$

are called jump operators.

**Definition 2.2.** (See [1].) A nonmaximal element $t \in \mathbb{T}$ is said to be right-scattered (rs) if $\sigma(t) > t$ and right-dense (rd) if $\sigma(t) = t$. A nonminimal element $t \in \mathbb{T}$ is called left-scattered (ls) if $\rho(t) < t$ and left-dense (ld) if $\rho(t) = t$.

**Definition 2.3.** (See [1].) The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(\sigma(t), t) = \sigma(t) - t.$$  

For convenience, we denote it by $\mu^*(t)$. When $\mathbb{T} = \mathbb{Z}$, $\mu^*(t) \equiv 1$ and $\mathbb{T} = \mathbb{R}$, $\mu^*(t) \equiv 0$.

**Definition 2.4.** (See [1].) The mapping $g : \mathbb{T} \to X$ where $X$ is a Banach space, is called rd continuous if

(i) it is continuous at each right-dense $t \in \mathbb{T}$,

(ii) at each left-dense point the left-sided limit $g(t^-)$ exists.

Let $C_{rd}[\mathbb{T}, X]$ denote the set of rd-continuous mappings from $\mathbb{T}$ to $X$. 
Definition 2.5. (See [1].) Let \( f \) be a mapping \( T \to X \). At \( t \in T \), \( f \) has the derivative \( f_t^\Delta \in X \) if for each \( \varepsilon > 0 \), there exists a neighborhood \( U \) of \( t \) such that for all \( s \in U \),

\[
| f(\sigma(t)) - f(s) - f_t^\Delta(\sigma(t) - s) | \leq \varepsilon |\sigma(t) - s|.
\]

\( f \) is called differentiable at \( t \in T \), if \( f \) has exactly one derivative \( f_t^\Delta \) in \( t \).

Definition 2.6. (See [1].) For each \( t \in T \), let \( N \) be a neighborhood of \( t \). Then, for \( V \in C_{rd}(T \times \mathbb{R}^n, \mathbb{R}_+) \), define \( D^+V^\Delta(t, x(t)) \) to mean that, given \( \varepsilon > 0 \), there exists a right neighborhood \( N_{\varepsilon} \subseteq N \) of \( t \) such that for each \( s \in N_{\varepsilon}, s > t \), where \( \mu(s,t) \equiv \sigma(t) - s \). If \( t \) is rs and \( V(t,x(t)) \) is continuous at \( t \), this reduces to

\[
D^+V^\Delta(t, x(t)) = \frac{V(\sigma(t), x(\sigma(t))) - V(t, x(t))}{\mu(\sigma(t), t)} + \varepsilon.
\]

Definition 2.7. (See [3].) A proper subset \( K \) of \( \mathbb{R}^n \) is called a cone if (i) \( \lambda K \subseteq K \), \( \lambda \geq 0 \); (ii) \( K + K \subseteq K \); (iii) \( K = \overline{K} \); (iv) \( K^0 \neq \emptyset \); (v) \( K \cap (-K) = \{0\} \), where \( \overline{K} \) and \( K^0 \) denote the closure and interior of \( K \), respectively, and \( \partial K \) denotes the boundary of \( K \).

Definition 2.8. (See [3].) The set \( K^* = \{ \phi \in \mathbb{R}^n \colon (\phi, x) \geq 0, x \in K \} \) is called the adjoint cone if it satisfies properties (i)–(v) of Definition 2.7,

\[
x \in \partial K \text{ iff } (\phi, x) = 0 \text{ for some } \phi \in K_0^*, \ K_0 = K - \{0\}.
\]

Definition 2.9. (See [3].) A function \( g : D \to \mathbb{R}^n \), \( D \subseteq \mathbb{R}^n \) is said to be quasi-monotone nondecreasing relative to the cone \( K \) if \( x, y \in D \) and \( y - x \in \partial K \) imply that there exists \( \phi_0 \in K_0^* \) such that \( (\phi_0, y - x) = 0 \) and \( (\phi_0, g(y) - g(x)) \geq 0 \).

Definition 2.10. (See [3].) A function \( a(\cdot) \) is said to belong to the class \( \mathcal{K} \) if \( a \in C[[0, \rho), \mathbb{R}_+] \), \( a(0) = 0 \), and \( a(r) \) is a strictly monotone increasing function in \( r \).

