Variational iteration method—Some recent results and new interpretations

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Abstract

This paper is an elementary introduction to the concepts of variational iteration method. First, the main concepts in variational iteration method, such as general Lagrange multiplier, restricted variation, correction functional, are explained heuristically. Subsequently, the solution procedure is systematically addressed, in particular, for nonlinear oscillators. Particular attention is paid throughout the paper to give an intuitive grasp for the method. The main motivation is to put things together in a convenient form for later reference and systematic use.
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1. Introduction

The variational iteration method [8–11,15] has been extensively worked out over a number of years by numerous authors. Starting from pioneering ideas going back to Inokuti–Sekine–Mura method [16], the method has matured into a fully fledged theory thanks to the effects of many researchers, notably, Wazwaz [27], Draganescu [6,7], Abdou and Soliman [1–4,23,24], Odibat and Momani [19–22], Bildik [5], Marinca [17], Moghimi [18], Dehghan [26], Sweilam [25] to mention only a few. For a relatively comprehensive survey on the method and its applications, the reader is referred to present author's review article [13] and monograph [14].

Recently, some rather extraordinary virtues of the method have been exploited, and wide applications have been found in various fields [1–7,17–26].

In this paper, we will survey some basic concepts, theory and applications of the variational iteration method.

2. Basic concepts in the variational iteration method

In the course of developing the variational iteration method, we introduced some new nomenclatures, such as restricted variation, correction functional. Heuristic interpretation of those concepts leads new comers to the field to

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start working immediately without the long search and preparation of advanced calculus and calculus of variations, at the same time those already familiar with variational iteration method will find the most recent new results.

2.1. General Lagrange multiplier

We know Lagrange multiplier well in optimization and calculus of variations. Inokuti et al. suggested a method of general Lagrange multiplier [16]. In order to understand the concept of the general Lagrange multiplier, we consider an algebraic equation

$$f(x) = 0, \quad x \in \mathbb{R}. \tag{1}$$

If x_n is an approximate root of the above equation, it follows:

$$f(x_n) \neq 0. \tag{2}$$

To improve its accuracy, we write the following correction equation:

$$x_{n+1} = x_n + \lambda f(x_n), \tag{3}$$

where λ is a general Lagrange multiplier, which can be identified optimally by setting

$$\frac{\mathrm{d}x_{n+1}}{\mathrm{d}x_n} = 0, \tag{4}$$

which leads to the well-known Newton iteration formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{5}$$

There are alternative approaches to construction of correction equation. We write another correction for x_n as follows:

$$x_{n+1} = x_n + \lambda g(x_n) f(x_n), \tag{6}$$

where $g(x)$ is an auxiliary function. After identification of the multiplier, we have a general iteration formulation:

$$x_{n+1} = x_n - \frac{g(x_n) f(x_n)}{g(x_n) f'(x_n) + g'(x_n) f(x_n)}. \tag{7}$$

The value of the auxiliary function should not be zero or small value during the all iteration steps, $|g(x_n)| > 1$. If we choose $g(x_n) = \mathrm{e}^{-\alpha x_n}$, the iteration formulation, Eq. (7), reduces to

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n) - \alpha f(x_n)}. \tag{8}$$

This iteration formulation is very effective when $f'(x_n)$ is small. As an example, we consider the equation

$$\sin x = 0. \tag{9}$$

If we begin with $x_0 = 1.6$, the Newton iteration is not valid for $\cos 1.6$ is a small value. Table 1 shows the iteration procedure, the nearest solution near $x_0 = 1.6$ is $x = \pi$.

Table 1

Iteration	$x \ (\alpha = 1)$	$x \ (\alpha = 2)$	$x \ (\alpha) = 0.97$
0	1.6	1.6	1.6
1	2.57	2.09	2.60
2	2.96	2.48	2.98
3	3.12	2.78	3.12
4	3.14	2.99	3.14

We also obtained some iteration formulae by general Lagrange multiplier [14]

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f''(x_n) f^2(x_n)}{2 f'^3(x_n)}, \tag{10}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f''(x_n) f^2(x_n)}{2 f'^3(x_n)} + \frac{[f'''(x_n) f'^3(x_n) - 3 f''^2(x_n) f'^2(x_n)] f^3(x_n)}{2 f'^7(x_n)}. \tag{11}$$

If Eq. (1) is replaced by a differential equation, then a correctional functional similar to Eq. (3) can be established.

2.2. Stationary conditions

The problem of optimization is ubiquitous in nature. The simplest problem of the calculus of variation [12] is to determine a function $y = f(x)$ for which the value of a given functional

$$J = \int_{x_1}^{x_2} F(y, y'; x) \, \mathrm{d}x + g_1(x) y|_{x=x_1} - g_2(x) y|_{x=x_2} \tag{12}$$

is a maximum or a minimum.

