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www.elsevier.com/locate/physletbQuark CP -phase and Froggatt–Nielsen mechanismChuichiro Hattori^a, Masahisa Matsuda^{b,1}, Mamoru Matsunaga^{c,d,*}, Takeo Matsuoka^d^a Science Division, General Education, Aichi Institute of Technology, Toyota 470-0392, Japan^b Management Office, Nara University of Education, Nara 630-8528, Japan^c College of Liberal Arts and Sciences, Mie University, Tsu 514-8507, Japan^d Department of Physics Engineering, Mie University, Tsu 514-8507, Japan

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ABSTRACT

On the basis of the Froggatt–Nielsen mechanism, we study quark flavor mixings in the $SU(6) \times SU(2)_R$ model. The characteristic structure of the CKM matrix is attributed to the hierarchical effective Yukawa couplings due to the Froggatt–Nielsen mechanism and also to the state-mixings beyond the MSSM. We elucidate the detailed form of the CKM matrix elements and find interesting relations between the CP violating phase and three mixing angles. Taking the existing data of three mixing angles, we estimate the quark CP -phase at $\delta = (75 \pm 3)^\circ$. This result is in accord with observations.

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1. Introduction

One of theoretically challenging issues is to understand characteristic features of quark mass patterns and the CKM-mixing matrix [1]. It seems that the important key to this issue is the state-mixing between quarks and extra particles beyond the minimal supersymmetric standard model (MSSM). In fact, it was shown in the context of $SU(6) \times SU(2)_R$ string-inspired model, which contains massless particles beyond the MSSM, that we were able to explain characteristic patterns of the observed mass spectra and mixing matrices of quarks and leptons [2–6]. In the model the Froggatt–Nielsen (F–N) mechanism [7] plays an important role. It is noticeable that doublet Higgs and color-triplet Higgs fields belong to different representations of $SU(6) \times SU(2)_R$. This situation is favorable to solve the triplet–doublet splitting problem. In addition, the longevity of the proton can be guaranteed under appropriate flavor symmetries [2]. In this paper we focus our attention on the detailed form of the CKM matrix elements in the above-mentioned model. It has been shown that in the model the hierarchical pattern of three mixing angles can be understood systematically [4]. We shed light on relations between the CP violating phase and three mixing angles in this paper.

In the present model, it is assumed that the hierarchical structure of fermion mass matrices is attributed to the F–N factors coming from the F–N mechanism. In the previous work [8], we derived the typical relations among CKM matrix elements

$$|V_{cd}| = |V_{us}|, \quad (1)$$

$$|V_{ts}| = |V_{cb}|, \quad (2)$$

$$|V_{td}| = |V_{us} V_{cb}|. \quad (3)$$

The CKM-matrix is defined as

$$V_{\text{CKM}} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} = \begin{pmatrix} c_{12} c_{13} & s_{12} c_{13} & s_{13} e^{-i\delta} \\ -s_{12} c_{23} - c_{12} s_{23} s_{13} e^{i\delta} & c_{12} c_{23} - s_{12} s_{23} s_{13} e^{i\delta} & s_{23} c_{13} \\ s_{12} s_{23} - c_{12} c_{23} s_{13} e^{i\delta} & -c_{12} s_{23} - s_{12} c_{23} s_{13} e^{i\delta} & c_{23} c_{13} \end{pmatrix} \quad (4)$$

in the standard representation of Particle Data Group (PDG) [9]. Due to unitarity condition on V_{CKM} , Eqs. (1) and (2) are equivalent to each other. From the independent relations (2) and (3), the CP -phase δ is expressed in terms of the three mixing angles as

$$\cos \delta = \frac{s_{23}^2 (s_{12}^2 - s_{13}^2) - s_{12}^2 c_{23}^2 s_{13}^2}{2 s_{12} c_{12} s_{23} c_{23} s_{13}}, \quad (5)$$

$$\cos \delta = s_{13} \frac{s_{12}^2 s_{23}^2 (1 + c_{13}^2) + c_{12}^2 c_{23}^2}{2 s_{12} c_{12} s_{23} c_{23}}, \quad (6)$$

* Corresponding author.

E-mail addresses: yama_houshi@ybb.ne.jp (C. Hattori), mmatsuda@nara-edu.ac.jp, mmatsuda@aeu.ac.jp (M. Matsuda), matsuna@phen.mie-u.ac.jp, matsumamo@mediacat.ne.jp (M. Matsunaga), t-matsu@siren.ocn.ne.jp (T. Matsuoka).

¹ Permanent address: Aichi University of Education, Kariya 448-8542, Japan.

respectively. Here if we input the experimental values of $s_{12} = 0.22536 \equiv \lambda$, $s_{23} \simeq \lambda^{2.1}$ and $s_{13} \simeq \lambda^{3.8}$ [9], Eqs. (5) and (6) exhibit

$$\cos \delta \simeq \frac{s_{12} s_{23}}{2 s_{13}} \simeq \lambda^{-0.7}/2, \quad (7)$$

$$\cos \delta \simeq \frac{s_{13}}{2 s_{12} s_{23}} \simeq \lambda^{0.7}/2, \quad (8)$$

respectively. These results are incompatible with each other.

