We identify some remnants of normality and call them rudimentary normality, generalize the concept of submetacompact spaces to that of a weakly subparacompact space and that of a weakly* subparacompact space, and make a simultaneous generalization of collectionwise normality and screenability with the introduction of what is to be called collectionwise $\sigma$-normality. With these weak properties, we show that,

1) on weakly subparacompact spaces, countable compactness = compactness, $\omega_1$-compactness = Lindelöfness;
2) on weakly subparacompact Hausdorff spaces with rudimentary normality, regularity = normality = countable paracompactness; and
3) on weakly subparacompact regular $T_1$-spaces with rudimentary normality, collectionwise $\sigma$-normality = screenability = collectionwise normality = paracompactness.

The famous Normal Moore Space Conjecture is thus given an even more striking appearance and Worrell and Wicke’s factorization of paracompactness (over Hausdorff spaces) along with Krajewski’s are combined and strengthened. The methodology extends itself to the factorization of paracompactness on locally compact, locally connected spaces in the manner of Gruenhage and on locally compact spaces in that of Tall, and to the factorization of subparacompactness and metacompactness in the genre of Katuta, Chaber, Junnila and Price and Smith and that of Boone, improving all of them.
The results are:

1. on weakly subparacompact Hausdorff spaces with rudimentary normality, regularity = normality = countable paracompactness;
2. on weakly subparacompact spaces, collectionwise $\sigma$-normality $\Rightarrow$ screenability, while strong collectionwise $\sigma$-normality $\Rightarrow$ Krajewski’s semiparacompactness;
3. weak subparacompactness + regularity + rudimentary normality + collectionwise $\sigma$-normality $\Rightarrow$ paracompactness;
4. on locally compact, locally connected Hausdorff spaces, weak $^\ast$ subparacompactness + rudimentary normality + countable collectionwise $\epsilon$-normality $\Rightarrow$ paracompactness;
5. on locally compact spaces, weak subparacompactness + regularity + rudimentary normality + collectionwise $\sigma$-normality with respect to compact subsets $\Rightarrow$ paracompactness;
6. weak subparacompactness + countable metacompactness + weak collectionwise $\sigma$-normality $\Rightarrow$ metacompactness;
7. weak subparacompactness + collectionwise $\delta^\ast$-normality $\Rightarrow$ subparacompactness;
8. on weakly subparacompact spaces, countable compactness = compactness, and weakly subparacompact, $\omega_1$-compact spaces are Lindelöf;
9. an abstraction of Wicke and Worrell’s result that, on weakly $\delta\theta$-refinable spaces, countable compactness = compactness.

0. Notations, terminology and preliminaries

0.1. No separation axioms are assumed. Thus, regular or normal spaces are not necessarily $T_1$. Paracompact (respectively, screenable and strongly screenable) spaces are those, every open cover of which has a locally finite (respectively, $\sigma$-disjoint and $\sigma$-discrete) open refinement, at variance with Michael’s [15] usage of the term.

0.2. Undefined terms can be found defined in [4] and [12].

0.3. A topological space $X$ is normal if (and only if) given disjoint closed subsets $A$ and $B$ of $X$, there is a sequence $\langle V_n \rangle$ of open subsets such that $B \subset \bigcup \{V_n: n \in \mathbb{N}\}$ and $\text{Cl} V_n \subset X \setminus A$, for every $n \in \mathbb{N}$, (part of Theorem 3 of Hung [7]).

0.4. A topological space $X$ is countably paracompact if, and only if, for a decreasing sequence $\langle F_n \rangle$ of closed sets with void intersection, there is an increasing sequence $\langle G_n \rangle$ of open sets such that $F_n \cap \text{Cl} G_n = \emptyset$ and $\bigcup \{G_n: n \in \mathbb{N}\} = X$ (Ishikawa [11], Theorem V.6 of [17]).

0.5. Screenable, countably paracompact, normal spaces are strongly screenable (Nagami [16]).

0.6. Regular, strongly screenable spaces are paracompact (Michael [15]).

1. Three weak covering properties

We first identify a small part of normality, itself absent in the property of regularity, which, under some circumstances, is enough to make normal of the regular.

Definition 1.1. A topological space $X$ is said to have rudimentary normality if, given a discrete family $C$ of closed subsets and, for every $C \in C$, given a neighborhood $U(C)$, disjoint from $C' \in C$, when $C' \neq C$, there is, for every $n \in \mathbb{N}$, such a family $V_n \equiv \{V_n(C): C \in C\}$ of open subsets that:

i) $C \subset \bigcup \{V_n(C): n \in \mathbb{N}\} \subset U(C)$ for every $C \in C$, and

ii) for every $n \in \mathbb{N}$, $\text{Cl} \bigcup \{V_n(C): C \in C\} \subset \bigcup \{\text{Cl} U(C): C \in C\}$.