The extremum condition (stationary condition) of the functional (12) requires that

$$\delta J = \delta \int_{x_1}^{x_2} F(y, y'; x) \, \mathrm{d}x + g_1 \delta y|_{x=x_1} - g_2 \delta y|_{x=x_2}$$

$$= \int_{x_1}^{x_2} \delta F(y, y'; x) \, \mathrm{d}x + g_1 \delta y|_{x=x_1} - g_2 \delta y|_{x=x_2}$$

$$= \int_{x_1}^{x_2} \left\{ \frac{\mathrm{d}F}{\mathrm{d}y} \delta y + \frac{\mathrm{d}F}{\mathrm{d}y'} \delta y' \right\} \mathrm{d}x + g_1 \delta y|_{x=x_1} - g_2 \delta y|_{x=x_2}$$

$$= \int_{x_1}^{x_2} \left\{ \frac{\mathrm{d}F}{\mathrm{d}y} \delta y + \frac{\mathrm{d}F}{\mathrm{d}y'} \frac{\mathrm{d}}{\mathrm{d}x} (\delta y) \right\} \mathrm{d}x + g_1 \delta y|_{x=x_1} - g_2 \delta y|_{x=x_2}$$

$$= \int_{x_1}^{x_2} \left\{ \left[\frac{\mathrm{d}F}{\mathrm{d}y} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\mathrm{d}F}{\mathrm{d}y'} \right) \right] \delta y + \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\mathrm{d}F}{\mathrm{d}y'} \delta y \right) \right\} \mathrm{d}x + g_1 \delta y|_{x=x_1} - g_2 \delta y|_{x=x_2}$$

$$= \int_{x_1}^{x_2} \left\{ \left[\frac{\mathrm{d}F}{\mathrm{d}y} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\mathrm{d}F}{\mathrm{d}y'} \right) \right] \delta y \right\} \mathrm{d}x + \left[\frac{\mathrm{d}F}{\mathrm{d}y'} \delta y \right]_{x_1}^{x_2} + g_1 \delta y|_{x=x_1} - g_2 \delta y|_{x=x_2}$$

$$= 0.$$

For arbitrary δy, from the above relation , we have

$$\frac{\mathrm{d}F}{\mathrm{d}y} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\mathrm{d}F}{\mathrm{d}y'} \right) = 0, \tag{13}$$

and the boundary conditions

$$\frac{\mathrm{d}F}{\mathrm{d}y'}(x_1) - g_1(x_1) = 0 \quad \text{and} \quad \frac{\mathrm{d}F}{\mathrm{d}y'}(x_2) - g_2(x_2) = 0. \tag{14}$$

Eq. (13) is called *Euler–Lagrange's differential equation*, or *Euler's equation*, and Eq. (14) is known as the *natural boundary conditions*.

2.3. Restricted variation

To illustrate how restricted variation works in variational iteration method, we consider a simple algebraic equation

$$x^2 - 3x + 2 = 0. \tag{15}$$

Table 2

Iteration	Eq. (18)	Newton iteration formulation
0	0.5	0.5
1	0.8	0.875
2	0.909	0.987
3	0.956	0.999
4	0.978	1.000
5	0.989	1.000
6	0.994	1.000
7	0.997	1.000
8	0.998	1.000
9	0.999	1.000

We re-write Eq. (15) in the form

$$\tilde{x} \cdot x - 3x + 2 = 0, \tag{16}$$

where $\tilde{x}$ is called restricted variable, the value of $\tilde{x}$ is assumed to be known (initial guess). Solving x from (16) leads to the result

$$x = \frac{2}{3 - \tilde{x}} \tag{17}$$

or an iteration form

$$x_{n+1} = \frac{2}{3 - x_n}. \tag{18}$$

This method is often very efficient for good prediction, see Table 2.

In variational iteration method, initial guess is always chosen with a possible unknown parameter, one iteration leads to highly accurate solutions. We illustrate the effectiveness of free parameter in the above example. Introducing a free parameter in the initial prediction:

$$x_0 = 0.5 + b, \tag{19}$$

where b is a small parameter to be further determined, and substituted (19) into (18), we have

$$x_1 = \frac{2}{3 - 0.5 - b} = \frac{2}{2.5 - b} = \frac{2}{2.5\left(1 - \frac{1}{2.5}b\right)} = 0.8\left(1 + \frac{1}{2.5}b\right) + \mathrm{O}(b^2) = 0.8 + 0.32b + \mathrm{O}(b^2). \tag{20}$$

To identify the value b we set

$$x_0 = x_1 \tag{21}$$

or

$$0.8 + 0.32b = 0.5 + b. \tag{22}$$

From above relation, we can immediately identify

$$b = 0.4412. \tag{23}$$

So the updated approximate root is $x_1 = 0.9412$.

Now we consider restricted variation in a variational functional. Consider temperature distribution in convective straight fins with temperature-dependent thermal conductivity, the dimensionless governing equation is [15]

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1 + \beta\theta)\frac{\mathrm{d}\theta}{\mathrm{d}x}\right] - \psi^2\theta = 0, \quad \theta'(0) = 0, \quad \theta(1) = 1, \tag{24}$$

where θ is dimensionless temperature, β and ψ are constants.

