Entropy and ergodic probability for differentiable dynamical systems and their bundle extensions

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Received 18 June 2004; received in revised form 4 September 2006; accepted 4 September 2006

Abstract

We answer a problem of Liao [S.T. Liao, Standard systems of differential equations and obstruction sets—from linearity to perturbations, in: System Researches, Proceedings Dedicated to the 85th Anniversary of Qian Xue-Sen, Zhejiang Education Press, Hangzhou, China, 1996, pp. 279–290 (in Chinese): A $C^1$ vector field or a $C^1$ diffeomorphism on an $n$-dimensional manifold has equal entropy with that of its bundle extensions. We also prove that each ergodic probability with simple Lyapunov spectrum has at most $2^n n!$ covering probabilities on each bundle extension.

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MSC: 37B38; 37D99

Keywords: Entropy equality; Ergodic probability; Bundle extension

1. Notions and a problem

We start from a $C^1$ vector field $S$ on a compact smooth $n$-dimensional Riemannian manifold $M$, and its induced flows $\phi_t : M \to M$, $t \in \mathbb{R}$ on the state manifold and $\Phi_t = d\phi_t : TM \to TM$, $t \in \mathbb{R}$ on the tangent bundle.

Fix some integer $\ell$, $1 \leq \ell \leq n$. Construct a bundle $U_\ell = \bigcup_{x \in M} U_\ell (x)$ of $\ell$-frames, where the fiber over $x$ is

$$U_\ell (x) = \{(u_1, \ldots, u_\ell) \in T_x M \times \cdots \times T_x M \mid u_1, u_2, \ldots, u_\ell \text{ are linearly independent}\}.$$

The vector field $S$ induces a flow on $U_\ell$, which we denote (with the same notation as the tangent map for the sake of simplicity) by $\Phi_t$, $t \in \mathbb{R}$, namely,

$$\Phi_t (u_1, u_2, \ldots, u_\ell) = (d\phi_t (u_1), d\phi_t (u_2), \ldots, d\phi_t (u_\ell)).$$

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1 Sun is supported by NNSFC (# 10231020, 10171004) and doctor research funds from Education Ministry of China (# 20040001036).
2 Vargas is supported by CNPq-Brazil.

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For $\alpha = (u_1, u_2, \ldots, u_\ell) \in U_\ell$ and a non-degenerate $\ell \times \ell$ matrix $B = (b_{ij})$ we write

$$\alpha \circ B = \left( \sum_{i=1}^{\ell} b_{i1} u_1, \sum_{i=1}^{\ell} b_{i2} u_2, \ldots, \sum_{i=1}^{\ell} a_{i\ell} u_\ell \right).$$

Then $\Phi_t(\alpha \circ B) = \Phi_t(\alpha) \circ B$. By the Gram–Schmidt orthogonalization process there exists a unique upper triangular matrix $\Gamma(\alpha)$ with diagonal elements 1 such that $\alpha \circ \Gamma(\alpha)$ is orthogonal.

Construct the bundle $\mathcal{F}_\ell = U_{x \in M} \mathcal{F}_\ell(x)$ of orthogonal $\ell$-frames, where the fiber over $x$ is

$$\mathcal{F}_\ell(x) = \{ (u_1, u_2, \ldots, u_\ell) \in U_\ell(x) \mid (u_i, u_j) = 0, \ 1 \leq i \neq j \leq \ell \}.$$

The vector field $S$ then induces a flow

$$\chi_\ell : \mathcal{F}_\ell \to \mathcal{F}_\ell, \quad \alpha \mapsto \Phi_t(\alpha) \circ \Gamma(\Phi_t(\alpha)).$$

If we define $\pi : U_\ell \to \mathcal{F}_\ell$ by $\alpha \mapsto \alpha \circ \Gamma(\alpha)$ then $\chi_\ell(\alpha) = \pi(\Phi_t(\alpha))$, $t \in \mathbb{R}$.

Construct a bundle $\mathcal{F}_{\ell}^\# = U_{x \in M} \mathcal{F}_{\ell}^\#(x)$ of orthonormal $\ell$-frames, where the fiber over $x$ is

$$\mathcal{F}_{\ell}^\#(x) = \{ (u_1, u_2, \ldots, u_\ell) \in \mathcal{F}_\ell(x) \mid \|u_i\| = 1, \ i = 1, 2, \ldots, \ell \}.$$

Then $\mathcal{F}_{\ell}^\#$ is a compact metric space. Let $\pi^\# : \mathcal{F}_\ell \to \mathcal{F}_{\ell}^\#$ be given by

$$\pi^\#(u_1, u_2, \ldots, u_\ell) = \left( \frac{u_1}{\|u_1\|}, \frac{u_2}{\|u_2\|}, \ldots, \frac{u_\ell}{\|u_\ell\|} \right).$$

Setting $\chi_{\ell}^\# = \pi^\# \circ (\chi_\ell | \mathcal{F}_\ell)$, we get a flow $\chi_{\ell}^\# : \mathcal{F}_{\ell}^\# \to \mathcal{F}_{\ell}^\#$, $t \in \mathbb{R}$.