However, the relations (2) and (3) are derived in the leading approximation. So, we need to accomplish more accurate calculation in order to discuss the quark CP -phase. For this reason, in this paper we carry out the analysis up to the next-to-leading approximation in the F–N scheme, which allows us to find more accurate relations among the CKM-matrix elements. For example, we obtain

$$|V_{cd}|^2 \simeq |V_{us}|^2 \times \left[1 - |V_{cb}|^2\right],$$

which yields an attractive relation between the CP -phase and three mixing angles. Using these relations, we are able to estimate the quark CP -phase without relying on a specific flavor symmetry.

This paper is organized as follows. In Sec. 2 we briefly explain Yukawa couplings in the $SU(6) \times SU(2)_R$ model together with the F–N mechanism. Solving the eigenvalue problem for the mass matrices of the up-type and down-type quark sectors, we derive the diagonalization matrices. In Sec. 3 the detailed form of the CKM matrix is presented and interesting equations among the CKM-matrix elements are found. In Sec. 4 it is shown that these yield attractive interrelations between the quark CP -phase and three mixing angles. Taking the existing data of three mixing angles, we estimate the quark CP -phase at $\delta = (75 \pm 3)^\circ$, which is in accord with the current data of δ . Section 5 is devoted to summary.

2. Yukawa couplings and F–N mechanism

Here we briefly summarize the parts of the model which are relevant to our analysis. For a more complete discussion, see Refs. [2–6,8]. In this model the unification gauge symmetry is assumed to be $SU(6) \times SU(2)_R$ at the underlying string scale M_5 . The gauge group $G = SU(6) \times SU(2)_R$ is a subgroup of E_6 . Within the framework of E_6 we assign three families and one vector-like multiplet to matter superfields, i.e.,

$$3 \times \mathbf{27}(\Phi_{1,2,3}) + (\mathbf{27}(\Phi_0) + \overline{\mathbf{27}}(\overline{\Phi})). \quad (9)$$

The superfields Φ are decomposed into two multiplets of G as

$$\Phi(\mathbf{27}) = \begin{cases} \phi(\mathbf{15}, \mathbf{1}) : & \{Q, L, g, g^c, S\}, \\ \psi(\overline{\mathbf{6}}, \mathbf{2}) : & \{U^c, D^c, (N^c, E^c), (H_u, H_d)\}, \end{cases} \quad (10)$$

where g, g^c and H_u, H_d represent colored Higgs and $SU(2)_L$ -doublet Higgs superfields, respectively. Doublet Higgs and color-triplet Higgs fields belong to different representations of G and this situation is favorable to solve the triplet–doublet splitting problem. The superfields N^c and S are R -handed neutrinos and $SO(10)$ -singlets, respectively. Although D^c and g^c as well as L and H_d have the same quantum numbers under the standard model gauge group $G_{SM} = SU(3)_c \times SU(2)_L \times U(1)_Y$, they belong to different irreducible representations of G . We assign odd (even) R -parity to superfields $\Phi_{1,2,3}$ (Φ_0 and $\overline{\Phi}$). Since ordinary Higgs doublets have even R -parity, they are contained in Φ_0 . It is assumed that R -parity remains unbroken down to the electroweak scale.

The gauge symmetry G gets spontaneously broken in two steps at the scales $\langle S_0 \rangle = \langle \overline{S} \rangle$ and $\langle N_0^c \rangle = \langle \overline{N}^c \rangle$ to G_{SM} as

$$G = SU(6) \times SU(2)_R \xrightarrow{\langle S_0 \rangle} SU(4)_{PS} \times SU(2)_L \times SU(2)_R \xrightarrow{\langle N_0^c \rangle} G_{SM},$$

where $SU(4)_{PS}$ represents the Pati–Salam $SU(4)$ [10]. The D -flatness conditions require $\langle S_0 \rangle = \langle \overline{S} \rangle$ and $\langle N_0^c \rangle = \langle \overline{N}^c \rangle$ at each step of the symmetry breakings. Hereafter it is supposed that the symmetry breaking scales are $\langle S_0 \rangle = 10^{17-18}$ GeV and $\langle N_0^c \rangle = 10^{15-17}$ GeV. Under the $SU(4)_{PS} \times SU(2)_L \times SU(2)_R$ the chiral superfields $\phi(\mathbf{15}, \mathbf{1})$ and $\psi(\overline{\mathbf{6}}, \mathbf{2})$ are decomposed as

$$(\mathbf{15}, \mathbf{1}) = (\mathbf{4}, \mathbf{2}, \mathbf{1}) + (\mathbf{6}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{1}),$$

$$(\overline{\mathbf{6}}, \mathbf{2}) = (\overline{\mathbf{4}}, \mathbf{1}, \mathbf{2}) + (\mathbf{1}, \mathbf{2}, \mathbf{2}).$$