Remarks. Clearly normal spaces have rudimentary normality. So does the denormalization of collectionwise normality (Definition 1.1 of [10]). In the property expandability of Katetov and Krajewski (see §4 of [12]), we can of course find rudimentary normality, although not normality itself. Clearly, $\omega_1$-compact spaces have this property.

To exploit the newly defined property of rudimentary normality, we define a class, as big as possible, in which regularity under its influence becomes normality.

Definitions 1.2. Given a topological space $X$. A sequence $\{P_\alpha: \alpha \in \omega^2\}$ of collections of subsets is said to be a $\tau$-sequence, if:

i) $P_{\omega m+n}$ restricted to any closed set $F$ disjoint from $C_{\omega m+n} = \bigcup \{P_\beta: \omega m \leq \beta < \omega m + n\}$ is discrete and closed, for all $0 \leq m, n < \omega$, and,

ii) $X = \bigcup \{P_\alpha: \alpha \in \omega^2\}$. 

Given an open cover $\mathcal{U}$ of $X$, if there is a $\tau$-sequence $\{P_\alpha : \alpha < \omega^2\}$ such that, for any $P \in \mathcal{P}_{\text{om}+q}$ and any closed set $\Gamma$ disjoint from $C_{\text{om}+q}$, there is such a $U \in \mathcal{U}$ that $P \cap \Gamma \subset U$, we say $\mathcal{U}$ is refined by the $\tau$-sequence. $X$ is said to be weakly subparacompact if every open cover $\mathcal{U}$ of $X$ is refined by some $\tau$-sequence.

**Remarks.** 1. The $\tau$-sequence can be shortened to one indexed by the ordinal $\omega$, with the device that shows the union of countably many countable sets is countable. We decide to keep it as is so that its genesis from submetacompactness is more on the surface. Note that the families $\{P_\alpha : \alpha < \omega\}, \{P_\alpha : \alpha \in [\omega, \omega_1)\}, \ldots$ are really independent, one of another, and they are strung together for easy recall.

2. The concept weak subparacompactness thus defined is formally weaker than $B(D, \omega)$, the strongest among a host of properties, $B(D, \lambda), B(L(F, \lambda)$ and $B(HCP, \lambda)$, identified by J.C. Smith [20] for the purpose of exploring the area populated by weak $\theta$-refinability, weak $\theta$-refinability, irreducibility and the like. We are looking for a class biggest possible where we are, with rudimentary normality, allowed to scale the Separation Axioms from $T_2$ to $T_4$ and beyond.

**Question.** Is it true that weakly subparacompact spaces have property $B(D, \omega)$?

3. We can add a third item to the requirements in Definitions 1.2 thus, 

iii) for every $P \in \bigcup\{P_\alpha : \alpha < \omega^2\}$, there is such a finite family $\mathcal{F}(P)$ of open neighborhoods of $P$ that: a) $P \subset V \subset U$, for some $U \in \mathcal{U}$, if $V \in \mathcal{F}(P)$, b) for every $0 \leq m < \omega$, $P \subset V_m$, $P \subset V$, $P \subset \mathcal{P}_{\text{om}+p}$. $Q \in \mathcal{P}_{\text{om}+q}$: either $Q \subset \bigcup \mathcal{F}(P)$ or $Q \cap \bigcup \mathcal{F}(P) = \emptyset$.

and have instead weak* subparacompactness. Note that submetacompactness $\Rightarrow$ weak* subparacompactness $\Rightarrow$ weak subparacompactness.

4. Clearly, subparacompact spaces are weakly subparacompact, as are submetacompact spaces, when we note that, if, for some open cover $V$ of $X$, we let $X_m \equiv \{x \in X : |\{V \in V : x \in V\}| = m\}$, we see that $y \notin X_m$, $m < |\{V \in V : y \in V\}|$. Weakly subparacompact spaces, however, are not necessarily submetacompact. Counter-examples can be found in 4.5 and 4.9(i) of [4].

With these two newly defined properties, we are going to use them to strengthen Worrell and Wicke’s and Krajewski’s factorization of paracompactness and introduce below a simultaneous generalization of the two properties Bing introduced in his effort in the same direction.