Construct a Grassmann bundle $\mathcal{L}_{\ell}(M) = \bigcup_{x \in M} \mathcal{L}_{\ell}(x)$, where the fiber $\mathcal{L}_{\ell}(x)$ over $x$ is the Grassmann manifold formed by all $\ell$-dimensional linear subspaces in $T_x M$. Such a linear subspace $H$, when regarded as a point of $\mathcal{L}_{\ell}(x)$, will be denoted by $[H]$. Then $\mathcal{L}_{\ell}(M)$ is a compact metric space. Defining $\varphi_t([H]) := [\Phi_t(H)]$ gives a flow $\varphi_t : \mathcal{L}_{\ell}(M) \to \mathcal{L}_{\ell}(M), t \in \mathbb{R}$. By $p_\ell$ we denote the canonical projections $\mathcal{F}_{\ell}^\# \to M$ and $\mathcal{L}_{\ell}(M) \to M$. The following commutability properties hold clearly

$$p_\ell \circ \chi_{\ell}^\# = \varphi_t \circ p_\ell = p_\ell \circ \varphi_t, \quad t \in \mathbb{R}, \quad \ell = 1, \ldots, n.$$

For more details of these flows see [5] or [12].

A probability $\mu$ on $M$ is $\phi$-invariant if $\mu(\Phi_t(B)) = \mu(B)$ for any $t \in \mathbb{R}$ for any Borel set $B \subset M$. A probability $\mu$ is $\phi$-ergodic if each Borel set $B$ that is $\phi_t$-invariant for any $t \in \mathbb{R}$ has $\mu$-probability 1 or 0. Denote by $E(M, \phi)$ the set of all $\phi$-invariant and ergodic probabilities on $M$. Similarly one defines $E(\mathcal{F}_{\ell}^\#, \chi^\#)$ and $E(\mathcal{L}_{\ell}(M), \varphi)$, $\ell = 1, \ldots, n$.

**Problem.** (Liao [7]) Let $1 \leq \ell \leq n$. Take arbitrarily $v \in E(M, \phi)$, $\mu_\ell \in E(\mathcal{F}_{\ell}^\#, \chi^\#)$ and $m_\ell \in E(\mathcal{L}_{\ell}(M), \varphi)$ so that $p_{\ell \times \ell} m_\ell = v = p_\ell m_\ell$. For measure-theoretic entropy, what is the relation among $h_1(\phi)$, $h_\mu(\chi^\#)$ and $h_m(\varphi)$? And what is the relation among topological entropies $h(\phi)$, $h(\chi^\#)$ and $h(\varphi)$?

For a given $C^1$ diffeomorphism $f : M \to M$, we replace the flows $\Phi_t : U_\ell \to U_\ell$, $\chi^\# : \mathcal{F}_{\ell}^\# \to \mathcal{F}_{\ell}^\#$ and $\varphi_t : \mathcal{L}_{\ell}(M) \to \mathcal{L}_{\ell}(M)$ by homeomorphisms $df : U_\ell \to U_\ell$, $F_{\ell}^\# : \mathcal{F}_{\ell}^\# \to \mathcal{F}_{\ell}^\#$ and $L_{\ell} : \mathcal{L}_{\ell}(M) \to \mathcal{L}_{\ell}(M)$, respectively, where $F_{\ell}^\#$ and $L_{\ell}$ are induced by $df$ in a corresponding process to that for the flow case. Clearly

$$p_\ell \circ F_{\ell}^\# = f \circ p_\ell = p_\ell \circ L_{\ell}, \quad \ell = 1, \ldots, n.$$

For more details of these homeomorphisms one sees [13]. Let $E(M, f)$ denote the set of all $f$-invariant ergodic probabilities. One can similarly define $E(\mathcal{L}_{\ell}(M), L_{\ell})$ and $E(\mathcal{F}_{\ell}^\#, F_{\ell}^\#)$. One can pose the Liao problem for diffeomorphisms $f : M \to M$.