Use the concept of restricted variation, an approximate variational functional can be established:

$$J(\theta) = \int_0^1 \left\{ (1 + \beta\tilde{\theta}) \left(\frac{d\theta}{dx} \right)^2 + \psi^2\theta^2 \right\} dx, \tag{25}$$

where $\tilde{\theta}$ is a restricted variation, i.e., $\delta\tilde{\theta} = 0$.

We write (24) in an iteration form

$$J(\theta_{n+1}) = \int_0^1 \left\{ (1 + \beta\theta_n) \left(\frac{d\theta_{n+1}}{dx} \right)^2 + \psi^2\theta_{n+1}^2 \right\} dx. \tag{26}$$

We begin with an initial guess satisfying the boundary condition $\theta_0'(0) = 0$ and $\theta_0(1) = 1$:

$$\theta_0 = 1 - a + ax^2, \tag{27}$$

where a is free parameter. We apply Ritz method to solving θ_1, the trial-function for θ_1 is assumed to be have the form

$$\theta_1 = 1 - b + bx^2, \tag{28}$$

where the unknown parameter b to be further determined. Substituting (27) into (25) yields

$$J = \int_0^1 \{4(1 + \beta(1 - a + ax^2))b^2x^2 + \psi^2(1 - b + bx^2)^2\} dx$$

$$= \tfrac{4}{3}(1 + \beta(1 - a))b^2 + \tfrac{4}{5}ab^2 + \psi^2(1 - b)^2 + \tfrac{2}{3}b(1 - b)\psi^2 + \tfrac{1}{5}b^2\psi^2. \tag{29}$$

Minimizing the functional, Eq. (25), with respect to θ is approximately equivalent to minimizing the above function, Eq. (28), with respect to b:

$$\frac{dJ}{db} = \frac{8}{3}(1 + \beta(1 - a))b + \frac{8}{5}ab - 2\psi^2(1 - b) + \frac{2}{3}(1 - 2b)\psi^2 + \frac{2}{5}b\psi^2 = 0. \tag{30}$$

We set $\theta_0 = \theta_1$, then we can identify

$$a = \frac{-(20 + 20\beta + 8\psi^2) + \sqrt{(20 + 20\beta + 8\psi^2)^2 + 40\psi^2(12 - 20\beta)}}{24 - 40\beta}. \tag{31}$$

3. Variational iteration method

The variational iteration method [2–4,23] has been shown to solve effectively, easily, and accurately a large class of nonlinear problems with approximations converging rapidly to accurate solutions.

To illustrate its basic idea of the technique, we consider following general nonlinear system:

$$L[u(t)] + N[u(t)] = g(t), \tag{32}$$

where L is a linear operator, N is a nonlinear operator, and $g(t)$ is a given continuous function.

The basic character of the method is to construct a correction function for system (2), which reads

$$u_{n+1}(t) = u_n(t) + \int_{t_0}^t \lambda\{Lu_n(s) + N\tilde{u}_n(s) - g(s)\} ds, \tag{33}$$

where λ is a general Lagrange multiplier which can be identified optimally via variational theory, u_n is the nth approximate solution, and $\tilde{u}_n$ denotes a restricted variation, i.e., $\delta\tilde{u}_n = 0$.

For linear problems, its exact solution can be obtained by only one iteration step due to the fact that the Lagrange multiplier can be exactly identified. Considering the following example.

The linear differential equation of first order

$$u' + a(t)u = b(t), \quad u(0) = c. \tag{34}$$

Its correction functional can be written in the form

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda \left\{ \frac{du_n}{ds} + a(s)u_n(s) - b(s) \right\} ds. \tag{35}$$

Making the above correct functional stationary with respect to u_n, noticing that $\delta u_n(0) = 0$,

$$\delta u_{n+1}(t) = \delta u_n(t) + \delta \int_0^t \lambda \left\{ \frac{du_n}{ds} + a(s)u_n(s) - b(s) \right\} ds$$

$$= \delta u_n(t) + \lambda \delta u_n(s)|_{s=t} + \int_0^t \left\{ -\frac{\partial \lambda}{\partial s} + a(s)\lambda \right\} \delta u_n(s) \, ds = 0. \tag{36}$$

Thus, we obtain the Euler–Lagrange equation

$$-\frac{\partial \lambda(t, s)}{\partial s} + a(s)\lambda(t, s) = 0, \tag{37}$$

and the natural boundary condition

$$1 + \lambda(t, t) = 0. \tag{38}$$

We, therefore, identify the Lagrange multiplier in the form

$$\lambda(t, s) = -\exp \left\{ \int_0^s a(\xi) \, d\xi - \int_0^t a(\xi) \, d\xi \right\}. \tag{39}$$