From the viewpoint of the string unification theory, it is probable that the hierarchical structure of Yukawa couplings is attributed to some kind of flavor symmetries at the string scale M_5 . If the flavor symmetry contains Abelian groups, the F–N mechanism works for the interactions among quarks, leptons and Higgs fields. The superpotential at the string scale is governed by the flavor symmetry as well as the gauge symmetry G . Aside from the flavor symmetry, we have two types of gauge invariant trilinear combinations

$$(\phi(\mathbf{15}, \mathbf{1}))^3 = Q Q g + Q g^c L + g^c g S,$$

$$\begin{aligned} \phi(\mathbf{15}, \mathbf{1})(\psi(\overline{\mathbf{6}}, \mathbf{2}))^2 &= Q H_d D^c + Q H_u U^c + L H_d E^c + L H_u N^c \\ &+ S H_u H_d + g N^c D^c + g E^c U^c + g^c U^c D^c. \end{aligned} \quad (11)$$

They must be multiplied by additional G -invariant factors suppressed by powers of $1/M_5$ to form flavor symmetric terms. Namely, the couplings arise from the nonrenormalizable terms controlled by the flavor symmetry [2,11,12].

We first consider the effective Yukawa couplings of up-type quark sector, which are given by

$$W_U = \sum_{i,j=1}^3 \mathcal{M}_{ij} Q_i U_j^c H_{u0}. \quad (12)$$

Due to the F–N mechanism, the dimensionless matrix \mathcal{M} takes the form

$$\mathcal{M} = f_M \Gamma_1 M \Gamma_2. \quad (13)$$

Our basic assumption is that the hierarchical structure of all 3×3 mass matrices is attributed to the F–N factors Γ_1 and/or Γ_2 . Hence, hierarchy of \mathcal{M}_{ij} stems only from Γ_1 and Γ_2 , and the dimensionless matrix M contains no hierarchical structure. Here we put a factor f_M in order to set $\det M = 1$. It means that all the elements of M are $\mathcal{O}(1)$. The F–N factors Γ_1 and Γ_2 are described as

$$\Gamma_1 = \text{diag}(x^{\alpha_1}, x^{\alpha_2}, 1), \quad \Gamma_2 = \text{diag}(x^{\beta_1}, x^{\beta_2}, 1) \quad (14)$$

with the hierarchy $x^{\alpha_1} \ll x^{\alpha_2} \ll 1$ and $x^{\beta_1} \ll x^{\beta_2} \ll 1$. To be specific, we take the F–N factors like $x^{\alpha_1} \sim \lambda^3$, $x^{\alpha_2} \simeq x^{\beta_2} \sim \lambda^2$ and $x^{\beta_1} \sim \lambda^{4-5}$ consonant to the experimental data.

The mass matrix \mathcal{M} is diagonalized via biunitary transformation as

$$\mathcal{V}_u^{-1} \mathcal{M} \mathcal{U}_u = \Lambda_u, \quad v_{u0} \Lambda_u = \text{diag}(m_u, m_c, m_t) \quad (15)$$

with $v_{u0} = \langle H_{u0} \rangle$. According to the standard procedure for diagonalizing $\mathcal{M} \mathcal{M}^\dagger$, we obtain mass eigenvalues

$$\begin{aligned} (m_u, m_c, m_t) \\ \simeq |v_{u0} f_M| \times \left(\frac{1}{|\overline{m}_{11}|} x^{\alpha_1 + \beta_1}, \frac{|\overline{m}_{11}|}{|m_{33}|} x^{\alpha_2 + \beta_2}, |m_{33}| \right), \end{aligned} \quad (16)$$

where $m_{ij} = (M)_{ij}$, $\overline{m}_{ij} = (M^{-1})_{ji}^* = \Delta(M)_{ij}^*$. The diagonalization matrix \mathcal{V}_u is described in terms of eigenvectors $\mathbf{w}_i^{(u)}$ of $\mathcal{M} \mathcal{M}^\dagger$ as

$$\mathcal{V}_u = (\mathbf{w}_1^{(u)}, \mathbf{w}_2^{(u)}, \mathbf{w}_3^{(u)}), \quad (17)$$

where $\mathbf{w}_i^{(u)}$ are expressed as

$$\begin{aligned} \mathbf{w}_1^{(u)} &= N_1^{(u)} \begin{pmatrix} 1 \\ u_1^{(u)} \\ v_1^{(u)} \end{pmatrix}, & \mathbf{w}_2^{(u)} &= N_2^{(u)} \begin{pmatrix} u_2^{(u)} \\ 1 \\ v_2^{(u)} \end{pmatrix}, \\ \mathbf{w}_3^{(u)} &= N_3^{(u)} \begin{pmatrix} u_3^{(u)} \\ v_3^{(u)} \\ 1 \end{pmatrix}. \end{aligned} \quad (18)$$