**Definition 1.3.** A topological space $X$ is said to have property collectionwise $\sigma$-normality (respectively, strong collectionwise $\sigma$-normality) if, given a discrete family $\mathcal{C}$ of closed subsets, there is, for every $n \in \mathbb{N}$, such a disjoint (respectively, locally finite) family $V_n \equiv \{V_n(C) : C \in \mathcal{C}\}$ of open subsets that $C \subset \bigcup\{V_n(C) : n \in \mathbb{N}\}$, for every $C \in \mathcal{C}$.

**Remarks.** 1. The property collectionwise $\sigma$-normality is so named because, here, instead of the separating disjoint neighborhoods in the case of collectionwise normality, we have countably many families of disjoint open sets in order to separate in some manner the members of the discrete family of closed sets. Clearly, it is a simultaneous generalization of screenability and collectionwise normality, both of Bing [2]. It was first introduced in [8]. Strong collectionwise $\sigma$-normality, on the other hand, is a simultaneous generalization of strong screenability and collectionwise normality of Bing [2] and a property studied by Katetov and Krajewski (Theorem 4.1 of [12]). Variations of the notion of collectionwise $\sigma$-normality appear below in various places near where they are invoked, in the remarks 2) and 3) on Corollary 2.6 and in Definitions 2.7 and 3.1.

2. A sufficient condition for collectionwise $\sigma$-normality is that given a discrete family $\mathcal{C}$ of closed subsets, there is, for each $C \in \mathcal{C}$, such a (decreasing) sequence of open neighborhoods $\langle V_n(C) \rangle$ of $C$ that, for every $\Gamma \in \mathcal{C}$, we have $\Gamma \cap \bigcap\{\text{Cl} \bigcup\{V_n(C) : C \in \mathcal{C}, C \neq \Gamma\} : n \in \mathbb{N}\} = \emptyset$.

2. Main results

The result 2.1 below is given first, not because it is the most important or the most profound but because it is the easiest to demonstrate. Its first part generalizes Worrell and Wicke [24] (Corollary 1.13 of [12]), though not Theorem 1.1 of Wicke and Worrell [23], which says essentially that, on a countably compact space, any open cover that can be arranged in the form of a weak $\delta\tau$-sequence (Definition analogous to that of $\tau$-sequences) has a (countable and therefore) finite subcover. In Theorem 2.2 below, an abstraction of the possibility of such an arrangement is made.

**Theorem 2.1.** Weakly subparacompact, countably compact spaces $X$ are compact. Weakly subparacompact, $\omega_1$-compact spaces are Lindelöf.

**Proof.** Let $\mathcal{U}$ be any open cover of $X$ refined by some $\tau$-sequence $\{P_\alpha : \alpha < \omega^2\}$. We note that $\mathcal{P}_\alpha$ is finite if $\alpha = \text{om}$, $0 < m < \omega$. It can be arranged that $P \subset U(P) \in \mathcal{U}$ for every $P \in \mathcal{P}_\alpha$. $\mathcal{U}_\alpha \equiv \{U(P) : P \in \mathcal{P}_\alpha\}$ is a finite subfamily of $\mathcal{U}$. Clearly, $\mathcal{P}_{\alpha+1}$, restricted to $X_1 = X \setminus U_{\mathcal{P}_\alpha}$, is discrete and closed in $X$ and therefore also finite. It can be arranged that $P \cap X_1 \subset
U(P) ∈ ℰ, for every P ∈ ℰ_{a+1} such that P ∩ X_1 = ∅. ℰ_{a+1} = \{U(P): P ∈ ℰ_{a+1}, P ∩ X_1 = ∅\} is a finite subfamily of ℰ. Clearly, ℰ_{a+2}, restricted to X_2 = X_1 \backslash ℰ_{a+1}, is discrete and closed in X and therefore finite. It can be arranged that P ∩ X_2 ⊂ U(P) ∈ ℰ, for every P ∈ ℰ_{a+2} such that P ∩ X_2 = ∅.

Clearly V = \bigcup\{U_{om+n}: 0 ≤ m, n ∈ ω\} is a countable subfamily of ℰ that covers X and there is a finite subcover F ⊂ V ⊂ ℰ of X. X is therefore compact. □

**Theorem 2.2.** A countably compact space X is compact, if (and only if), (*) for every open cover ℰ of X, there is such a countable cover V = \{V_n: n ∈ N\} that, for every n ∈ N, there is an open neighborhood V_n of V_n so that C ⊂ V_n can be covered by a countable subfamily of ℰ, if C ⊂ V_n.