In Liao’s problem the entropy of a flow indicates the entropy defined by the time one homeomorphism, e.g. $h_1(\phi) := h_t(\phi_1), h(\phi) := h(\phi_1)$. For the entropy of a flow defined by the whole flow itself, instead of by the time one homeomorphism, see Thomas [16,17], Sun [11] and Sun and Vargas [15]. We point out that the above problem is basic in Liao theory [5], the fundamental concepts of which, qualitative functions (see Section 4), standard systems of differential equations (see [5]), and obstruction sets (see [5]), are established on or through the induced flows $\chi_{\ell}^\# : \mathcal{F}_{\ell}^\# \times R \to \mathcal{F}_{\ell}^\#$ and $\varphi : \mathcal{L}_{\ell}(M) \times R \to \mathcal{L}_{\ell}(M)$. We answer Liao’s problem for both cases of a $C^1$ vector field and a $C^1$ diffeomorphism by the following.
Theorem 1.1. Let \( \dim M = n \) and let \( 1 \leq \ell \leq n \).

1. If \( v \in E(M, \phi) \) and \( m_\ell \in E(L_\ell(M), \varphi) \) and \( \mu_\ell \in E(F^\#, \chi^\#) \) satisfy \( p_{\ell,\#}(m_\ell) = v = p_{\ell,\#}(\mu_\ell) \), then for measure-theoretic entropy
   \[
   h_{m_\ell}(\varphi) = h_v(\phi) = h_{\mu_\ell}(\chi^\#).
   \]
   For topological entropy
   \[
   h(\varphi) = h(\phi) = h(\chi^\#).
   \]

2. If \( v \in E(M, f) \) and \( m_\ell \in E(L_\ell(M), L_\ell) \) and \( \mu_\ell \in E(F^\#, F^\#) \) satisfy \( p_{\ell,\#}(m_\ell) = v = p_{\ell,\#}(\mu_\ell) \), then for measure-theoretic entropy
   \[
   h_{m_\ell}(L_\ell) = h_v(f) = h_{\mu_\ell}(F^\#).
   \]
   For topological entropy
   \[
   h(L_\ell) = h(f) = h(F^\#).
   \]

A special case of Liao’s problem was solved by Sun in [14], where the cardinality of pre-images (under the canonical bundle projection) of \( v \) almost every point is finite, namely, each quasi-regular point (for definition see Section 4) for \( v \) has finitely many pre-images that are quasi-regular with respect to covering probabilities. The difficulty faced in the general case, Theorem 1.1, is that one quasi-regular point for \( v \) has uncountable many pre-images that are quasi-regular with respect to covering probabilities. We will prove Theorem 1.1 by using an approach different from that in [14]. In Section 2, we will present a probability version of the Bowen entropy inequality (his inequality is originally for topological entropy [2, Theorem 17]) for general semi-conjugate systems, by which Liao’s problem can be reduced to show that the set of pre-images (under the natural bundle projection) of \( v \) almost every point contributes no entropy, even though the set is uncountable. In order to show that the set contributes no entropy, Sacksteder–Shub’s argument in [9] on topological entropy for a differentiable dynamical system and its unit sphere bundle will be adapted. We will solve Liao’s problem completely in Section 3.

The next theorem shows that an ergodic probability with a simple Lyapunov spectrum has finitely many covering probabilities on bundle extensions.

Theorem 1.2. Let \( n = \dim M \) and let \( 1 \leq \ell \leq n \).

1. Let \( v \in E(M, \phi) \) have a simple Lyapunov spectrum, namely, all Lyapunov exponents of \( v \) have multiplicity 1. Let
   \[
   A = \{ \mu \in E(F^\#, \chi^\#) \mid p_{\ell,\#}(\mu) = v \} \text{ and } B = \{ m \in E(L_\ell, \varphi) \mid p_{\ell,\#}(m) = v \}.
   \]
   Then
   \[
   A_n^\ell \leq \text{Card } A \leq 2^n A_n^\ell \quad \text{and} \quad 1 \leq \text{Card } B \leq 2^n A_n^\ell,
   \]
   where \( A_n^\ell = n(n-1) \cdots (n - \ell + 1) \).

2. Let \( v \in E(M, f) \) have a simple Lyapunov spectrum, namely, all Lyapunov exponents of \( v \) have multiplicity 1. Let
   \[
   A = \{ \mu \in E(F^\#, F^\#) \mid p_{\ell,\#}(\mu) = v \} \text{ and } B = \{ m \in E(L_\ell, L_\ell) \mid p_{\ell,\#}(m) = v \}.
   \]
   Then
   \[
   A_n^\ell \leq \text{Card } A \leq 2^n A_n^\ell \quad \text{and} \quad 1 \leq \text{Card } B \leq 2^n A_n^\ell,
   \]
   where \( A_n^\ell = n(n-1) \cdots (n - \ell + 1) \).