Substituting the identified Lagrange multiplier into (2) results in the following iteration formulation:

$$u_{n+1}(t) = u_n(t) - \int_0^t \exp \left\{ \int_0^s a(\xi) \, d\xi - \int_0^t a(\xi) \, d\xi \right\} \left\{ \frac{du_n}{ds} + a(s)u_n(s) - b(s) \right\} ds. \tag{40}$$

If we begin with $u_0 = c \exp\{-\int_0^t a(\xi) \, d\xi\}$, the solution the homogeneous equation, $u' + a(t)u = 0$ with initial condition $u(0) = c$, then

$$u_1(t) = c \exp \left\{ -\int_0^t a(\xi) \, d\xi \right\} - \int_0^t b(s) \exp \left\{ \int_0^s a(\xi) \, d\xi - \int_0^t a(\xi) \, d\xi \right\} ds$$

$$= c \exp \left\{ -\int_0^t a(\xi) \, d\xi \right\} + \exp \left\{ \int_0^t a(s) \, ds \right\} \int_0^t b(s) \exp \left\{ \int_0^s a(\xi) \, d\xi \right\} ds, \tag{41}$$

which is the exact explicit solution. From the solution all properties can be quickly and effortlessly deduced.

Now consider the linear oscillator with a forced term:

$$u''(t) + \omega^2 u(t) = f(u, u', u''). \tag{42}$$

Its correction functional can be written in the form

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda \{ u_n''(s) + \omega^2 u_n(s) - \tilde{f}(u_n, u_n', u_n'') \} \, ds. \tag{43}$$

Calculating variation with respect to u_n, noticing that $\delta u_n(0) = 0$, yields

$$\delta u_{n+1}(t) = \delta u_n(t) + \delta \int_0^t \lambda \left\{ u_n''(s) + \omega^2 u_n(s) - f(s) \right\} ds$$

$$= \delta u_n(t) + \lambda(s) \delta u_n'(s)|_{s=t} - \frac{\partial \lambda}{\partial s} \delta u_n(s)|_{s=t} + \int_0^t \left\{ \frac{\partial^2 \lambda}{\partial s^2} + \omega^2 \lambda(s) \right\} \delta u_n(s) \, ds$$

$$= \left(1 - \frac{\partial \lambda}{\partial s} \right) \delta u_n(s)|_{s=t} + \lambda(s) \delta u_n'(s)|_{s=t} + \int_0^t \left\{ \frac{\partial^2 \lambda}{\partial s^2} + \omega^2 \lambda(s) \right\} \delta u_n(s) \, ds = 0. \tag{44}$$

We, therefore, have the following stationary conditions:

$$\frac{\partial^2 \lambda(t, s)}{\partial s^2} + \omega^2 \lambda(t, s) = 0, \tag{45}$$

$$1 - \left. \frac{\partial \lambda(t, s)}{\partial s} \right|_{t=s} = 0, \tag{46}$$

$$\lambda(t, s)|_{t=s} = 0. \tag{47}$$

The Lagrange multiplier, therefore, can be readily identified

$$\lambda = \frac{1}{\omega} \sin \omega(s - t). \tag{48}$$

As a result, we obtain the following iteration formula:

$$u_{n+1}(t) = u_n(t) + \frac{1}{\omega} \int_0^t \sin \omega(s - t) \left\{ u_n''(s) + \omega^2 u_n(s) - f(u_n, u_n', u_n'') \right\} ds. \tag{49}$$

If we use its complementary solution $u_0(t) = C_1 \cos \omega t + C_2 \sin \omega t$ with suitable constants C_1 and C_2 as an initial approximation, by the iteration formula (18), we have

$$u_1(t) = C_1 \cos \omega t + C_2 \sin \omega t - \frac{1}{\omega} \int_0^t f(s) \sin \omega(s - t) \, ds, \tag{50}$$

which is the general solution of (11).

For nonlinear equation, the Lagrange multiplier is difficult to be identified. To overcome the difficulty, we apply restricted variations to nonlinear terms. In order to best illustrate the restricted variations, we consider the following linear equations:

$$u''(t) + \omega^2 u(t) = 0. \tag{51}$$

We write the correction functional as follows:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda \left\{ u_n''(s) + \omega^2 \tilde{u}_n(s) \right\} ds. \tag{52}$$

Herein $\tilde{u}_n$ is considered as a restricted variation, i.e., $\delta \tilde{u}_n = 0$. Under this condition, its stationary conditions of the above correction functional (21) can be expressed as follows:

$$\frac{\partial^2 \lambda(t, s)}{\partial s^2} = 0, \tag{53}$$

$$1 - \left. \frac{\partial \lambda(t, s)}{\partial s} \right|_{t=s} = 0, \tag{54}$$

$$\lambda(t, s)|_{t=s} = 0. \tag{55}$$

The Lagrange multiplier, therefore, can be easily identified as

$$\lambda = s - t, \tag{56}$$

leading to following iteration formula:

$$u_{n+1}(t) = u_n(t) + \int_0^t (s - t)\{u_n''(s) + \omega^2 u_n(s)\} \, ds. \tag{57}$$