3. Comparison result

We consider the following hybrid impulsive dynamic system

\[
\begin{align*}
x' &= f(t, x, y_k), & t \neq t_k, \\
x(t^+) &= I_k(x(t)), & t = t_k, \\
x(t_0^+) &= x_0^+, \\
y^\Delta &= F(t, y, x_k^+), & y(t_k) = y_k, \\
y(t_0) &= y_0, & k = 1, 2, 3, \ldots,
\end{align*}
\]

(3.1)

under the following assumptions \((A_0)\):

(i) \( t_k \in \mathbb{T} \) for each \( k \), where \( 0 \leq t_0 < t_1 < t_2 < \cdots < t_k < \cdots \) with \( t_k \to \infty \) as \( k \to \infty \).
(ii) $f: \mathbb{R}^+ \times K \times K \to K$ is continuous in $(t_k, t_{k+1}] \times K \times K$ and for each $x, y$, \( \lim f(t, \xi, \eta) = f(t_k^+, x, y) \) exists as $(t, \xi, \eta) \to (t_k^+, x, y)$.

(iii) $I_k: K \to K$.

(iv) $F \in C_{rd}[\mathbb{T} \times K \times K, K]$.

By a solution $x(t, t_0, x_0), y(t, t_0, y_0)$ of the system (3.1) we mean the following:

\[
x(t) = x(t, t_0, x_0) = \begin{cases} 
  x_0(t, t_0, x_0^+, y_0), & t_0 < t \leq t_1, \\
  x_k(t, t_k, x_k^+, y_k), & t_k < t \leq t_{k+1}, 
\end{cases} \\
y(t) = y(t, t_0, y_0) = \begin{cases} 
  y_0(t, t_0, x_1^+), & t_0 \leq t \leq t_1, \\
  y_k(t, t_k, y_k, x_{k+1}^+), & t_k \leq t \leq t_{k+1}, 
\end{cases}
\]

where $x_k(t) = x_k(t, t_k, x_k^+, y_k)$ is the solution of

\[
x'(t) = f(t, x(t), y_k), \quad x(t_k) = x_k^+, \quad t_k < t \leq t_{k+1}, \tag{3.2}
\]

and $y_k(t) = y_k(t, t_k, y_k, x_{k+1}^+)$ is the solution of

\[
y^\Delta(t) = F(t, y(t), x_{k+1}^+), \quad y(t_k) = y_k, \quad t_k \leq t \leq t_{k+1}, \\
x_{k+1}^+ = x_k(t_k^+) = I_k(x_k(t_k)), \quad y_{k+1} = y_k(t_{k+1}). \tag{3.3}
\]

for each $k = 1, 2, 3, \ldots$. We assume that the solution $x_k(t), y_k(t)$ exists and is unique on each interval $t_k \leq t \leq t_{k+1}$. It should be noted that the solution $x(t, t_0, x_0)$ are piecewise continuous functions with points of discontinuity of the first type at $t = t_k$ at which they are supposed to be left continuous and $y(t, t_0, x_0)$ are rd-continuous for $t \in \mathbb{T}$.

We need the scalar comparison hybrid impulsive dynamic system

\[
u' = g(t, u, v_k), \quad t \neq t_k, \\
u(t_k^+) = \psi_k(u(t)), \quad t = t_k, \\
u(t_0^+) = u_0^+, \\
v^\Delta = H(t, v, u_{k+1}^+), \quad v(t_k) = v_k, \\
v(t_0) = v_0, \quad k = 1, 2, 3, \ldots, \tag{3.4}
\]

under the following conditions (B0):

(i) (A0)(i) holds;

(ii) $g: \mathbb{R}^+ \times K \times K \to K$ is continuous in $(t_k, t_{k+1}] \times K \times K$ and for each $u, v$, \( \lim g(t, q, s) = g(t_k^+, u, v) \) exists as $(t, q, s) \to (t_k^+, u, v)$;

(iii) $\psi_k: K \to K$ and $\psi_k$ is quasi-monotone nondecreasing relative to $K$;

(iv) $H \in C_{rd}[\mathbb{T} \times K \times K, K]$.

The maximal solution $r(t, t_0, u_0), R(t, t_0, v_0)$ of (3.4), which we can define similar to $x(t), y(t)$ of (3.1). We omit it here.
Let $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$. Then $V$ is said to belong to the class $V_0$ if

(i) $V$ is continuous in $(t_k, t_{k+1}] \times \mathbb{R}^n$ and for each $x \in \mathbb{R}^n$,

$$\lim_{(t,y) \to (t_k^+,x)} V(t,y) = v(t_k^+,x)$$

exists;

(ii) $V(t,x)$ is locally Lipschitzian in $x$ relative to the cone $K$.

Then we define for $(t,x,y) \in (t_k, t_{k+1}] \times K \times K$,

$$D^+ V(t,x,y) = \lim sup_{h \to 0^+} \frac{1}{h} \left[ V(t+h,x+hf(t,x,y)) - V(t,x) \right].$$

Also, let $W \in C_{rd}[T \times \mathbb{R}^d, K]$. Then $W$ is said to belong to the class $W_0$ if $W(x,y)$ is locally Lipschitzian in $y$ relative to the cone $K$ for each $t \in T$ which is rd and $D^+ W(t,y_k(t))$ exists where $y_k(t)$ is the solution of (3.3).