In order to estimate the upper bound of covering probabilities for a given ergodic probability \( v \), we will establish a general criterion Lemma 4.8, by which it suffices to estimate the cardinality of quasi-regular points in one fiber over a quasi-regular point for \( v \). The techniques in [14] could be applied to get the cardinality. However, we will present a more natural proof as Appendix A, by which an interesting relation between the usual Lyapunov exponent \( \lim_{t \to \infty} \frac{1}{t} \log \| \Phi_t(uk) \| \) and the limit \( \lim_{t \to \infty} \frac{1}{t} \log \zeta_{\alpha k}(t) \), where \( u_k \) is the \( k \)th vector in \( \alpha \), defined in Liao theory becomes clear. The argument on the lower bound is quite direct: we take the number of the covering probabilities given in Liao’s reordering lemma [6] as the lower bound. We complete the proof of Theorem 1.2 in Section 4.

The notations employed in the present paper take the same form as in a series of papers by Liao collected in the book in [5].
2. Bowen inequality of measure-theoretic entropy

We will recall in this section an inequality of measure-theoretic entropy for maps and proceed to deduce the inequality for flows. All these inequalities for maps and flows could be regarded as a probability version of Bowen inequality of topological entropy [2, Theorem 17] (see also [10, Theorem 1]).

Let \((X, d)\) be a compact metric space and let \(f : X \to X\) be a continuous and surjective map. For \(x \in X\), \(n > 1\) and \(\varepsilon > 0\) put

\[
D(x, n, \varepsilon, f) := \{ y \in X \mid d(f^i x, f^i y) < \varepsilon, \ 0 \leq i \leq n - 1 \},
\]

and call it an \((n, \varepsilon, f)\)-box. Take a probability \(\mu\) from \(E(X, f)\), the set of all \(f\)-invariant ergodic probabilities. For \(0 < \delta < 1\), let \(R(\delta, \varepsilon, n, f)\) denote the smallest number of \((n, \varepsilon, f)\)-boxes needed to cover a set of \(\mu\)-probability bigger than \(1 - \delta\). According to Katok [3], the measure-theoretic entropy \(h_{\mu}(f)\) can be defined by

\[
h_{\mu}(f) := \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log R(\delta, \varepsilon, n, f),
\]

which is independent of the choice of \(\delta\).

Let \(\alpha\) be an open cover of \(X\). Let \(A\) be a subset of \(X\). Set

\[
N_A(\alpha) := \min \left\{ \text{Card}(\beta) \mid \beta \subset \alpha, \bigcup_{B \in \beta} B \supseteq A \right\}.
\]

Define \(h(f, A, \alpha) := \lim_{\alpha \to 0} \frac{1}{n} \log N_A(\bigcup_{i=0}^{n-1} f^{-i} \alpha)\) and \(h(f, A) := \sup_{\alpha} h(f, A, \alpha)\). According to Adler–Konheim–McAndrew [1], the topological entropy \(h(f)\) of the whole system \((X, f)\) coincides with \(h(f, X)\).

**Lemma 2.3.** Let \(f : X \to X\), \(g : Y \to Y\), and \(p : X \to Y\) be continuous surjective maps on compact metric spaces satisfying \(p \circ f = g \circ p\). Take \(\mu \in E(X, f)\) and \(m \in E(Y, g)\), with \(p_*(\mu) = m\). Then for any \(f\)-invariant and \(\mu\) full probability subset \(W \subseteq X\) and any \(g\)-invariant and \(m\) full probability subset \(W' \subseteq Y\) satisfying \(p(W) = W'\), we have

\[
h_{\mu}(f) \leq h_{m}(g) + \sup_{y \in \Lambda} h(f, \rho^{-1}(y)),
\]

where \(\rho = p|W : W \to \Lambda\), and \(\rho^{-1}(\Lambda) \subseteq W\).

**Proof.** This result appears in other places, for example it is a special case (put \(f = 0\)) of Proposition 3.5 in [4].

We now present an alternative proof by using Katok’s entropy definition (2.1). Take \(\varepsilon > 0\) and \(0 < \delta < 1\). Let \(\alpha\) be a cover consisting of open balls \(B(x, \varepsilon)\) of radius \(\varepsilon\). Set

\[
a := \sup \inf_{y \in \Lambda, n} \frac{1}{n} \log N_{\rho^{-1}(y)}(\bigcup_{i=0}^{n-1} f^{-i} \alpha).
\]

If \(a = \infty\), the formula holds automatically. Hence we assume that \(a < \infty\). For each \(y \in \Lambda\) there exists a positive integer \(m_y\) such that

\[
N_{\rho^{-1}(y)}(\bigcup_{i=0}^{m_y-1} f^{-i} \alpha) \leq e^{m_y(a+\varepsilon)}.
\]