Here $N_i^{(u)}$ are normalization factors. The phase factors are so chosen that the diagonal elements of \mathcal{V}_u are real. Explicit forms of $u_i^{(u)}$ and $v_i^{(u)}$ ($i = 1, 2, 3$) are

$$\begin{aligned} u_1^{(u)} &= x^{\alpha_1 - \alpha_2} \left[\frac{\overline{m}_{21}}{\overline{m}_{11}} + \mathcal{O}(x^{2(\beta_1 - \beta_2)}) \right], \\ v_1^{(u)} &= x^{\alpha_1} \left[\frac{\overline{m}_{31}}{\overline{m}_{11}} + \mathcal{O}(x^{2(\beta_1 - \beta_2)}) \right], \\ u_2^{(u)} &= -x^{\alpha_1 - \alpha_2} \left[\frac{\overline{m}_{21}^*}{\overline{m}_{11}^*} + \mathcal{O}(x^{2\alpha_2}, x^{2(\beta_1 - \beta_2)}) \right], \\ v_2^{(u)} &= -x^{\alpha_2} \left[\frac{m_{23}^*}{m_{33}^*} - x^{2(\alpha_1 - \alpha_2)} \frac{m_{13}^* \overline{m}_{21}}{m_{33}^* \overline{m}_{11}^*} + \mathcal{O}(x^{2\beta_2}) \right], \\ u_3^{(u)} &= x^{\alpha_1} \left[\frac{m_{13}}{m_{33}} + \mathcal{O}(x^{2\beta_2}) \right], \\ v_3^{(u)} &= x^{\alpha_2} \left[\frac{m_{23}}{m_{33}} + \mathcal{O}(x^{2\beta_2}) \right], \end{aligned} \quad (19)$$

where $x^{\alpha_1 - \alpha_2} \sim \lambda$, $x^{\alpha_1} \sim \lambda^3$ and $x^{\alpha_2} \sim \lambda^2$.

Note that $x^{2(\beta_1 - \beta_2)}$, $x^{2\alpha_2}$ and $x^{2\beta_2}$ are $\mathcal{O}(\lambda^4)$ or less than $\mathcal{O}(\lambda^4)$. The normalization factors are given by

$$\begin{aligned} N_1^{(u)} &= 1 - x^{2(\alpha_1 - \alpha_2)} \frac{|\overline{m}_{21}|^2}{2|\overline{m}_{11}|^2} + \mathcal{O}(x^{4(\alpha_1 - \alpha_2)}), \\ N_2^{(u)} &= 1 - x^{2(\alpha_1 - \alpha_2)} \frac{|\overline{m}_{21}|^2}{2|\overline{m}_{11}|^2} + \mathcal{O}(x^{4(\alpha_1 - \alpha_2)}, x^{2\alpha_2}), \\ N_3^{(u)} &= 1 + \mathcal{O}(x^{2\alpha_2}), \end{aligned} \quad (20)$$

and we have the relation $N_1^{(u)} N_3^{(u)} = N_2^{(u)} (1 + \mathcal{O}(\lambda^6))$.

We next proceed to study the effective Yukawa couplings of down-type quark sector, which are of the form

$$W_D = \sum_{i,j=1}^3 \left[\mathcal{Z}_{ij} g_i g_j^c S_0 + \mathcal{M}_{ij} \left(g_i D_j^c N_0^c + Q_i D_j^c H_{d0} \right) \right], \quad (21)$$

where $\mathcal{Z} = f_Z \Gamma_1 Z \Gamma_1$ and $\det Z = 1$. It is assumed that there is no hierarchical structure in Z . The mass matrix of down-quark sector is given by the 6×6 matrix

$$\widehat{\mathcal{M}}_d = \begin{pmatrix} g^c & D^c \\ \rho_S \mathcal{Z} & \rho_N \mathcal{M} \\ 0 & \rho_d \mathcal{M} \end{pmatrix}, \quad (22)$$

where $\rho_S = \langle S_0 \rangle / M_S$, $\rho_N = \langle N_0^c \rangle / M_S$ and $\rho_d = \langle H_{d0} \rangle / M_S = v_{d0} / M_S$. It is noticeable that D^c - g^c mixings occur in down-type quark sector. Diagonalization is accomplished via biunitary transformation as

$$\widehat{\mathcal{V}}_d^{-1} \widehat{\mathcal{M}}_d \widehat{\mathcal{U}}_d = \text{diag}(\Lambda_d^{(0)}, \epsilon_d \Lambda_d^{(2)}), \quad (23)$$

where $\epsilon_d = \rho_d / \rho_N = v_{d0} / \langle N_0^c \rangle = \mathcal{O}(10^{-15})$. $\Lambda_d^{(0)}$ means the heavy modes with the GUT scale masses. To solve the eigenvalue problem, we deal with $\widehat{\mathcal{M}}_d \widehat{\mathcal{M}}_d^\dagger$, which are expressed as

$$\widehat{\mathcal{M}}_d \widehat{\mathcal{M}}_d^\dagger = \begin{pmatrix} A_d + B_d & \epsilon_d^* B_d \\ \epsilon_d B_d & |\epsilon_d|^2 B_d \end{pmatrix} \quad (24)$$

with the notation $A_d = |\rho_S|^2 \mathcal{Z} \mathcal{Z}^\dagger$ and $B_d = |\rho_N|^2 \mathcal{M} \mathcal{M}^\dagger$. Within $\mathcal{O}(\epsilon_d^2)$ mass eigenvalues $\Lambda_d^{(2)}$ are given as

$$(\Lambda_d^{(2)})^2 = \mathcal{V}_d^{-1} (A_d^{-1} + B_d^{-1})^{-1} \mathcal{V}_d \quad (25)$$