If, for example, the initial conditions are $u(0) = 1$ and $u'(0) = 0$, we begin with $u_0(0) = u(0) = 1$, by the above iteration formula (26) we have following approximate solutions:

$$u_1(t) = 1 - \frac{1}{2!}\omega^2 t^2, \tag{58}$$

$$u_2(t) = 1 - \frac{1}{2!}\omega^2 t^2 + \frac{1}{4!}\omega^4 t^4, \tag{59}$$

and

$$u_n(t) = 1 - \frac{1}{2!}\omega^2 t^2 + \cdots + (-1)^n \frac{1}{(2n)!}\omega^{2n} t^{2n}. \tag{60}$$

From the above solution procedure, we can see clearly that the approximate solutions converge to its exact solution $\cos \omega t$ relatively slowly due to the approximate identification of the multiplier. It should be specially pointed out that the more accurate the identification of the multiplier, the more fast the approximations converge to its exact solution.

Now consider the following nonlinear equation:

$$u' + a(t)u = b(t) + N(u), \quad u(0) = c. \tag{61}$$

Here N is a nonlinear function of u. Following the above notation, we may set

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda \left\{ \frac{du_n}{ds} + a(s)u_n(s) - b(s) - N(\tilde{u}_n) \right\} ds, \tag{62}$$

where $\tilde{u}_n$ is a restricted variation, i.e., $\delta \tilde{u}_n = 0$.

Proceeding along the lines indicated above, we obtain the following iteration formulation:

$$u_{n+1}(t) = u_n(t) - \int_0^t \exp\left\{ \int_0^s a(\xi) \, d\xi - \int_0^t a(\xi) \, d\xi \right\} \left\{ \frac{du_n}{ds} + a(s)u_n(s) - b(s) - N(u_n) \right\} ds. \tag{63}$$

Consider the following second-order nonlinear equation

$$u'' + a(t)u' + b(t)u + N(u) = 0, \quad u(0) = c_1, \quad u'(0) = c_2. \tag{64}$$

Its correction functional can be written in the form

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda \left\{ u_n''(s) + a(s)u_n'(s) + b(s)u_n(s) + N(\tilde{u}_n(s)) \right\} ds. \tag{65}$$

Making the above correct functional stationary with respect to u_n, noticing that $\delta u_n(0) = 0$,

$$\delta u_{n+1}(t) = \delta u_n(t) + \delta \int_0^t \lambda \left\{ u_n''(s) + au_n'(s) + bu_n(s) + N(\tilde{u}_n(s)) \right\} ds$$

$$= \delta u_n(t) + \lambda \delta u_n'(s)|_{s=t} - \frac{\partial \lambda}{\partial s} \delta u_n(s)|_{s=t} + a\lambda(s)\delta u_n(s)|_{s=t}$$

$$+ \int_0^t \left\{ \frac{\partial^2 \lambda}{\partial s^2} - \frac{\partial}{\partial s}(a\lambda) + b(s)\lambda(s) \right\} \delta u_n(s) \, ds = 0, \tag{66}$$

we have the following stationary conditions:

$$\frac{\partial^2 \lambda}{\partial s^2} - \frac{\partial}{\partial s}(a\lambda) + b(s)\lambda(t, s) = 0, \tag{67}$$

$$1 - \frac{\partial \lambda}{\partial s}(t, t) + a(t)\lambda(t, t) = 0, \tag{68}$$

$$\lambda(t, t) = 0. \tag{69}$$

From the above relations, Eqs. (40)–(42), we can identify the Lagrange multiplier as follows:

$$\lambda = \exp\left\{\int_0^s a(\xi)\,d\xi\right\}[u_1(s)u_2(t) - u_2(s)u_1(t)]. \tag{70}$$

If w_1 and w_2 are chosen to be a fundamental set of solutions specified by the initial conditions

$$w_1(0) = 1, \quad w_1'(0) = 0,$$

$$w_2(0) = 0, \quad w_2'(0) = 1. \tag{71}$$

To first approximation, we have

$$u_1(t) = c_1 w_1 + c_2 w_2 + \int_0^t N(u_0(s))\lambda(t, s)\,ds. \tag{72}$$

Consider the following inhomogeneous equation:

$$u'' + a(t)u' + b(t)u = f(u), \tag{73}$$

where the solution is specified by values at two points, say

$$u(0) = c_1, \quad u(1) = c_2. \tag{74}$$

We shall let $u_{(1)}$ and $u_{(2)}$ to the two principal solutions; that is

$$u_{(1)}(0) = 1, \quad u_{(1)}'(0) = 0, \tag{75}$$

$$u_{(2)}(0) = 0, \quad u_{(2)}'(0) = 1, \tag{76}$$

and write

$$u = c_1 u_{(1)} + c_2 u_{(2)}. \tag{77}$$

Its correction functional can be written in the form

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda\{u_n''(s) + au_n'(s) + bu_n(s) - \tilde{f}_n\}\,ds. \tag{78}$$