We extend \(\rho : W \to \Lambda\) to \(\tilde{\rho} : \tilde{W} \to \tilde{\Lambda}\) (we denote the extended map by \(\rho\) for notational simplicity). Denote

\[
b := \sup \inf_{y \in \tilde{\Lambda}, n} \frac{1}{n} \log N_{\rho^{-1}(y)}(\bigcup_{i=0}^{n-1} f^{-i} \alpha).
\]

Then \(b \geq a\) and for each \(y \in \tilde{\Lambda}\) there exists \(m'_y, m''_y = m_y\) when \(y \in \Lambda\), such that

\[
N_{\rho^{-1}(y)}(\bigcup_{i=0}^{m'_y-1} f^{-i} \alpha) \leq e^{m'_y(b+\varepsilon)}.
\]
For $y \in \tilde{A}$, take $\alpha_y \in \bigcup_{i=0}^{n-1} f^{-i}\alpha$ such that $\text{Card}(\alpha_y) = N_{\rho^{-1}(y)}(\bigcup_{i=0}^{n-1} f^{-i}\alpha)$ and $\bigcup_{\Lambda \in \alpha_y} \Lambda \supset \rho^{-1}(y)$. Set $O_y = \bigcup_{\Lambda \in \alpha_y} \Lambda$. Denote by $C(y)$ the set of all neighborhoods of $y$ in $\tilde{A}$. Then there exists $K(y) \in C(y)$ such that $\rho^{-1}(K(y)) \subset O_y$. Let $U_y = \text{int} \ K(y)$, the interior of $K(y)$, and construct an open cover $\{U_y; y \in \tilde{A}\}$. Choose a Lebesgue number $\varepsilon_1$ for this cover with $0 < \varepsilon_1 < \varepsilon$. Choose $(n, \varepsilon_1, g)$-boxes in the compact subset $\tilde{A}$

$$D(y_1, n, \varepsilon_1, g), \ldots, D(y_R, n, \varepsilon_1, g),$$

where $R = R(\delta, \varepsilon_1, n, g)$, so that their union covers a subset of $Y$ of $m$-probability bigger than $1 - \delta$. For a fixed $i$,

$$D(y_i, n, \varepsilon_1, g) = B(y_i, \varepsilon_1) \cap g^{-1}B(g(y_i), \varepsilon_1) \cap \cdots \cap g^{-(n-1)}B(g^{n-1}(y_i), \varepsilon_1).$$

(2.2)

Since $\Lambda$ is dense in $\tilde{A}$, without loss of generality we suppose $y_i \in \Lambda, i = 1, 2, \ldots, R$. Write $m_k := m_{g^k(y_i)}$, $k = 0, 1, 1, \ldots, n - 1$. From the choice of $\varepsilon_1$ it is easy to see that

$$N_{\rho^{-1}(B(g^k(y_i), \varepsilon_1))} \left( \bigvee_{i=0}^{m_k-1} f^{-i}\alpha \right) \leq e^{m_k(a+\varepsilon)}, \quad k = 1, \ldots, n - 1.$$

Rewrite (2.2) as

$$A = A(0) \cap g^{-1}A(1) \cap \cdots \cap g^{-(n-1)}A(n-1),$$

where

$$A = D(y_i, n, \varepsilon_1, g), \quad A(j) = B(g^j(y_i), \varepsilon_1).$$

for $j = 0, \ldots, n - 1$. By using the collection $\{A(j)\}_{j=0}^{n-1}$ we define recursively a sequence $\{i_s\}$ as follows:

$$i_0 = 0,$$

$$i_{s+1} = i_s + m_k, \quad \text{where } A(i_s) = B(g^k(y_i), \varepsilon_1).$$

Let $q$ denote the least integer such that $i_{q+1} \geq n$ and put $n_s = m_k$, if $A(i_s) = B(g^k(y_i), \varepsilon_1)$ for $0 \leq s \leq q$. Then $i_{q+1} = n_0 + n_1 + \cdots + n_q$. Since $i_{q+1} \geq n$,

$$N_{\rho^{-1}(A)} \left( \bigvee_{i=0}^{n-1} f^{-i}\alpha \right) \leq N_{\rho^{-1}(A)} \left( \bigvee_{i=0}^{i_{q+1}-1} f^{-i}\alpha \right).$$