Now we give a comparison result.

**Theorem 3.1.** Assume $K \subseteq \mathbb{R}^n$ is a cone and that

(A1) $V \in V_0$ and for $t_k < t \leq t_{k+1}$, $x, y \in S(h, \rho)$,

$$D^+ V(t,x,y_k) \leq g(t,V(t,x),W(t,y_k)), \quad t \neq t_k,$$

$$V(t_k^+,I_k(x)) \leq \psi_k(V(t,x)), \quad t = t_k,$$

where $g$ satisfies the conditions (B0)–(iii), $S(h, \rho) = \{(t,x) \in T \times \mathbb{R}^n: h(t,x) < \rho, \rho > 0\}$;

(A2) $W \in W_0$ and for $t_k \leq t \leq t_{k+1}$, $t \in T$,

$$D^+ W(t,y) \leq H(t,W(t,x),V(t_{k+1},x_{k+1}^+)),$$

where $H$ satisfies the condition (B0)(iv), $H(t,v,u)\mu^*(t) + v$ is nondecreasing in $v$ for each $(t,u)$ and $H(t,v,u)$ is quasi-monotone nondecreasing in $u$ relative to $K$ for each $(t,v)$;

(A3) $r(t), R(t)$ are the maximal solutions of (3.4) existing for $t \geq t_0$.

Then $V(t_0,x_0^+) \leq u_0$, $W(t_0,y_0) \leq v_0$ imply

$$V(t,x(t,t_0,x_0)) \leq r(t,t_0,u_0), \quad t \geq t_0,$$

$$W(t,y(t,t_0,y_0)) \leq R(t,t_0,v_0), \quad t \geq t_0, \quad t \in T.$$

The proof is similar to the proof of Theorem 2.1 in [8].

4. **Main results**

We firstly give some definitions below.

**Definition 4.1.** The zero solution of (3.1) is said to be

(S1) $\phi_0$-equistable, if, for each $\varepsilon > 0$, there exists $\delta = \delta(t_0, \varepsilon)$ continuous in $t_0$ for each $\varepsilon$, such that the inequality
\[(\phi_0, x_0) < \delta \quad \text{implies} \quad (\phi_0, x^*(t)) < \varepsilon, \quad t \geq t_0,\]
and
\[(\phi_0, y_0) < \delta \quad \text{implies} \quad (\phi_0, y^*(t)) < \varepsilon, \quad t \geq t_0, \quad t \in \mathbb{T},\]
where and in the rest of this paper \(x^*(t)\) denotes the maximal solution of (3.1) relative to the cone \(K \subseteq \mathbb{R}^n;\)
(S2) uniformly \(\phi_0\)-stable, if \(\delta\) in (S1) is independent of \(t_0;\)
(S3) asymptotically \(\phi_0\)-stable, if (S1) holds and for given \(\varepsilon > 0\) there exists \(T = T(t_0, \varepsilon) > 0\) such that \((\phi_0, x_0) < \delta\) implies \((\phi_0, x^*(t)) < \varepsilon, \quad t \geq t_0 + T,\) and \((\phi_0, y_0) < \delta\) implies \((\phi_0, y^*(t)) < \varepsilon, \quad t \geq t_0 + T \in \mathbb{T};\)
(S4) uniformly asymptotically \(\phi_0\)-stable, if (S2) holds and \(T\) in (S3) is independent of \(t_0.\)

**Theorem 4.1.** Assume \(K \subseteq \mathbb{R}^n\) is a cone and that

(A1) \(V \in V_0, \ W \in W_0\) and for \(t_k \leq t \leq t_{k+1},\)
\[
D^+(\phi_0, V(t, x)) \leq 0,
\]
\[
D^+(\phi_0, W(t, x)) \leq 0,
\]
(A2) there exist \(b_1, b_2 \in K\) such that
\[
b_1(\phi_0, x(t)) \leq (\phi_0, V(t, x)),
\]
\[
b_2(\phi_0, y(t)) \leq (\phi_0, V(t, y)).
\]

(A3) \(x \in S(h, \rho)\) implies \(I_k(x) \in S(h, \rho).\)

Then the zero solution of (3.1) is \(\phi_0\)-equistable.

**Proof.** From (A1), it is clear that
\[
(\phi_0, V(t, x^*(t))) \leq (\phi_0, V(t_0, x_0)), \tag{4.1}
\]
\[
(\phi_0, W(t, y^*(t))) \leq (\phi_0, W(t_0, y_0)). \tag{4.2}
\]

where \(x^*(t), y^*(t)\) is the maximal solution of (3.1).