and

$$M_S |\epsilon_d \rho_N| \Lambda_d^{(2)} = \text{diag}(m_d, m_s, m_b), \quad (26)$$

where \mathcal{V}_d is unitary within $\mathcal{O}(\epsilon_d)$ as seen in Eq. (24). It turns out that down-type quark masses are

$$\begin{aligned} (m_d, m_s, m_b) &\simeq |v_{d0} f_M| \times \left(\frac{1}{\sqrt{l_{11}}} x^{\alpha_1 + \beta_1}, \sqrt{\frac{l_{11}}{g}} x^{\alpha_2 + \beta_1}, \sqrt{\frac{g}{h}} x^{\beta_1} \right), \end{aligned} \quad (27)$$

where

$$\begin{aligned} l_{ij} &= \xi_d^2 \bar{z}_{i1} \bar{z}_{j1}^* + \bar{m}_{i1} \bar{m}_{j1}^*, \\ g &= \xi_d^2 |D_3^{11}|^2, \\ h &= \xi_d^4 x^{2(\alpha_1 - \alpha_2)} |(\mathbf{z}_3 \cdot \bar{\mathbf{m}}_1^*)|^2 + \xi_d^2 x^{2(\beta_1 - \beta_2)} |(\mathbf{m}_3 \cdot \bar{\mathbf{z}}_1^*)|^2, \\ \xi_d^2 &= \left| \frac{\rho_N f_M}{\rho_S f_Z} \right|^2 x^{2(\beta_1 - \alpha_1)}, \quad D_k^{ij} = (\bar{\mathbf{z}}_i \times \bar{\mathbf{m}}_j)_k. \end{aligned}$$

Here we use the notations $z_{ij} = (Z)_{ij}$, $\bar{z}_{ij} = \Delta(Z)_{ij}^*$ and

$$\begin{aligned} \mathbf{m}_i &= (m_{1i}, m_{2i}, m_{3i})^T, & \bar{\mathbf{m}}_i &= (\bar{m}_{1i}, \bar{m}_{2i}, \bar{m}_{3i})^T, \\ \mathbf{z}_i &= (z_{1i}, z_{2i}, z_{3i})^T, & \bar{\mathbf{z}}_i &= (\bar{z}_{1i}, \bar{z}_{2i}, \bar{z}_{3i})^T. \end{aligned}$$

The diagonalization matrix \mathcal{V}_d is expressed as

$$\mathcal{V}_d = (\mathbf{w}_1^{(d)}, \mathbf{w}_2^{(d)}, \mathbf{w}_3^{(d)}) \quad (28)$$

with

$$\begin{aligned} \mathbf{w}_1^{(d)} &= N_1^{(d)} \begin{pmatrix} 1 \\ u_1^{(d)} \\ v_1^{(d)} \end{pmatrix}, & \mathbf{w}_2^{(d)} &= N_2^{(d)} \begin{pmatrix} u_2^{(d)} \\ 1 \\ v_2^{(d)} \end{pmatrix}, \\ \mathbf{w}_3^{(d)} &= N_3^{(d)} \begin{pmatrix} u_3^{(d)} \\ v_3^{(d)} \\ 1 \end{pmatrix}. \end{aligned} \quad (29)$$

Here the phase factors are so taken that the diagonal elements of \mathcal{V}_d are real. Each element of \mathcal{V}_d is of the form

$$\begin{aligned} u_1^{(d)} &= x^{\alpha_1 - \alpha_2} \left[\frac{l_{21}}{l_{11}} + x^{2(\alpha_1 - \alpha_2)} \frac{\xi_d^2}{(l_{11})^2} \left(\bar{z}_{12}^* n_{13}^* + \frac{l_{21}}{l_{11}} |D_3^{11}|^2 \right) \right. \\ &\quad \left. + \mathcal{O}(\lambda^4) \right], \\ v_1^{(d)} &= x^{\alpha_1} \left[\frac{l_{31}}{l_{11}} - x^{2(\alpha_1 - \alpha_2)} \frac{\xi_d^2}{(l_{11})^2} \left(\bar{z}_{12}^* n_{12}^* + \frac{l_{21}}{l_{11}} D_2^{11} D_3^{11*} \right) \right. \\ &\quad \left. + \mathcal{O}(\lambda^4) \right], \\ u_2^{(d)} &= -x^{\alpha_1 - \alpha_2} \left[\frac{l_{12}}{l_{11}} + x^{2(\alpha_1 - \alpha_2)} \frac{\xi_d^2}{(l_{11})^2} \left(\bar{z}_{12} n_{13} \right) \right. \\ &\quad \left. + \mathcal{O}(\lambda^4) \right], \end{aligned} \quad (30)$$