Now we have

$$N_{\rho^{-1}(A)} \left( \bigvee_{i=0}^{n-1} f^{-i}\alpha \right) \leq N_{\rho^{-1}(A(0))} \left( \bigvee_{i=0}^{n_0-1} f^{-i}\alpha \right) N_{\rho^{-1}(g^{-n_0}A(n_0))} \left( f^{-n_0} \bigvee_{i=0}^{n_1-1} f^{-i}\alpha \right) \cdots$$

$$\times N_{\rho^{-1}(g^{-n_0-\cdots-n_q-1}A(n_0+\cdots+n_q-1))} \left( f^{-n_0-\cdots-n_q-1} \bigvee_{i=0}^{n_q-1} f^{-i}\alpha \right)$$

$$= N_{\rho^{-1}(A(0))} \left( \bigvee_{i=0}^{n_0-1} f^{-i}\alpha \right) N_{f^{-n_0}\rho^{-1}A(n_0)} \left( f^{-n_0} \bigvee_{i=0}^{n_1-1} f^{-i}\alpha \right) \cdots$$

$$\times N_{f^{-(n_0+\cdots+n_q-1)}\rho^{-1}A(n_0+\cdots+n_q-1)} \left( f^{-(n_0+\cdots+n_q-1)} \bigvee_{i=0}^{n_q-1} f^{-i}\alpha \right)$$

$$= N_{\rho^{-1}(A(0))} \left( \bigvee_{i=0}^{n_0-1} f^{-i}\alpha \right) N_{\rho^{-1}(A(n_0))} \left( \bigvee_{i=0}^{n_1-1} f^{-i}\alpha \right) \cdots$$

$$\times N_{\rho^{-1}(A(n_0+\cdots+n_q-1))} \left( \bigvee_{i=0}^{n_q-1} f^{-i}\alpha \right)$$

$$\leq e^{(a+\varepsilon)(n_0+\cdots+n_q)}$$

$$\leq e^{(a+\varepsilon)(n+H)}.$$
where $H = \max\{n_0, \ldots, n_q\}$. Observe that
\[
\mu \left( \cup_{i=1}^R D(y_i, n, \epsilon_1, g) \right) = m \left( \cup_{i=1}^R D(y_i, n, \epsilon_1, g) \right) > 1 - \delta.
\]
Thus
\[
R(\delta, n, \epsilon, f) \leq R(\delta, n, \epsilon_1, g) e^{(n+H)(a+\epsilon)}.
\]
Therefore,
\[
\lim_{n \to \infty} \frac{1}{n} \log R(\delta, n, \epsilon, f) \leq \lim_{n \to \infty} \frac{1}{n} \log R(\delta, n, \epsilon_1, g) + a + \epsilon.
\]
So by (2.1), $h_\mu(f) \leq h_m(g) + a$.

Now we present a parallel Bowen entropy inequality for flows. Observe that an ergodic probability for $\phi$ is not necessarily ergodic for $\phi_1$, the time one homeomorphism for $\phi$, one cannot deduce the inequality for flows automatically from Lemma 2.3.

**Lemma 2.4.** Let $\tilde{\psi} : X \times \mathbb{R} \to X$, $\psi : Y \times \mathbb{R} \to Y$ be two continuous flows on compact metric spaces and let $p : X \to Y$ be a continuous surjective map with $p \circ \tilde{\psi}_t = \psi_t \circ p$, for $t \in \mathbb{R}$. If $\mu \in E(X, \tilde{\psi})$ covers $m \in E(Y, \psi)$, namely, $p^*\mu = m$, then for any $\tilde{\psi}_1$ invariant and $\mu$ full probability subset $W \subset X$ and any $\psi_1$ invariant and $m$ full probability subset $\Lambda \subset Y$ with $p(W) = \Lambda$, we obtain
\[
h_\mu(\tilde{\psi}_1) \leq h_m(\psi_1) + \sup_{y \in \Lambda} h(\tilde{\psi}_1, \eta_1^{-1}(y)),
\]
where $\eta = p|_W : W \to \Lambda, \eta^{-1}(\Lambda) \subset W$.

**Proof.** Denote by $I(X, \tilde{\psi}_1)$ and $I(Y, \psi_1)$ the set of $\tilde{\psi}_1$-invariant probabilities and the set of $\psi_1$-invariant probabilities, respectively. Then $\mu \in I(X, \tilde{\psi}_1)$ and $m \in I(Y, \psi_1)$. By ergodic decomposition theorem, there exist uniquely probabilities $\tau$ on $I(X, \tilde{\psi}_1)$ and $\sigma$ on $I(Y, \psi_1)$ so that $\tau(E(X, \tilde{\psi}_1)) = 1$ and $\sigma(E(Y, \psi_1)) = 1$ and $\mu = \int_{E(X, \tilde{\psi}_1)} \mu' d\tau(\mu')$ and $m = \int_{E(Y, \psi_1)} m' d\sigma(m')$.