Thus for \(\phi_0 \in K_0^*\)
\[
\|\phi_0\| \|V(t, x^*(t))\| \leq \|\phi_0\| \|V(t_0, x_0)\|,
\]
\[
\|\phi_0\| \|W(t, y^*(t))\| \leq \|\phi_0\| \|W(t_0, y_0)\|.
\]

Thus from the continuity of \(V(t_0, x_0), W(t_0, y_0)\) in \(t_0\), it follows that given \(\varepsilon_1, \varepsilon_2 > 0,\ t_0 \in \mathbb{T},\) there exist \(\delta_1 = \delta_1(t_0, \varepsilon) > 0,\ \delta_2 = \delta_2(t_0, \varepsilon) > 0\) such that
\[
\|x_0\| < \delta_1 \quad \text{implies} \quad \|V(t_0, x_0)\| < \varepsilon_1,
\]
\[
\|y_0\| < \delta_2 \quad \text{implies} \quad \|W(t_0, y_0)\| < \varepsilon_2.
\]

Now, if \(\phi_0 \in K_0^*,\) then we get
\[
\|\phi_0\| \|x_0\| < \|\phi_0\| \delta_1 = \delta \quad \text{implies} \quad \|\phi_0\| \|V(t_0, x_0)\| < \|\phi_0\| \varepsilon_1 = \varepsilon,
\]
\[
\|\phi_0\| \|y_0\| < \|\phi_0\| \delta_2 = \delta^* \quad \text{implies} \quad \|\phi_0\| \|W(t_0, y_0)\| < \|\phi_0\| \varepsilon_2 = \varepsilon^*.
\]
Thus

\[(\phi_0, x_0) < \delta \quad \text{implies} \quad (\phi_0, V(t_0, x_0)) < \epsilon,\]
\[(\phi_0, y_0) < \delta^* \quad \text{implies} \quad (\phi_0, W(t_0, y_0)) < \epsilon^*.\]

From (4.1) and (4.2), we get

\[(\phi_0, x_0) < \delta \quad \text{implies} \quad (\phi_0, V(t, x^*(t))) < \epsilon,\]
\[(\phi_0, y_0) < \delta^* \quad \text{implies} \quad (\phi_0, W(t, y^*(t))) < \epsilon^*.\]

Choose \(\delta^0 = \min[\delta, \delta^*], \epsilon^0 = \min[\epsilon, \epsilon^*]\), then the zero solution of (3.1) is \(\phi_0\)-equistable.

**Theorem 4.2.** Let the hypotheses of Theorem 4.1 be satisfied, except the condition (A2) being replaced by

(A4)

\[
\begin{align*}
    b_1(\phi_0, x(t)) &\leq (\phi_0, V(t, x)) \leq a_1(\phi_0, x(t)), \\
    b_2(\phi_0, y(t)) &\leq (\phi_0, W(t, y)) \leq a_2(\phi_0, y(t)),
\end{align*}
\]

where \(a_i, b_i \in \mathcal{K}, i = 1, 2. \phi_0 \in \mathcal{K}_0^*\).

Then the zero solution of (3.1) is uniformly \(\phi_0\)-stable.

**Proof.** As in the proof of Theorem 4.1 and from (A4), we have

\[
\begin{align*}
    b_1(\phi_0, x^*(t)) &\leq (\phi_0, V(t, x^*(t))) \leq (\phi_0, V(t_0, x_0)) \leq a_1(\phi_0, x_0), \\
    b_2(\phi_0, y^*(t)) &\leq (\phi_0, W(t, y^*(t))) \leq (\phi_0, W(t_0, y_0)) \leq a_2(\phi_0, y_0).
\end{align*}
\]

For given \(\epsilon > 0\), let \(\delta_1 = a_1^{-1}b_1(\epsilon) > 0, \delta_2 = a_2^{-1}b_2(\epsilon) > 0\) independent of \(t_0\), choose \(\delta = \min[\delta_1, \delta_2]\), such that \((\phi_0, x_0) < \delta, (\phi_0, y_0) < \delta\). Then for any solution \(x(t), y(t)\) of (3.1)

\[
\begin{align*}
    b_1(\phi_0, x^*(t)) &\leq a_1(\phi_0, x_0) < a_1\delta_1 = b_1(\epsilon), \\
    b_2(\phi_0, y^*(t)) &\leq a_2(\phi_0, y_0) < a_2\delta_2 = b_2(\epsilon).
\end{align*}
\]

So

\[(\phi_0, x^*(t)) < \epsilon, \quad (\phi_0, y^*(t)) < \epsilon.\]

Hence system (3.1) is uniformly \(\phi_0\)-stable.