Making the above correct functional stationary with respect to u_n, noticing that $\delta u_n(0) = 0$,

$$\delta u_{n+1}(t) = \delta u_n(t) + \delta \int_0^t \lambda\{u_n''(s) + au_n'(s) + bu_n(s) - \tilde{f}_n\}\,ds$$

$$= \delta u_n(t) + \lambda(s)\delta u_n'(s)|_{s=t} - \lambda'(s)\delta u_n(s)|_{s=t} + a\lambda(s)\delta u_n(s)|_{s=t}$$

$$+ \int_0^t \left\{\frac{d^2\lambda}{ds^2} - \frac{d}{ds}(a\lambda) + b(s)\lambda(s)\right\}\delta u_n(s)\,ds = 0,$$

we have the following stationary conditions:

$$\begin{cases} \delta u_n(s): & \dfrac{\mathrm{d}^2\lambda}{\mathrm{d}s^2} - \dfrac{\mathrm{d}}{\mathrm{d}s}(a\lambda) + b(s)\lambda(t,s) = 0, \\[2mm] \delta u_n(t): & 1 - \dfrac{\mathrm{d}\lambda}{\mathrm{d}s} + a(s)\lambda(s,s) = 0, \\[2mm] \delta u_n'(t): & \lambda(s,s) = 0. \end{cases} \tag{79}$$

The Lagrange multiplier can be identified in the form

$$\lambda = -\exp\left\{\int_0^s a(\xi)\,\mathrm{d}\xi\right\}[u_{(1)}(s)u_{(2)}(t) - u_{(2)}(s)u_{(1)}(t)]. \tag{80}$$

So we obtain the following iteration formulation:

$$u_{n+1}(t) = u_n(t) - \int_0^t \exp\left\{\int_0^s a(\xi)\,\mathrm{d}\xi\right\}[u_{(1)}(s)u_{(2)}(t) - u_{(2)}(s)u_{(1)}(t)]$$

$$\{u_n''(s) + au_n'(s) + bu_n(s) - f_n\}\,\mathrm{d}s. \tag{81}$$

Example 1 (*Mathematical pendulum*). When friction is neglected, the differential equation governing the free oscillation of the mathematical pendulum is given by

$$u'' + \omega^2 \sin u = 0, \tag{82}$$

with initial conditions $u(0) = A$ and $u'(0) = 0$.

Herein u designates the deviation angle from the vertical equilibrium position, $\omega^2 = g/l$, where g is the gravitational acceleration, l the length of the pendulum.

System (82) seems to be very simple, but what appears to be a simple problem is really not that simple unless the approximation $\sin u \approx u$ is used, thus the nonlinear differential equation (82) can be replaced by following linear one:

$$u'' + \omega^2 u = 0. \tag{83}$$

For small u this should lead to good results. To obtain a more accurate approximate solution, we expand $\sin u \approx u - u^3/6$, as a result, we get

$$u'' + \omega^2 u - \tfrac{1}{6}\omega^2 u^3 = 0. \tag{84}$$

In order to give a somewhat more general treatment, in Ref. [22] the investigation is based on the Duffing equation

$$u'' + \omega^2 u + \varepsilon u^3 = 0, \tag{85}$$

where $\varepsilon = -\omega^2/6$.

If the angular frequency of system (82) is Ω, system (82) can be equivalently written in the form

$$u'' + \Omega^2 u + F(u) = 0, \tag{86}$$

where $F(u) = \omega^2 \sin u - \Omega^2 u$.

Its correction functional can be constructed as follows:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda\{u_n''(s) + \Omega^2 u_n(s) + \tilde{F}_n\}\,\mathrm{d}s. \tag{87}$$

By the same manipulation, the multiplier can be identified and the following iteration formula can be obtained as

$$u_{n+1}(t) = u_n(t) + \frac{1}{\Omega}\int_0^t \sin\Omega(s-t)\{u_n''(s) + \omega^2 \sin u_n(s)\}\,\mathrm{d}s. \tag{88}$$

If we begin with $u_0(t) = A \cos \Omega t$, we have

$$u_1(t) = A \cos \Omega t + \frac{1}{\Omega} \int_0^t \sin \Omega(s-t)\{-A\Omega^2 \cos \Omega s + \omega^2 \sin(A \cos \Omega s)\} \, ds. \tag{89}$$

Generally speaking, the second term of the right-hand side of Eq. (89) is not equal to zero, it would vanish if $u_0(t)$ were to be a solution of (82). We may, however, at least assure

$$\int_0^T \sin \Omega(s-t) \left\{-A\Omega^2 \cos \Omega s + \omega^2 \sin(A \cos \Omega s)\right\} \, ds = 0, \tag{90}$$

with $T = 2\pi/\Omega$. Thus, we have

$$\Omega^2 = \frac{\omega^2 \int_0^T \sin \Omega s \sin(A \cos \Omega s) \, ds}{\int_0^T A \sin \Omega s \cos \Omega s \, ds} = \frac{2\omega^2 J_1(A)}{A}, \tag{91}$$

where $J_1(A)$ is the first-order Bessel function of the first kind:

$$J_1(A) = \tfrac{1}{2} A - \tfrac{1}{16} A^3 + \tfrac{1}{384} A^5 + \cdots . \tag{92}$$