The map $p_* : I(X, \tilde{\psi}_1) \to I(Y, \psi_1)$, $\mu_0 \to \mu_0 \circ p^{-1}$ induced by $p$ is continuous. Thus $(p_*)_* (\tau)$, denoted by $p_{**}(\tau)$ for notational simplicity, is a Borel probability on $I(Y, \psi_1)$. We assert that
\[
\sigma = p_{**}(\tau).
\]

Indeed,
\[
(p_{**}(\tau))(E(Y, \psi_1)) = \tau \circ p_{**}^{-1} E(Y, \psi_1) \\
\geq \tau\{E(X, \tilde{\psi}_1)\} = 1.
\]

For a given $g \in C^0(Y)$, we have
\[
\int_{E(Y, \psi_1)} \left( \int_Y g(y) d\mu' \right) d\mu = \int_{E(Y, \psi_1)} \left( \int_Y g(y) d\mu' \right) \circ p_* d\tau \\
= \int_{E(X, \tilde{\psi}_1)} \left( \int_X g \circ p(x) d\mu' \right) d\tau \\
= \int_X g \circ p d\mu = \int_Y g d\mu.
\]
This implies $m = \int_{E(Y, \psi_1)} m' d\mu_{**}(\tau)$ and thus by the ergodic decomposition theorem $p_{**}(\tau) = \sigma$. 


From the Jacob theorem (see [18, Theorem 8.4])

\[ h_\mu(\tilde{\psi}_1) = \int_{E(X,\tilde{\psi}_1)} h_{\mu'}(\tilde{\psi}_1) d\tau(\mu'), \]

\[ h_\sigma(\psi_1) = \int_{E(Y,\psi_1)} h_{\sigma'}(\psi_1) d\sigma(m'). \]

The above assertion together with Lemma 2.3 gives rise to

\[ h_\mu(\tilde{\psi}_1) \leq h_\sigma(\psi_1) + \sup_{y \in A} h(\tilde{\psi}_1, \eta^{-1}(y)). \quad \square \]

Due to the same observation as for Lemma 2.4, the following variational principle for flows is not an automatic corollary of the variation principle for the discrete case.

**Lemma 2.5.** Let \( \psi : X \times R \to X \) be a continuous flow on a compact metric space. Then

\[ h(\psi_1) = \sup\{ h_\mu(\psi_1) \mid \mu \in E(X, \psi) \}. \]

**Proof.** This is Theorem A in [14]. \( \square \)

### 3. Proof of Theorem 1.1

Based on Bowen inequality of topological entropy [2, Theorem 17] Sacksteder and Shub established in [9] an equality of topological entropy between a diffeomorphism and its sphere bundle. By using Lemma 2.3, we obtain a probability version of Sacksteder–Shub equality and then prove Theorem 1.1 in this section.

Let \( X_0, X_1, \ldots, \) be a sequence of metric spaces with metrics \( d_0, d_1, \ldots, \) and let \( \tau = \{\tau_i : i = 1, 2, \ldots, \} \) be a sequence of continuous maps \( \tau_i : X_{i-1} \to X_i. \) Let \( \Sigma = \{\sigma_i : i = 0, 1, 2, \ldots, \} \) be a sequence of continuous maps \( \sigma_i : X_0 \to X_i \) defined by \( \sigma_i = \prod_{j=1}^{i} \tau_j \) for \( i \geq 1. \) We call \( \Sigma \) a compositional representation of \( \tau. \) If \( \delta > 0 \) and \( K \subset X_0 \) is compact, a subset \( W \subset K \) is said to \( (n, \delta) \)-span \( K \) if for every \( y \in K \) there is an \( x \) in \( W \) such that for \( 0 \leq j \leq n, \) \( d_j(\sigma_j(x), \sigma_j(y)) < \delta. \) Such a set \( W \) is said to be \( (n, \delta) \) separated in \( K \) if for all \( x, y \in W \) with \( x \neq y, d_j(\sigma_j(x), \sigma_j(y)) > \delta \) for some \( j, 0 \leq j \leq n. \) Let \( r_n(\delta, K) \) denote the minimal cardinality of any \( (n, \delta) \) separated set for \( K \) and \( s_n(\delta, K) \) the maximal cardinality of any \( (n, \delta) \) separated set of \( K. \) As in Bowen [2] it follows for any compact \( K, \)

\[ \lim_{\delta \to 0} \limsup_{n \to \infty} \frac{1}{n} \log r_n(\epsilon, K) = \lim_{\delta \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s_n(\epsilon, K). \]

We denote this equality by \( h(\Sigma, K) \) and define the entropy of \( \Sigma \) by

\[ h(\Sigma) := \sup_{K \text{ compact}} h(\Sigma, K). \]

Let \( E \) be a vector bundle (over a compact metric space \( X \)) with a Finsler structure, that is, with a norm \( \| \cdot \| \) on the fibers, and let \( S(E) \) be the corresponding unit sphere sub-bundle of \( E. \) If \( A : E \to E \) is a vector bundle map, one can define a map \( S(A) : S(E) \to S(E) \) by

\[ S(A)(v) = A(v) \setminus \| A(v) \|. \quad (3.1) \]

The following lemma is from [9].