**Theorem 4.3.** Let the hypotheses of Theorem 4.1 be satisfied, except condition (A1) being replaced by

(A5)

\[
\begin{align*}
    D^+(\phi_0, V(t, x)) &\leq -c_1(\phi_0, V(t, x)), \\
    D^+(\phi_0, W(t, y)) &\leq -c_2(\phi_0, W(t, y)), \quad c_1, c_2 \in \mathcal{K}.
\end{align*}
\]

Then the zero solution of (3.1) is asymptotically \(\phi_0\)-stable.
Proof. Since the condition (A5) implies (A3), from Theorem 4.1, it follows that the zero solution of (3.1) is \( \phi_0 \)-equistable. By condition (A5), \( V(t, x) \), \( W(t, y) \) are monotone nonincreasing functions, thus the limits

\[
V^* = \lim_{t \to \infty} V(t, x), \quad W^* = \lim_{t \to \infty} W(t, y),
\]

exist. Now, we prove that \( V^* = 0 \) and \( W^* = 0 \). Suppose these are false, i.e., \( V^* \neq 0 \), \( W^* \neq 0 \), then \( c_1(V^*) \neq 0 \), \( c_2(W^*) \neq 0 \). Since \( c_1(r) \), \( c_2(r) \) are monotone increasing functions, then

\[
c_1(\phi_0, V(t, x)) \geq c_1(\phi_0, V^*), \quad c_2(\phi_0, W(t, y)) \geq c_2(\phi_0, W^*),
\]

and so from (A5), we get

\[
D^+(\phi_0, V(t, x)) \leq -c_1(\phi_0, V^*), \quad (4.3)
\]

\[
D^+(\phi_0, W(t, y)) \leq -c_2(\phi_0, W^*). \quad (4.4)
\]

Integrating (4.3), (4.4) on \([t_0, t], t \in (t_k, t_{k+1}]\), we obtain

\[
\begin{align*}
(\phi_0, V(t, x(t))) &\leq -c_1(\phi_0, V^*)(t - t_0) + (\phi_0, V(t_0, x_0)), \\
(\phi_0, W(t, y(t))) &\leq -c_2(\phi_0, W^*)(t - t_0) + (\phi_0, W(t_0, y_0)).
\end{align*}
\]

Thus, as \( k \to \infty \), we get

\[
(\phi_0, V(t, x(t))) \to -\infty, \quad (\phi_0, W(t, y(t))) \to -\infty.
\]

These contradict condition (A2). Therefore, \( V^*, W^* \) must be equal to zero. Hence

\[
(\phi_0, V(t, x)) \to 0, \quad (\phi_0, W(t, y)) \to 0 \quad \text{as} \quad k \to \infty, \quad t \in (t_k, t_{k+1}]. \quad (4.5)
\]

From (4.5) and condition (A2), we get

\[
(\phi_0, x(t)) \to 0, \quad (\phi_0, y(t)) \to 0 \quad \text{as} \quad k \to \infty, \quad t \in (t_k, t_{k+1}].
\]

Thus for given \( \varepsilon > 0 \), \( t_0 \in \mathbb{T} \), there exist \( \delta = \delta(t_0, \varepsilon) \) and \( T = T(t_0, \varepsilon) \) such that

\[
\begin{align*}
(\phi_0, x_0) < \delta &\implies (\phi_0, x^*(t)) < \varepsilon, \quad t \geq t_0 + T, \\
(\phi_0, y_0) < \delta &\implies (\phi_0, y^*(t)) < \varepsilon, \quad t \geq t_0 + T.
\end{align*}
\]

Then the system (3.1) is asymptotically \( \phi_0 \)-stable. \( \square \)

**Theorem 4.4.** Let the hypotheses of Theorem 4.2 be satisfied, and condition (A1) be replaced by

\[
D^+(\phi_0, V(t, x)) \leq -c_1(\phi_0, x(t)),
\]

\[
D^+(\phi_0, W(t, y)) \leq -c_2(\phi_0, y(t)),
\]

where \( c_1, c_2 \in \mathcal{K} \).