**Proof.** Given ℰ and V, we define an increasing sequence \{W_n: n ∈ N\} of open subsets as follows. If V_1 can be covered by some countable subfamily V_1 of ℰ, let W_1 = \bigcup V_1. Otherwise, let W_1 = V_1. If V_2 \bigcup W_1 can be covered by some countable subfamily V_2 of ℰ, let W_2 = W_1 \bigcup V_2. Otherwise let W_2 = W_1 \bigcup W_2, etc. Clearly, \{W_n: n ∈ N\} is an increasing open cover of X and W_n = X for some n ∈ N. Let μ be the largest natural number ≤ v such that V_μ \bigcup W_μ−1 = \bigcup W_μ = ∅ cannot be covered by any countable subfamily of ℰ. If μ < v, let W’ = \bigcup\{V_n: μ < n ≤ v\}. If μ = v, let W’ = ∅. Either way W’ is (at most) countable and the set X \bigcup\{W_μ−1 \bigcup W’: μ < v ≤ v\} is closed. It follows that V_μ \bigcup W_μ−1 = \bigcup W_μ = ∅ and therefore V_μ \bigcup W_μ−1 can be covered by some countable subfamily of ℰ, contradicting the definition of μ.

Therefore W_2 = W_{n−1} \bigcup V_2 for 0 < n < v, X = \bigcup\{V_n: n ≤ v\} and there is a finite ℬ ⊂ \bigcup\{V_n: n ≤ v\} such that X = \bigcup ℬ, i.e., X is compact. □

**Remarks.** Clearly, if X is countably compact and weakly δθ-refinable [23], we can identify the \bigcup V_n’s of Wicke and Worrell with our V_n and their |x ∈ X: 0 < ord(V_n, x) ≤ ω| with our V_n and arrive at the conclusion of Theorem 1.1 of [23].

**Corollary 2.3.** A countably compact space X is compact, if, for every open cover ℰ, there is a countable open cover \{V_n: n ∈ N\} such that every closed subset C ⊂ X, for some n ∈ N, can be covered by a countable subfamily of ℰ.

**Theorem 2.4.** On weakly subparacompact Hausdorff spaces X with rudimentary normality, regularity = normality = countable paracompactness.

**Proof.** 1) We are to prove that weakly subparacompact, countably paracompact Hausdorff spaces X, with rudimentary normality, are regular. Let W be an open neighborhood of ξ ∈ X. For every x ∈ W, let U(x) be such a neighborhood of x that ξ ∈ Cl(U(x)). Let ℰ = \{U(x): x ∈ X\}. The family ℰ ∪ \{W\} is an open cover of X, and is refined by some τ-sequence \{P_α: α ∈ ω^2\}. For all α ∈ ω^2, let Q_α = \{P ∈ ℰ: P \cap W = ∅\}. We note Q_α is a discrete and closed family in X, if α = ωm, 0 ≤ m < ωm. Choose U(Q) ∈ ℰ, so that Q ⊂ U(Q), for every Q ∈ Q_α, and let U’(Q) = U(Q) \bigcap \{R: R ∈ Q_α\}. Because of rudimentary normality on X, there is, for every n ∈ N and every Q ∈ Q_α, an open subset V_n(Q), as described in Definition 1.1, so that, for every n ∈ N,

\[
Cl\bigcup\{V_n(Q): Q ∈ Q_α\} ⊂ Cl\bigcup\{U’(Q): Q ∈ Q_α\} ⊂ Cl\bigcup\{U(Q): Q ∈ Q_α\} ⊂ Cl\bigcup\{X\{ξ\} = ∅.
\]

and if we write V_{α,n} = \bigcup\{V_n(Q): Q ∈ Q_α\}, we can see that U(Q_α) ⊂ U(V_{α,n}: n ∈ N). Clearly, Q_{α+1}, restricted to X_1 = X \backslash V_{α,n} is discrete and closed in X, and there is, for every n ∈ N and every Q ∈ Q_{α+1}, Q \backslash V_{α,n} = ∅, an open subset V_{α,n} = \bigcup\{V_n(Q): Q ∈ Q_α\}, Q \backslash V_{α,n} = ∅, and there is V_{α+1} = \bigcup\{V_{α+1,n}: n ∈ N\} = \bigcup\{W_n: α ∈ ω^2, n ∈ N\}. Clearly, V_{α,n} = \bigcup\{W_n: α ∈ ω^2, n ∈ N\} ∪ \{W\} is a countable open cover of X and there is a locally finite open refinement G. Clearly, X \backslash W ⊂ Cl\bigcup\{G: G ∈ G\} \backslash W = ∅ ∩ Cl\bigcup\{G: G ∈ G\} \backslash W = ∅} ⊂ Cl\bigcup\{X\{ξ\} = ∅. That is, X is regular.