$$\begin{aligned}
& + \frac{l_{12}}{l_{11}} |D_3^{11}|^2 \Big) + \mathcal{O}(\lambda^4) \Big], \\
v_2^{(d)} &= -x^{\alpha_2} \left[\frac{D_3^{11}}{D_3^{11*}} + x^{2(\alpha_1 - \alpha_2)} \left(f_\alpha^* n_{13} - \frac{l_{12} D_3^{11}}{l_{11} D_3^{11*}} \right) + \mathcal{O}(\lambda^4) \right], \\
u_3^{(d)} &= x^{\alpha_1} \left[\frac{D_3^{11*}}{D_3^{11*}} - x^{2(\alpha_1 - \alpha_2)} f_\alpha n_{23}^* + \mathcal{O}(\lambda^4) \right], \\
v_3^{(d)} &= x^{\alpha_2} \left[\frac{D_3^{11*}}{D_3^{11*}} + x^{2(\alpha_1 - \alpha_2)} f_\alpha n_{13}^* + \mathcal{O}(\lambda^4) \right],
\end{aligned}$$

where

$$n_{ij} = \xi_d^2 \bar{z}_{i1} z_{j3} - \bar{m}_{i1} D_j^{21*}, \quad f_\alpha = \frac{(\mathbf{z}_3 \cdot \bar{\mathbf{m}}_1^*)}{D_3^{11*} |D_3^{11}|^2}.$$

The normalization factors become

$$\begin{aligned}
N_1^{(d)} &= 1 - x^{2(\alpha_1 - \alpha_2)} \frac{|l_{21}|^2}{2(l_{11})^2} + \mathcal{O}(x^{4(\alpha_1 - \alpha_2)}), \\
N_2^{(d)} &= 1 - x^{2(\alpha_1 - \alpha_2)} \frac{|l_{21}|^2}{2(l_{11})^2} + \mathcal{O}(x^{4(\alpha_1 - \alpha_2)}, x^{2\alpha_2}), \\
N_3^{(d)} &= 1 + \mathcal{O}(x^{2\alpha_2})
\end{aligned} \quad (31)$$

with

$$N_1^{(d)} N_3^{(d)} = N_2^{(d)} (1 + \mathcal{O}(\lambda^6)). \quad (32)$$

3. The CKM matrix

In the present framework the CKM matrix is given by

$$V_{\text{CKM}} = \mathcal{V}_u^{-1} \mathcal{V}_d = \mathcal{V}_u^\dagger \mathcal{V}_d \quad (33)$$

and each element of V_{CKM} becomes

$$(V_{\text{CKM}})_{ij} = \mathbf{w}_i^{(u)*} \cdot \mathbf{w}_j^{(d)}. \quad (34)$$

Using approximate analytic expressions of $\mathbf{w}^{(u)}$ and $\mathbf{w}^{(d)}$ given in the preceding section, we are in a position to exhibit each element of V_{CKM} explicitly. Thus V_{us} and V_{cd} are expressed as

$$\begin{aligned}
V_{us} &= x^{\alpha_1 - \alpha_2} N_1^{(u)} N_2^{(d)} \left[\frac{\xi_d^2 \bar{z}_{11} D_3^{11*}}{\bar{m}_{11}^* l_{11}} - x^{2(\alpha_1 - \alpha_2)} \frac{\xi_d^2}{(l_{11})^2} \right. \\
& \quad \left. \times \left(\bar{z}_{12} n_{13} + \frac{l_{12}}{l_{11}} |D_3^{11}|^2 \right) + \mathcal{O}(\lambda^4) \right], \\
\end{aligned} \quad (35)$$

$$\begin{aligned}
V_{cd} &= -x^{\alpha_1 - \alpha_2} N_2^{(u)} N_1^{(d)} \left[\frac{\xi_d^2 \bar{z}_{11} D_3^{11}}{\bar{m}_{11} l_{11}} - x^{2(\alpha_1 - \alpha_2)} \frac{\xi_d^2}{(l_{11})^2} \right. \\
& \quad \left. \times \left(\bar{z}_{12}^* n_{13}^* + \frac{l_{21}}{l_{11}} |D_3^{11}|^2 \right) + \mathcal{O}(\lambda^4) \right], \\
\end{aligned} \quad (36)$$

where $x^{\alpha_1 - \alpha_2} \sim \lambda$. The relation of $V_{cd} = -(V_{us})^* \times (1 + \mathcal{O}(\lambda^4))$ is hold in the present phase convention, in which the diagonal elements of \mathcal{V}_u and \mathcal{V}_d are chosen to be real. Also the other elements are given as

$$\begin{aligned}
V_{cb} &= x^{\alpha_2} N_2^{(u)} N_3^{(d)} \left[\frac{\bar{m}_{11}^* (\bar{\mathbf{z}}_1 \cdot \mathbf{m}_3)}{m_{33} D_3^{11*}} + x^{2(\alpha_1 - \alpha_2)} \right. \\
& \quad \left. \times \left(f_\alpha n_{13}^* + \frac{|\bar{m}_{21}|^2 (\bar{\mathbf{z}}_1 \cdot \mathbf{m}_3)}{m_{33} \bar{m}_{11} D_3^{11*}} \right) + \mathcal{O}(\lambda^4) \right], \\
\end{aligned} \quad (37)$$