Then the system (3.1) is uniformly asymptotically \( \phi_0 \)-stable.
Proof. For given \( \varepsilon > 0 \), choose \( \delta > 0 \) independent of \( t_0 \). Suppose that \( (\phi_0, x_0) < \delta, (\phi_0, y_0) < \delta \), then by Theorem 4.2 the system (3.1) is uniformly \( \phi_0 \)-stable. We choose

\[
V^* = \left\{ \sup_{t_0} (\phi_0, V(t_0, x_0)) : (\phi_0, x_0) < \delta \right\},
\]

\[
W^* = \left\{ \sup_{t_0} (\phi_0, W(t_0, y_0)) : (\phi_0, y_0) < \delta \right\},
\]

and \( T_1(\varepsilon) = \frac{V^*}{c_1(\varepsilon)}, T_2(\varepsilon) = \frac{W^*}{c_2(\varepsilon)} \). Let \( T = \max\{T_1, T_2\} \), we prove that

\[
(\phi_0, x_0) < \delta \quad \text{implies} \quad (\phi_0, x(t)) < \varepsilon, \quad t \geq t_0 + T,
\]

\[
(\phi_0, y_0) < \delta \quad \text{implies} \quad (\phi_0, y(t)) < \varepsilon, \quad t \geq t_0 + T.
\]

Suppose that this is not true, then there exists at least one \( t \geq t_0 + T, t \in (t_k, t_{k+1}] \), \( k = 1, 2, 3, \ldots \), such that

\[
(\phi_0, x_0) < \delta \quad \text{implies} \quad (\phi_0, x^*(t)) \geq \varepsilon, \quad t \geq t_0 + T, \quad (4.6)
\]

\[
(\phi_0, y_0) < \delta \quad \text{implies} \quad (\phi_0, y^*(t)) \geq \varepsilon. \quad (4.7)
\]

From (4.6), (4.7), condition (A6) and the monotonicity of \( c_1, c_2 \), we get

\[
D^+(\phi_0, V(t, x(t))) \leq -c_1(\varepsilon), \quad (4.8)
\]

\[
D^+(\phi_0, W(t, y(t))) \leq -c_2(\varepsilon). \quad (4.9)
\]

Integrating (4.8), (4.9) on \([t_0, t] \), \( t \in (t_k, t_{k+1}] \), we get

\[
(\phi_0, V(t, x(t))) \leq -c_1(\varepsilon)(t - t_0) + (\phi_0, V(t_0, x_0)), \quad t \geq t_0 + T.
\]

\[
(\phi_0, W(t, y(t))) \leq -c_2(\varepsilon)(t - t_0) + (\phi_0, W(t_0, y_0)), \quad t \geq t_0 + T.
\]

Thus, as \( n \to \infty \), we get

\[
(\phi_0, V(t, x(t))) \to -\infty,
\]

\[
(\phi_0, W(t, y(t))) \to -\infty,
\]

which contradict condition (A4). Hence for each \( \varepsilon > 0, t_0 \in \mathbb{T} \), there exist \( \delta > 0 \) and \( T > 0 \) such that

\[
(\phi_0, x_0) < \delta \quad \text{implies} \quad (\phi_0, x^*(t)) < \varepsilon, \quad t \geq t_0 + T,
\]

\[
(\phi_0, y_0) < \delta \quad \text{implies} \quad (\phi_0, y^*(t)) < \varepsilon. \quad t \geq t_0 + T.
\]

Then the system (3.1) is uniformly asymptotically \( \phi_0 \)-stable. \( \square \)

Now we give a result via comparison principle.

Theorem 4.5. Assume \( K \subseteq \mathbb{R}^n \) is a cone and that

(i) \( V \in V_0 \) and for \( t_k < t \leq t_{k+1}, x, y \in S(h, \rho) \)

\[
D^+ V(t, x, y) \leq K g(t, V(t, x), W(t_k, y_k)), \quad t \neq t_k,
\]

\[
V(t^+, I_k(x)) \leq \psi_k(V(t, x)), \quad t = t_k,
\]

where \( \psi_k \) satisfies (B0)(iii);
(ii) $W \in W_0$ and for $t_k \leq t \leq t_{k+1}$
\[ D^+ W(t, y) \leq H(t, W(t, x), V(t_{k+1}, x^+_{k+1})) \]
where $H$ satisfies $(B_0)(iv)$, $H(t, v, u) \mu^*(t) + v$ is nondecreasing in $v$ for each $(t, u)$ and $H(t, v, u)$ is quasi-monotone nondecreasing in $u$ relative to $K$ for each $(t, v)$;

(iii) 
\[ b_1(\phi_0, x) \leq \left( \phi_0, V(t, x) \right) \leq a_1(\phi_0, x), \]
\[ b_2(\phi_0, y) \leq \left( \phi_0, W(t, y) \right) \leq a_2(\phi_0, y), \]
where $a_i, b_i \in K, i = 1, 2$.

Then the $\phi_0$-stability properties of the trivial solution of the hybrid impulsive system (3.4) imply the corresponding $\phi_0$-stability properties of the trivial solution of (3.1).