2) We are to prove that weakly subparacompact regular (not necessarily Hausdorff) spaces X, with rudimentary normality, are normal. If, in 1) above, we let U(x) be replaced by a closed subset A, we see that A \bigcap Cl V_{α,n} = ∅ for every α ∈ ω^2, n ∈ N, and in view of 0.3, without recourse to countable paracompactness.

3) We are to prove that weakly subparacompact regular (not necessarily Hausdorff) spaces X, with rudimentary normality are countably paracompact. With 0.4 in mind, let there be a decreasing sequence (F_n) of closed sets with void intersection. For every point x ∈ X, let U(x) be such that F_n \bigcap Cl U(x) = ∅, where n ∈ N is the smallest possible. The family ℰ = \{U(x): x ∈ X\} clearly covers X and is refined by some τ-sequence \{P_α: α ∈ ω^2\}. For each α ∈ ω^2, α = ωm, 0 ≤ m < ωm, and each P ∈ ℰ, choose U(P) ∈ ℰ so that P ⊂ U(P) and F_n \bigcap Cl U(P) = ∅ for some n ∈ N. For every n ∈ N, let P_{α,n} = \{P: P ∈ ℰ_{α,n}, F_n \bigcap Cl U(P) = ∅\}. Clearly, P_{α,n} is discrete and closed and \bigcup\{P_{α,n}: n ∈ N\} = P_α. For every P ∈ P_{α,n}, let U’(P) = U(P) \bigcup\{R: R ∈ P_{α,n}\} = R = P. Because of countable paracompactness on X, there is, for every l ∈ N and every P ∈ P_{α,n}, an open subset V_l(P), as described in Definition 1.1, so that, for every l ∈ N,
and if we write \( V(\alpha, n, l) \equiv \bigcup \{ V(\alpha, n, l) : P \in \mathcal{P}_{\alpha, n} \} \), we see that \( F_0 \cap \operatorname{Cl} V(\alpha, n, l) = \emptyset \). If we write \( V(\alpha) \equiv \bigcup \{ V(\alpha, n, l) : n, l \in \mathbb{N} \} \), we see that \( P_{\alpha+1} \) restricted to \( X \setminus V(\alpha) \) is discrete and closed in \( X \), and there is, for every \( n \in \mathbb{N} \), \( P_{\alpha+1,n} \), there is, for every \( P \in P_{\alpha+1,n} \) and every \( l \in \mathbb{N} \), \( V(\alpha) \setminus V(\alpha + 1, n, l) = \emptyset \).....

If, for every \( n \in \mathbb{N} \), we let \( G_n \equiv \bigcup \{ V(\alpha, n, l) : 0 < l \leq n, 0 \leq m, p \leq n \} \), we can see that \( F_n \setminus \operatorname{Cl} G_n = \emptyset \) and \( \bigcup \{ G_n : n \in \mathbb{N} \} = X \).

And, by 0.4, \( X \) is countably paracompact. \( \Box \)

Remarks. In view of the above, the famous Normal Moore Space Conjecture takes on an even more striking form: Moore spaces with rudimentary normality (1.1) are metrizable. Indeed, it was the realization of this possibility that prompted my looking for a biggest class of spaces possible to exploit the potential of the concept of rudimentary normality, the present paper being the result. While the properties of countable paracompactness and normality are in general distinct even among perfect spaces, they are indistinguishable among weakly subparacompact \( T_2 \) spaces with rudimentary normality. Thus the countably paracompact non-normal Moore spaces asserted to exist in Corollaries 2 and 3 of [22] actually fail to be normal at a fundamental level, in a specific area. Note also that pseudo-normal spaces do not always have rudimentary normality (see Example 3 of [22]) and \( T_1 \) Dowker Spaces cannot be weakly subparacompact. And a question of Younglove takes a seemingly more plausible form.

Question. Is it consistent that countably paracompact Moore spaces have rudimentary normality (cf. III, S[a] of Tall [21])?

Theorem 2.5. Weakly subparacompact, collectionwise \( \sigma \)-normal spaces \( X \) are screenable. Weakly subparacompact, strongly collectionwise \( \sigma \)-normal spaces \( X \) are semiparacompact (Krajewski’s [14] terminology).