$$\begin{aligned}
V_{ts} &= -x^{\alpha_2} N_3^{(u)} N_2^{(d)} \left[\frac{\bar{m}_{11} (\bar{\mathbf{z}}_1 \cdot \mathbf{m}_3^*)}{m_{33}^* D_3^{11}} + x^{2(\alpha_1 - \alpha_2)} \right. \\
& \quad \left. \times \left(f_\alpha^* n_{13} + \frac{\bar{m}_{21} l_{12} (\bar{\mathbf{z}}_1 \cdot \mathbf{m}_3^*)}{m_{33}^* l_{11} D_3^{11}} \right) + \mathcal{O}(\lambda^4) \right], \\
\end{aligned} \quad (38)$$

where $x^{\alpha_2} \sim \lambda^2$. These equations yield the relation

$$|V_{ts}|^2 = |V_{cb}|^2 \times [1 - |V_{us}|^2 + p \lambda^4] \quad (39)$$

with $p = \mathcal{O}(1)$, which is free from the phase convention. From the unitarity condition on V_{CKM} Eq. (39) can be rewritten as

$$|V_{cd}|^2 = |V_{us}|^2 \times [1 - |V_{cb}|^2 + \mathcal{O}(\lambda^6)]. \quad (40)$$

Further we obtain

$$\begin{aligned}
V_{td} &= x^{\alpha_1} N_3^{(u)} N_1^{(d)} \left[\frac{\xi_d^2 \bar{z}_{11}^* (\mathbf{m}_3^* \cdot \bar{\mathbf{z}}_1)}{m_{33}^* l_{11}} + x^{2(\alpha_1 - \alpha_2)} \frac{\xi_d^2}{m_{33}^* (l_{11})^2} \right. \\
& \quad \times \left(\xi_d^2 \bar{z}_{11}^* \bar{z}_{12}^* (\mathbf{m}_3 \times \mathbf{z}_3)_1^* + \bar{z}_{12}^* |\bar{m}_{11}|^2 (\mathbf{m}_3^* \cdot \bar{\mathbf{z}}_2) \right. \\
& \quad \left. \left. - \frac{l_{21}}{l_{11}} \bar{m}_{11} D_3^{11*} (\mathbf{m}_3^* \cdot \bar{\mathbf{z}}_1) \right) + \mathcal{O}(\lambda^4) \right], \\
\end{aligned} \quad (41)$$

$$V_{ub} \simeq x^{3\alpha_1 - 2\alpha_2} N_1^{(u)} N_3^{(d)} \frac{\xi_d^2 z_{33}^* (\mathbf{z}_3 \cdot \bar{\mathbf{m}}_1^*)}{\bar{m}_{11}^* |D_3^{11}|^2}, \quad (42)$$

where $x^{\alpha_1} \sim \lambda^3$ and $x^{3\alpha_1 - 2\alpha_2} \sim \lambda^5$. In the above equation (42) the element V_{ub} is rather small compared to the element V_{td} due to the cancellation of leading term. The above expression of V_{td} leads us to the relation

$$|V_{td}|^2 = |V_{us} V_{cb}|^2 \times (1 + q \lambda^2) \quad (43)$$

with $q = \mathcal{O}(1)$.

4. The quark CP-phase

Here we pay attention to Eqs. (39) and (43), which yield interesting relations between the CP-phase and three mixing angles. It has been shown that in the present model the hierarchical magnitude of three mixing angles can be understood systematically [4]. The recent values of the CKM matrix elements have been summarized by PDG [9] as

$$\begin{aligned}
|V_{ts}| &= 0.0405 \pm 0.0012, & |V_{cb}| &= 0.0414 \pm 0.0012, \\
|V_{us}| &= 0.22536 \pm 0.00061, & |V_{cd}| &= 0.22522 \pm 0.00061, \\
|V_{td}| &= 0.00886 \pm 0.00033.
\end{aligned} \quad (44)$$

As seen in Eq. (4), the values of $|V_{ts}|$ and $|V_{cb}|$ are mainly determined by s_{23} due to the hierarchical structure of the mixing angles. So it is noted that the double-sign in them corresponds in the same order. Then, with the aid of these data the relations (39) and (43) lead to

$$p = 3.0 \pm 0.6, \quad q = -1.9 \pm 0.4, \quad (45)$$

which are consistent with $p, q = \mathcal{O}(1)$. The relation (40) is also in good agreement with the data. These results are in support of the present analyses up to the next-to-leading approximation in the F–N scheme.

In the standard representation of PDG [9] for V_{CKM} the relation (39) becomes

$$\begin{aligned}
& \left| c_{12} s_{23} + s_{12} c_{23} s_{13} e^{i\delta} \right|^2 \\
& = (s_{23} c_{13})^2 \times [1 - (s_{12} c_{13})^2 + p \lambda^4], \\
\end{aligned} \quad (46)$$

which is rewritten as

$$\cos \delta = \frac{s_{23} (c_{13})^2}{2 s_{12} c_{12} c_{23} s_{13}} p \lambda^4 - \frac{s_{13}}{2 s_{12} c_{12} s_{23} c_{23}} \left((s_{12} c_{23})^2 + (s_{23})^2 - (s_{23} s_{12})^2 (1 + (c_{13})^2) \right). \quad (47)$$

It is worth noting that in Eq. (46) the leading terms in the l.h.s. and the r.h.s. are canceled out and that Eq. (47) represents a relation between the non-leading terms. When we take $p = 3.0 \pm 0.6$ and the experimental values [9]

$$s_{12} = 0.22536 \pm 0.00061, \quad s_{23} = 0.0414 \pm 0.0012, \\ s_{13} = 0.00355 \pm 0.00015,$$

the above equation (47) results in $\cos \delta = 0.20 \pm 0.04$. It might be thought that we can simply derive $\cos \delta$ from $|V_{ts}| = |c_{12} s_{23} + s_{12} c_{23} s_{13} e^{i\delta}|$. However, it is impossible for us to get information about $\cos \delta$ because the magnitude of the experimental error of $|V_{ts}|$ is larger than the coefficient of $e^{i\delta}$.