The period then can be calculated as follows:

$$T = \frac{2\pi}{\omega \sqrt{2 J_1(A)/A}} = \frac{2\pi}{\omega \sqrt{1 - \tfrac{1}{8} A^2 + \tfrac{1}{192} A^4 + \cdots}}. \tag{93}$$

The mathematical pendulum, system (82), can be approximated by

$$u'' + \omega^2 u - \frac{1}{3!} \omega^2 u^3 + \frac{1}{5!} \omega^2 u^5 = 0. \tag{94}$$

The iteration formula can be written in the form

$$u_{n+1}(t) = u_n(t) + \frac{1}{\Omega} \int_0^t \sin \Omega(s-t) \left[u_n''(s) + \omega^2 u_n(s) - \frac{1}{3!} \omega^2 u_n^3(s) + \frac{1}{5!} \omega^2 u_n^5(s) \right] \, ds. \tag{95}$$

We begin with $u_0(t) = A \cos \Omega t$, by formula (14) and requirement of no secular terms, we obtain

$$u_1(t) = A \cos \Omega t + \frac{(-16A^3 + A^5)\omega^2}{1792\Omega^2} (\cos 3\Omega t - \cos \Omega t) + \frac{A^5 \omega^2}{15\,360\Omega^2} (\cos 5\Omega t - \cos \Omega t), \tag{96}$$

where the angular frequency is determined as

$$\Omega = \omega \sqrt{1 - \tfrac{1}{8} A^2 + \tfrac{1}{192} A^4}. \tag{97}$$

Example 2 (*A ball-bearing oscillator*). In this example we consider the motion of a ball-bearing oscillating in a smooth tube that is bent into a curve such that the restoring force depends upon the cube of the displacement. The governing equation, which can be readily obtained, is

$$u'' + \varepsilon u^3 = 0, \tag{98}$$

with initial conditions $u(0) = A$, $u'(0) = 0$.

The iteration formula for the system can be constructed as follows:

$$u_{n+1}(t) = u_n(t) + \frac{1}{\omega} \int_0^t \sin \omega(s-t)\{u_n''(s) + \varepsilon u_n^3(s)\} \, ds, \tag{99}$$

where ω is the angular frequency of the system.

We begin with $u_0(t) = A \cos \omega t$, by formula (99), we have

$$u_1(t) = A \cos \omega t - \tfrac{1}{32} A^3 (\cos 3\omega t - \cos \omega t), \tag{100}$$

where the angular frequency ω is identified with the physical understanding that no secular terms should be appeared in $u_1(t)$, which leads to

$$\omega = \tfrac{\sqrt{3}}{2} \varepsilon^{1/2} A. \tag{101}$$

The approximate period can be expressed in the form

$$T = \frac{4\sqrt{3}\pi}{3\varepsilon^{1/2}A}, \tag{102}$$

while the exact period can be readily obtained, which reads

$$T_{\text{ex}} = 2\sqrt{2} \int_0^A \frac{\mathrm{d}x}{\sqrt{\int_x^A g(u)\,\mathrm{d}u}}, \tag{103}$$

where $g(u) = \varepsilon u^3$. For any $A > 0$ and $\varepsilon > 0$, Eq. (103) can be approximated as

$$T_{\text{app}} = \frac{\pi^{3/2} A^{1/2}}{\sqrt{\int_0^{\pi/2} g(A\cos x)\cos x\,\mathrm{d}x}} = \frac{\pi^{3/2} A^{1/2}}{\sqrt{\int_0^{\pi/2} \varepsilon A^3 \cos^4 x\,\mathrm{d}x}} = \frac{4\pi}{\sqrt{3}\varepsilon^{1/2}A}. \tag{104}$$

So the results obtained by the present theory are valid for $A > 0$ and $\varepsilon > 0$.

Example 3. We consider an oscillator, which is governed by

$$u'' + \omega^2 u + 4\varepsilon u^2 u'' + 4\varepsilon u u'^2 = 0, \quad u(0) = A, \quad u'(0) = 0. \tag{105}$$

Supposing system (105) has the angular frequency Ω, we obtain the following iteration formula:

$$u_{n+1}(t) = u_n(t) + \frac{1}{\Omega} \int_0^t \sin \Omega(s-t) \left\{ u_n''(s) + \omega^2 u_n(s) \right\} \mathrm{d}s$$

$$+ \frac{1}{\Omega} \int_0^t \sin \Omega(s-t) \{ 4\varepsilon u_n^2(s) u''(s) + 4\varepsilon u_n(s) u_n'^2(s) \} \mathrm{d}s. \tag{106}$$