Proof. Let \( \mathcal{U} \) be an open cover of \( X \) that is refined by a \( \tau \)-sequence \( \{ P_\alpha : \alpha \in \omega \} \). \( P_\alpha \) being a discrete and closed collection, if \( \alpha = \text{om}, 0 \leq m < \omega \), there is, for every \( n \in \mathbb{N} \) and every \( P \in P_\alpha \), an open set \( V_n(P) \), as described in Definition 1.3, such that \( \{ V_n(P) : P \in P_\alpha \} \), for all \( n \in \mathbb{N} \), is disjoint and \( P \subset \bigcup \{ V_n(P) : n \in \mathbb{N} \} \). Choose \( U(P) \in \mathcal{U} \) so that \( P \subset U(P) \) and let \( W_n(P) = U(P) \setminus V_n(P) \), for every \( P \in P_\alpha \) and every \( n \in \mathbb{N} \). Clearly, \( \{ W_n(P) : P \in P_\alpha \} \) is disjoint for all \( n \in \mathbb{N} \). If we let \( W_\alpha = \bigcup \{ W_n(P) : P \in P_\alpha, n \in \mathbb{N} \} \), we see that \( P_{\alpha+1} \) restricted to \( X \setminus W_\alpha \) is discrete and closed in \( X \), and there is, for every \( n \in \mathbb{N} \) and every \( P \in P_{\alpha+1} \), \( P \setminus W_\alpha \neq \emptyset \), an open set \( V_n(P) \), as described in Definition 1.3, such that \( \{ V_n(P) : P \in P_{\alpha+1} \} \), \( P \setminus W_\alpha \neq \emptyset \), for all \( n \in \mathbb{N} \), is disjoint and \( P \setminus W_\alpha \subset \bigcup \{ W_n(P) : n \in \mathbb{N} \} \). Choose \( U(P) \in \mathcal{U} \) so that \( P \setminus W_\alpha \subset U(P) \) and let \( W_n(P) = U(P) \cap V_n(P) \), for all \( P \in P_{\alpha+1} \), \( P \setminus W_\alpha \neq \emptyset \) and all \( n \in \mathbb{N} \). If we let \( W_{\alpha+1} = W_\alpha \cup \bigcup \{ W_n(P) : P \in P_{\alpha+1}, P \setminus W_\alpha \neq \emptyset, n \in \mathbb{N} \} \).

Clearly \( \bigcup \{ W_n(P) : P \in P_\alpha, P \setminus W_{\alpha-1} \neq \emptyset \text{ if } \alpha \text{ has an immediate predecessor} \} : n \in \mathbb{N}, P \in P_\alpha \subset \omega^2 \) is a \( \sigma \)-disjoint refinement of \( \mathcal{U} \), i.e., \( X \) is screenable. The second statement is similarly proved. \( \Box \)

Remarks. The second statement strengthens Theorem 2.11 of Krajewski [14] (Theorem 4.1 of Junnila [12]). Note that expandable spaces are countably paracompact (Corollary 2.5.1 of [14]) and, in the presence of countable paracompactness, semiparacompactness = paracompactness.

Corollary 2.6. Regular, weakly subparacompact, collectionwise \( \sigma \)-normal spaces with rudimentary normality are paracompact, i.e., on weakly subparacompact regular spaces with rudimentary normality, collectionwise \( \sigma \)-normality = screenability = collectionwise normality = paracompactness. Regular, weakly subparacompact, strongly collectionwise \( \sigma \)-normal spaces are paracompact.

Proof. Items 2 and 3 in the proof of Theorem 2.4 + Nagami (0.5) + Michael (0.6) + Theorem 2.5 yields the first result. Michael (0.6) + Theorem 2.5 yields the second. \( \Box \)

Remarks. 1. Clearly in the above, we have an improvement of both Worrell and Wicke (Theorem 4.16 of [4]) and Krajewski (Theorem 2.11 of [14]). One consequence is, when viewed with the factorization of monotone developability of [9] into three factors, we have the property of metrizability in seven factors, a large number of small pieces.

Question. Is it well known [24] that on a submetacompact \( T_1 \)-space, monotone developability = developability. On weakly subparacompact \( T_1 \)-spaces?
2. If we define a weak form of collectionwise σ-normality by requiring \( \mathcal{V}_n \) in Definition 1.3 to be only point-finite, we can say: Countably metacompact, weakly subparacompact, weakly collectionwise σ-normal spaces are metacompact (strengthening Boone, Theorem 3.3 of [12]).

3. If we restrict the \( C \) in Definition 1.3 to discrete families of compact closed subsets, we can speak of collectionwise σ-normality with respect to compact subsets in the manner Tall [21] does of collectionwise normality with respect to compact subsets, and say: On locally compact, weakly subparacompact regular spaces with rudimentary normality, collectionwise σ-normality with respect to compact subsets ⇒ paracompactness (strengthening Tall, Theorem 1.7 of I of [21]).

The question of by how much the weakly subparacompact is short of being subparacompact arises immediately and naturally upon the definition of the former. By way of an answer we are to strengthen Junnila’s collectionwise δ-normality in the direction towards subparacompactness itself.