Proof. Let us first suppose that the system (3.4) is $\phi_0$-equistable. Let $\varepsilon > 0$ and $t_0 \in \mathbb{T}$ be given. Then for given $b_1(\varepsilon) > 0$ and $b_2(\varepsilon) > 0$, there exist $\delta_1, \delta_2 > 0$ such that
\[ (\phi_0, u_0) < \delta_1 \quad \text{implies} \quad (\phi_0, u(t, t_0, u_0)) < b_1(\varepsilon), \quad t \geq t_0, \]
\[ (\phi_0, v_0) < \delta_2 \quad \text{implies} \quad (\phi_0, v(t, t_0, v_0)) < b_2(\varepsilon), \quad t \geq t_0, \quad t \in \mathbb{T}, \]
where $u(t, t_0, u_0), v(t, t_0, v_0)$ are any solution of (3.4). Choose $\delta = \min\{a_1^{-1}(\delta_1), a_2^{-1}(\delta_2)\}$ and let $(\phi_0, x_0) < \delta, (\phi_0, y_0) < \delta$, then we claim that $(\phi_0, x(t)) < \varepsilon, t \geq t_0$ and $(\phi_0, y(t)) < \varepsilon, t \geq t_0, t \in \mathbb{T}$, where $x(t), y(t)$ are the solutions of (3.1). If this is not true, there would exist solutions $x(t), y(t)$ of (3.1) with $(\phi_0, x_0) < \delta, (\phi_0, y_0) < \delta$ such that either

(i) there exists $t^* > t_0$ with $t_k < t^* \leq t_{k+1}$ for some $k$, satisfying $(\phi_0, x(t^*)) \geq \varepsilon$ and $(\phi_0, x(t)) < \varepsilon, t_0 \leq t \leq t_k$, or
(ii) there exists $t_1 > t_0, t_1 \in \mathbb{T}$, satisfying $(\phi_0, x(t_1)) \geq \varepsilon$ and $(\phi_0, y(t)) < \varepsilon, t_0 < t < t_1, t \in \mathbb{T}$.

Suppose (i) holds. Then we can find $t^0$ such that $t_k < t^0 \leq t^*$ satisfying $(\phi_0, x(t^0)) \geq \varepsilon$. Choose $u_0 = V(t_0, x_0^+)$. Then by Theorem 3.1, we arrive at
\[ V(t, x(t)) \leq r(t, t_0, u_0), \quad t_0 \leq t \leq t^0. \]

But then we would have
\[ (\phi_0, u_0) = (\phi_0, V(t_0, x_0^+)) \leq a_1(\phi_0, x_0) < a_1(\delta) < \delta_1, \]
\[ b_1(\varepsilon) \leq b_1(\phi_0, x(t^0)) \leq (\phi_0, V(t^0, x(t^0))) \leq (\phi_0, r(t^0, t_0, u_0)) < b_1(\varepsilon), \]
which is a contradiction. If, on the other hand, (ii) holds, we get by Theorem 3.1
\[ W(t, y(t)) \leq R(t, t_0, v_0), \quad t_0 \leq t \leq t_1, \quad t \in \mathbb{T}, \quad v_0 = W(t_0, y_0). \]

In this case, it follows that
\[ b_2(\varepsilon) \leq b_2(\phi_0, y(t_1)) \leq (\phi_0, W(t_1, y(t_1))) \leq (\phi_0, R(t_1, t_0, v_0)) < b_2(\varepsilon), \]
where
\[ (\phi_0, v_0) = (\phi_0, W(t_0, y_0)) \leq a_2(\phi_0, y_0) < a_2(\delta) < \delta_2, \]
which is also a contradiction. Other $\phi_0$-stability can be prove similarly. The proof is completed.
5. Example

Consider a simple hybrid impulsive differential system

\[
x' = -14x - x^2 \exp(x) + 2y_k - y_k^2 \exp(x), \quad t \neq t_k,
\]

\[
x(t_k^+) = \beta_k(x(t_k)), \quad 0 < \beta_k \leq 1,
\]

\[
x(t_0^+) = x_0^+,
\]

\[
y^\Delta = -18\beta_k x_k(t_k) - y^2 \exp(y) + 2y - \beta_k^2 x_k^2(t_k) \exp(y), \quad y(t_k) = y_k,
\]

\[
y(t_0) = y_0.
\]

(5.1)

We now consider a vector Lyapunov function

\[
V(t, x, y_k) = \left( V(t, x), W(t_k, y_k) \right)^T,
\]

where \( V(t, x) = \max_{t \in \mathbb{T}} \| x \| \) and \( W(t, y) = \| y \| \). Here \( \mathbb{T} = [t_0, \infty) \). Then we have

\[
D^+ V(t, x) \leq -7 \max \| x \| + 2 \| y_k \| = -7V(t, x) + 2W(t_k, y_k),
\]

and

\[
D^+ W(t_k, y_k) \leq -9 \| x_k(t_k) \| + 2 \| y_k \| \leq -9V(t, x) + 2W(t_k, y_k).
\]