Further Eq. (43) is translated into

$$\left| s_{12} s_{23} - c_{12} c_{23} s_{13} e^{i\delta} \right|^2 = (s_{12} s_{23} c_{13}^2)^2 \times (1 + q \lambda^2), \quad (48)$$

which contains

$$\cos \delta = \frac{c_{12} c_{23} s_{13}}{2 s_{12} s_{23}} - \frac{s_{12} s_{23}}{2 c_{12} c_{23} s_{13}} \left((c_{13})^4 q \lambda^2 - (s_{13})^2 (1 + (c_{13})^2) \right). \quad (49)$$

Substituting the experimental values of the three mixing angles and $q = -1.9 \pm 0.4$ to Eq. (49), we obtain $\cos \delta = 0.32 \pm 0.03$. Note that the uncertainty of $\cos \delta$ obtained here is rather small compared with that ($\cos \delta = 0.32 \pm 0.11$) determined directly from $|V_{td}| = |s_{12} s_{23} - c_{12} c_{23} s_{13} e^{i\delta}|$.

Within 2σ there is no discrepancy in the above two values of $\cos \delta$. Consequently, we conclude with $\cos \delta = 0.26 \pm 0.05$ and $\delta = (75 \pm 3)^\circ$ in this analysis. The current experimental data show that $\gamma = (68 \pm 8)^\circ$ [9], where $\gamma = \arg(-(V_{ud} V_{ub}^*) / (V_{cd} V_{cb}^*)) \simeq \delta$. In addition, the recent averaged value is $\gamma = (73_{-10}^{+9})^\circ$ [13]. So our result is consistent with the data.

5. Summary

In the present model, the characteristic structure of the CKM matrix is attributed to the hierarchical effective Yukawa couplings due to the Froggatt–Nielsen mechanism and also to the state-mixings beyond the MSSM. The D^c – g^c mixings as well as generation mixings take place in the down-type quark sector. On the other hand, in the up-type quark sector we have no such mixings. These differences cause the nontrivial structure in CKM matrix. Specifically, the down-type mass matrix is described in terms of \mathcal{M} and \mathcal{Z} matrices in contrast with the up-type mass matrix of \mathcal{M} itself. As a result, all off-diagonal elements of the CKM matrix are expressed as the products of \mathcal{M} and \mathcal{Z} elements.

In the D^c – g^c mixings, since D^c and g^c are both $SU(2)_L$ -singlets, the disparity between the diagonalization matrices for up-type quarks and down-type quarks in $SU(2)_L$ -doublets is rather small. Accordingly, V_{CKM} exhibits small mixing. In this study we have found interesting relations between the CP -phase and three mixing angles without relying on a specific flavor symmetry. Taking the current data of three mixing angles, we estimate the quark CP -phase at $\delta = (75 \pm 3)^\circ$. This result is in accord with the current data of δ .

The relations between the CP -phase and three mixing angles stem from the fact that the CKM matrix comprises only two matrices \mathcal{M} and \mathcal{Z} . This is because all matter fields belong to either of $(\mathbf{15}, \mathbf{1})$ or $(\bar{\mathbf{6}}, \mathbf{2})$ representations in the gauge group $SU(6) \times SU(2)_R$. If the gauge group is chosen to be smaller than the above group, the number of irreducible representations for matter fields becomes larger than two. In such a case, there appear more parameters and hence we have no interesting relations between the CP -phase and three mixing angles. We expect that the present model gives a comprehensive explanation of fermion mass spectra and mixing matrices together with the longevity of the proton and gauge coupling unification [2–6].

Finally, we touch upon the study of the MNS matrix [14]. The observed features of the MNS matrix differ considerably from those of the CKM matrix. In the present model the L – H_d mixings occur in the lepton sector. Since L and H_d are both $SU(2)_L$ -doublets, there appears no disparity between the diagonalization matrices for charged leptons and neutrinos unless the seesaw mechanism does not take place. As a matter of fact, however, the seesaw mechanism is at work and an additional transformation is required to diagonalize the neutrino mass matrix. This additional transformation matrix yields nontrivial V_{MNS} . The seesaw mechanism brings about the cancellation of the F–N factors in the neutrino mass matrix [6,8]. As a consequence, there is no hierarchical structure in neutrino mass matrix and eventually V_{MNS} exhibits large mixing. It does not seem that the MNS matrix elements are connected to the CKM matrix elements in an uncomplicated way. For this reason it is difficult for us to find a simple interrelation among the quark CP -phase and the leptonic CP -phase.

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