**Definition 2.7.** A topological space is collectionwise \( \delta^+ \)-normal, if, for every discrete family \( C \) of closed subsets on \( X \), there is a sequence \( \{ \mathcal{V}_n(C): C \in \mathcal{C} \} \) of collections of open neighborhoods of members of \( C \), such that, for every \( x \in X \), there is such a \( v \in \mathbb{N} \) that \( |\mathcal{V}_v(x) = \{ V \in \mathcal{V}_v: x \in V \} | \leq 1 \) (cf. 3.7(ii) of [4]).

We can of course assume that, for every \( n \in \mathbb{N} \), \( C \in \mathcal{C} \),

i) \( C \cap \mathcal{V}_n(C') = \emptyset \), if \( C' \neq C \), and

ii) \( \mathcal{V}_{n+1} \subset \mathcal{V}_n(C) \).

Note that, in particular, if \( x \in \bigcap \{ \mathcal{V}_n(C): n \in \mathbb{N} \} \), \( |\mathcal{V}_n(x)| = 1 \) for large enough \( v \)’s.

We give the theorem in the following without proof, it being so very straightforward once the idea is grasped.

**Theorem 2.8.** (Cf. Katuta [13], Chaber [5], Theorem 2.7 of Junnila [12] and Price and Smith [18].) Weak subparacompactness +
collectionwise \( \delta^+ \)-normality = subparacompactness.

**Remarks.** 1. A pivot of the proof is provided below, in lieu of a full proof. Note that the \( C \) and the \( V \)’s in Definition 2.7 beget, for every \( n, k \in \mathbb{N} \), open sets \( W_k(n) \supset X \bigcup \{ \mathcal{V}_n(C): C \in \mathcal{C} \} \) and \( E_k(n) \supset \bigcup C \) such that, for every \( x \in X \), \( n \in \mathbb{N} \), there is such a \( k \in \mathbb{N} \) that \( x \notin E_k(n) \cap W_k(n) \). If, for every \( n, k \in \mathbb{N} \), \( C \in \mathcal{C} \), we let

\[ C(n, k) \equiv \{ x \in X: x \notin \mathcal{V}_n(C), x \notin \mathcal{V}_n(C') \text{ if } C' \neq C, x \notin W_k(n) \}, \]

we see that \( \{ C(n, k): C \in \mathcal{C} \} \) for every \( n, k \in \mathbb{N} \) is a discrete closed family and

\[ \bigcap \{ C(n, k): n, k \in \mathbb{N} \} \subset \bigcup \{ C(n, k): C \in \mathcal{C}, n, k \in \mathbb{N} \}. \]

2. It is not difficult to see that the difference between collectionwise \( \delta^+ \)-normality and collectionwise δ-normality can be accounted for with the notion of collectionwise \( \epsilon \)-normality to be introduced in 3.1 below, provided we enlarge its range to include all discrete families of closed subsets (not merely the compact closed ones), a property clearly found in submetacompactness.

3. The question of the subparacompactness of submetacompact spaces has many solutions. Katuta [13] offered subexpandability, Chaber [5] collectionwise sub-normality and Junnila [12] collectionwise δ-normality. Chaber’s collectionwise subnormality, equivalent to our collectionwise \( \delta^+ \)-normality, is in fact so strong that it allows the submetacompactness in Chaber’s result to weaken to \( B(D, \omega) \) (Theorem 3 of [18]), and here we further weaken it to weak subparacompactness.

3. **Main results on locally compact, locally connected spaces**

Given local compactness and local connectedness, normal Moore spaces are metrizable (Reed and Zenor [19]), normal, submetacompact spaces are paracompact (Gruenhage [6]). Thus collectionwise normality is dispensed with and simple normality suffices with the assumption of local compactness and local connectedness. We show in the following our weak covering properties and weak separation axioms can sharpen Gruenhage further. We introduce a simultaneous generalization of submetacompactness and the collectionwise \( \delta^+ \)-normality of Definition 2.7 above.

**Definition 3.1.** A topological space is collectionwise \( \epsilon \)-normal, if, for every discrete family \( C \) of compact closed subsets on \( X \), there is a sequence \( \{ \mathcal{V}_n(C): C \in \mathcal{C} \} \) of collections of open neighborhoods of members of \( C \), such that, for every \( x \in X \), there is such a \( v \in \mathbb{N} \) that \( |\mathcal{V}_v(x) = \{ V \in \mathcal{V}_v: x \in V \} | < \omega \).