Therefore

\[
D^+ V(t, x, y_k) \leq \begin{pmatrix} -7 & 2 \\ -9 & 2 \end{pmatrix} \begin{pmatrix} V(t, x) \\ W(t_k, y_k) \end{pmatrix} = g(t, V(t, x), W(t_k, y_k)), \quad t \neq t_k,
\]

and

\[
V(t_k^+, \beta_k(x(t_k))) = \max_{t \in \mathbb{T}} \| \beta_k(x(t_k)) \| \leq \beta_k \max_{t \in \mathbb{T}} \| x \| = \beta_k V(t, x).
\]

But in the comparison system

\[
u' = g(t, u, v_k) = Au, \quad A = \begin{pmatrix} -7 & 2 \\ -9 & 2 \end{pmatrix},
\]

\[
g \text{ is not quasi-monotone nondecreasing in } u. We now seek to construct a cone } K \subset \mathbb{R}^2_+ \text{ relative to which the system (5.2) is quasi-monotone. The eigenvalues of } A \text{ in (5.2) are given by the roots of the equation}
\]

\[
\lambda^2 + 5\lambda + 4 = 0 \quad \Rightarrow \quad \lambda_1 = -1, \quad \lambda_2 = -4.
\]

The eigenvectors are \((1, 3)^T\) and \((1, 3/2)^T\) corresponding to \(\lambda_1 = -1\) and \(\lambda_2 = -4\), respectively. Choose \(B = \begin{pmatrix} 1/3 & 1/2 \\ 1/3 & 1/2 \end{pmatrix}\), then \(B^{-1} = \begin{pmatrix} 2/3 & -2 \\ 1 & 1 \end{pmatrix}\) and \(B^{-1}AB = \begin{pmatrix} -4 & 0 \\ 0 & -1 \end{pmatrix}\). Thus the off-diagonal elements of \(B^{-1}AB\) are nonnegative. Clearly \(B\) is a nonnegative, nonsingular \(2 \times 2\) matrix with which the mapping \(u = Bm\) transforms (5.2) into

\[
m' = B^{-1}ABm.
\]

Then as in [3], there exists a cone \(K = \{ \Sigma_{i=1}^2 u_i b_i; \ u_i \geq 0, \ i = 1, 2 \} \subset \mathbb{R}_+^2\), generated by the 2 linearly independent column vectors of \(B\) relative to which (5.2) is quasi-monotone. As in [3], we choose

\[
V(t, x, y_k) = x(t, 0, \sigma_w(x(0, t, x)))
\]

(5.3)

as a cone-valued Lyapunov function for (5.2).
In a similar way, we obtain
\[ D^+ W(t, y) \leq \begin{pmatrix} -7 & 2 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} W(t, x) \\ V(t_k, x_k) \end{pmatrix} = H(t, W(t, x), V(t_k, x_k)), \]
\( t_k \leq t \leq t_{k+1}. \)

Then for system (5.1), we have another comparison system
\[ v^\Delta = H(t, v, u_{k+1}^+) = Cv, \quad C = \begin{pmatrix} -7 & 2 \\ 5 & 2 \end{pmatrix}. \] (5.4)

In the cone we have chosen above, we choose another cone-valued Lyapunov function
\[ W(t, y) = y(t, 0, \sigma_w(y(0, t, y))). \] (5.5)

It is easy to check that the right side of (5.1) satisfies the conditions of Theorem 4.1 in [3] and so (5.2) and (5.4) have the properties of (i)–(iii) of Theorem 4.5.

Thus
\[ r(t) = u_0 \exp(tA) \quad \Rightarrow \text{ } (\phi_0, r(t)) = (\phi_0, u_0) \exp(tA) \]
and
\[ R(t) = v_0 \exp(tC) \quad \Rightarrow \text{ } (\phi_0, R(t)) = (\phi_0, v_0) \exp(tC). \]

Now given \( \varepsilon > 0, \) there exist \( \delta_1, \delta_2 \) such that \( (\phi_0, u_0) < \delta_1 \) and \( (\phi_0, v_0) < \delta_2. \) Choose \( \delta_1 = \varepsilon \exp(-tA) \) and \( \delta_2 = \varepsilon \exp(-tC). \) Then for \( \delta = \min(\delta_1, \delta_2), \) we have \( (\phi_0, r(t)) < \varepsilon \) and \( (\phi_0, R(t)) < \varepsilon. \) This shows that \( u = 0 \) of (5.2) and \( v = 0 \) of (5.4) is \( \phi_0 \)-equistable. Thus Theorem 4.5 implies that the trivial solution of (5.1) is equistable.

Acknowledgments

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References