We begin with the initial approximation $u_0(t) = A \cos \Omega t$, by the above iteration formula, we have

$$u_1(t) = A \cos \Omega t - \frac{\varepsilon A^3}{4} (\cos 3\Omega t - \cos \Omega t), \tag{107}$$

where the angular frequency is identified with the understanding that no secular terms occur, which leads to

$$\Omega = \frac{\omega}{\sqrt{1 + 2\varepsilon A^2}}. \tag{108}$$

The period is

$$T = T_0 \sqrt{1 + 2\varepsilon A^2}, \quad T_0 = 2\pi/\omega, \tag{109}$$

while the exact period reads

$$T_{\text{ex}} = \frac{2}{\pi} T_0 \sqrt{1 + 4\varepsilon A^2} \int_0^{\pi/2} \sqrt{1 - k \sin^2 t}\,\mathrm{d}t, \tag{110}$$

where $k = 4\varepsilon A^2 / (1 + 4\varepsilon A^2)$.

It is obvious the obtained result (109) is valid for all $\varepsilon > 0$. Even in case $\varepsilon \to \infty$, we have

$$\lim_{\varepsilon \to \infty} \frac{T_{\mathrm{ex}}}{T} = 0.9003.$$

Example 4 (*Schrödinger equation*). The Schrödinger equation can be written in the form

$$Lf = -f'' + q(x)f = k^2 f. \tag{111}$$

In case $q(x) = 0$, we have two general solutions: $\mathrm{e}^{\mathrm{i}kx}$ and $\mathrm{e}^{-\mathrm{i}kx}$. We assume that $q(x) \to 0$ when $|x| \to \infty$.

In case $x \to +\infty$, the correction functional can be constructed as follows:

$$f_{n+1}(x) = f_n(x) + \int_x^{+\infty} \lambda \{ f_n''(\xi) - q(\xi)\tilde{f}_n(\xi) + k^2 f_n(\xi) \} \, \mathrm{d}\xi. \tag{112}$$

where $\tilde{f}_n$ is considered as a restricted variation.

Making (112) stationary, we obtain the following stationary conditions

$$\begin{cases} \lambda''(\xi) + k^2 f(\xi) = 0, \\ \lambda(\xi)|_{\xi=x} = 0, \\ 1 + \lambda'(\xi)|_{\xi=x} = 0. \end{cases} \tag{113}$$

The Lagrange multiplier can be identified as

$$\lambda = \frac{1}{k} \sin k(x - \xi). \tag{114}$$

We obtain the following iteration formulation:

$$f_{n+1}(x) = f_n(x) + \int_x^{+\infty} \frac{\sin k(x - \xi)}{k} \{ f_n''(\xi) - q(\xi)f_n(\xi) + k^2 f_n(\xi) \} \, \mathrm{d}\xi. \tag{115}$$

We begin with

$$f_0(x) = f_+(x) = \mathrm{e}^{\mathrm{i}kx}. \tag{116}$$

By (11), we have

$$f_1(x) = \mathrm{e}^{\mathrm{i}kx} - \int_x^{+\infty} \frac{\sin k(x - \xi)}{k} q(\xi)f_+(\xi) \, \mathrm{d}\xi. \tag{117}$$

Now we consider another case when $x \to -\infty$. Under such a case, we construct a correction functional in the form

$$f_{n+1}(x) = f_n(x) + \int_{-\infty}^x \lambda \{ f_n''(\xi) - q(\xi)\tilde{f}_n(\xi) + k^2 f_n(\xi) \} \, \mathrm{d}\xi. \tag{118}$$

Its stationary conditions are

$$\begin{cases} \lambda''(\xi) + k^2 f(\xi) = 0, \\ \lambda(\xi)|_{\xi=x} = 0, \\ 1 - \lambda'(\xi)|_{\xi=x} = 0. \end{cases} \tag{119}$$

The multiplier can be identified as

$$\lambda = -\frac{1}{k} \sin k(x - \xi). \tag{120}$$

So we obtain the following iteration formulation:

$$f_{n+1}(x) = f_n(x) - \int_x^{+\infty} \frac{\sin k(x - \xi)}{k} \{f_n''(\xi) - q(\xi)f_n(\xi) + k^2 f_n(\xi)\} \, \mathrm{d}\xi. \tag{121}$$

We begin with

$$f_0(x) = f_-(x) = \mathrm{e}^{-\mathrm{i}kx}. \tag{122}$$

By (121), we have

$$f_1(x) = \mathrm{e}^{\mathrm{i}kx} + \int_x^{+\infty} \frac{\sin k(x - \xi)}{k} q(\xi) f_-(\xi) \, \mathrm{d}\xi. \tag{123}$$

Eqs. (117) and (121) are actually Jost solutions.

4. Conclusions

The variational iteration method can be easily comprehended with only a basic knowledge of Advanced Calculus, even the reader has no knowledge of calculus of variations in pure mathematics.

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