We can of course assume that, for every \( n \in \mathbb{N} \), \( C \in \mathcal{C} \),

i) \( C \cap \mathcal{V}_n(C') = \emptyset \), if \( C' \neq C \), and

ii) \( \mathcal{V}_{n+1} \subset \mathcal{V}_n(C) \).
If the C’s are singletons, then we write $V_n(x)$ rather than $V_n(\{x\})$ and speak of the collectionwise $\epsilon$-Hausdorff property. If the cardinality of the family $C$ is always $\leq \epsilon$, then we attach the prefix $\epsilon$- to the word collectionwise.

Propositions 3.2 and 3.3 below demonstrate the interaction between these notions and that of rudimentary* normality, a stronger version of rudimentary normality, with $V_n(C) = V_1(C)$, in Definition 3.1, whatever $C$ and whatever $n$.

Given, on a $T_1$ space $X$, a closed discrete subset $A$ of cardinality $c$, there is a sequence $(P_n)$ of ever finer, finite, point-separating partitions of $A$. If $X$ is regular and has rudimentary* normality, then, for every $x \in A$, there is a decreasing sequence $(U(x, n))$ of open neighborhoods such that

1) $\text{cl}(U(x, 0)) \cap A = \{x\}$, and, for any $n > 0$,
2) $\text{cl}(U(x, n)) \subset \bigcup \{\text{cl}(U(x, n-1)) : x \in P\}$, for all $P \in \mathcal{P}_n$.

Thus, for every $n > 0$, $\mathcal{P}_n$ can be separated by disjoint open subsets, i.e., every $P \in \mathcal{P}_n$ has an open neighborhood $U(P)$ such that $\{U(P) : P \in \mathcal{P}_n\}$ is a disjoint family of open sets. We can of course assume that $U(x, n) \subset U(P)$ if $x \in P \in \mathcal{P}_n$. If, further, $X$ is locally connected and rim-compact, and $\epsilon$-collectionwise $\epsilon$-Hausdorff, then we can assume that $\partial U(x, 0)$ is compact for every $x \in A$ and all the $U(x, n)$’s are connected open neighborhoods of $x$ such that $(\ast)$ for every $x \in A$ there is a point $n(x)$ such that $U(x, n(x)) \cap U(y, n(x)) = \emptyset$ for all $y \neq x$ (which implies that the members of $A$ are separated by a family of disjoint open subsets). The negation of $(\ast)$ means that there are distinct $x, y, z, \ldots \in A$ such that $U(x, n) \cap U(y, n) \neq \emptyset$ whatever $n$ and there is $z \in \partial U(x, 0) \cap U(y, n)$ for every $n \in \omega$ clustering to $z \in \partial U(x, 0)$. But then, there is a $\nu \in \omega$ such that $|V_n(z)| < \omega$ and a $\mu \in \omega$ such that $x \in P \in \mathcal{P}_\mu$, $z, z_n \notin \text{cl}(U(P))$, for some large enough $n$, and $y_n \notin P$, i.e., $U(x, \mu) \cap U(y_n, \mu) = \emptyset$, a contradiction. Therefore, we have,

**Proposition 3.2.** On locally connected, rim-compact $T_3$ spaces, $\epsilon$-collectionwise $\epsilon$-Hausdorffness with rudimentary* normality $\Rightarrow \epsilon$-collectionwise Hausdorffness.

And, mutatis mutandis, we have,

**Proposition 3.3.** On locally connected, locally compact Hausdorff spaces $X$, $\epsilon$-collectionwise $\epsilon$-normality with rudimentary* normality $\Rightarrow \epsilon$-collectionwise normality with respect to compact subsets. Consequently, on $X$, the closure of any union of at most countably many open sets each with a compact closure cannot intersect every member of an uncountable discrete family of compact closed sets.

The arguments advanced in the proof of Lemma 3 of [6] and that of Lemma 8.13 of [4] remaining good, *mutatis mutandis,* for the reduced circumstances of weak* subparacompactness (see third item of Remarks on Definition 1.2), we have,

**Proposition 3.4.** If the spaces $X$ of Proposition 3.3 above are in addition connected and weakly* subparacompact, then any cover with open sets each with a compact closure has a subcover of cardinality $\leq \omega_1$.

**Theorem 3.5.** On locally connected, locally compact Hausdorff spaces, weak* subparacompactness $+ \epsilon$-collectionwise $\epsilon$-normality $\Rightarrow$ paracompactness (cf. Corollary 2.6 above).

**Question.** Is it true that weak* subparacompactness $+ \epsilon$-normality (with its range enlarged to include all discrete families of closed subsets) $\Rightarrow$ submetacompactness?

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**References**