**Lemma 3.6.** Let \( A_i : R^n \to R^n \) \( (i = 1, 2, \ldots; n \geq 2) \) be linear isomorphisms such that for some \( \lambda > 0, \| A_i \| \leq \lambda \) and \( \| A_i^{-1} \| \leq \lambda. \) If \( S(A_i) : S^n \to S^n \) is defined as in (3.1) for \( i = 1, 2, \ldots, \) and \( \Sigma = \{\sigma_i = \prod_{j=1}^{i} S(A_j) : i = 1, 2, \ldots\} \) is a compositional representation of \( \{S(A_i) : i = 1, 2, \ldots\}, \) then \( h(\Sigma) = 0. \)
Set
\[ P(E) := \{ [u] | v \in [u] \text{ iff } v = au, a \neq 0 \}. \]

Let \( A: E \to E \) be a vector automorphism over a given diffeomorphism \( f: X \to X \). \( A \) induces a bundle map \( P(A) : P(E) \to P(E), [u] \to [Au] \). The following theorem could be regarded as a probability version of Sacksteder–Shub equality [9].

**Theorem 3.7.** Let \( E \) be a vector bundle over the compact metric space \( X \) with a Finsler structure. Suppose that \( A: E \to E \) is a vector bundle endomorphism of \( E \) over a homeomorphism \( f: X \to X \). Let \( m \in E(X, f) \) and \( \mu \in E(P(E), P(A)) \) be probabilities with \( \pi_s(\mu) = m \), where \( \pi : P(E) \to X \) denotes the canonical bundle projection. Then \( h_\mu(P(A)) = h_m(f) \).

**Proof.** By \( \pi \) we denote both projections \( S(E) \to X \) and \( P(E) \to X \) without confusion. Define a map \( q : S(E) \to P(E) \) by \( u \to [u] \), then \( q \) is a two-to-one map with \( q \circ S(A) = P(A) \circ q \). Take \( \tilde{\mu} \in E(S(E), S(A)) \) with \( q_\ast \tilde{\mu} = \mu \). Then \( h_{\tilde{\mu}}(S(A)) = h_\mu(P(A)) \) by Lemma 2.3. Note \( \pi_s \circ q_\ast \tilde{\mu} = m \). To show the equality \( h_{\tilde{\mu}}(S(A)) = h_m(f) \), it suffices to show \( h_m(f) \geq h_{\tilde{\mu}}(S(A)) \).

Let \( C^0(X) \) be the set of continuous functions on \( X \) and define

\[ \Lambda := \left\{ x \in X \mid \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \xi(f^i x) = \int \xi \, dm, \, \xi \in C^0(X) \right\}. \]

By the Birkhoff ergodic theorem, \( f(A) = \Lambda \) and \( m(A) = 1 \). By Lemma 2.3,

\[ h_{\tilde{\mu}}(S(A)) \leq h_m(f) + \sup_{x \in A} h(S(A), \pi^{-1}(x)). \]

Fix \( x \in \Lambda \) and set

\[ X_0 = \pi^{-1}(x), \quad X_1 = S(A)X_0 = \pi^{-1}(fx), \quad X_2 = S(A)^2 X_0 = \pi^{-1}(f^2x), \ldots. \]

Clearly \( \pi^{-1}(f^j x) \) is homeomorphic to \( S^{n-1} \), where \( n \) indicates the fiber dimension of \( E \). Set \( \Sigma := \{ S(A)^j : X_0 \to X_i, \, i = 1, 2, \ldots \} \). Since \( A : E \to E \) is a vector bundle endomorphism and nonsingular, then \( \| S(A) \| \leq \lambda, \| S(A)^{-1} \| \leq \lambda \) for a constant \( \lambda \). Thus \( h(\Sigma) = 0 \) by Lemma 3.6. It is then simple to show by definition that

\[ h(S(A), \pi^{-1}(x)) = h(\Sigma), \quad \forall x \in \Lambda. \]

Thus \( h_{\tilde{\mu}}(S(A)) \leq h_m(f) \). \( \Box \)

**Proof of Theorem 1.1.** We prove (2) and leave (1) to readers.

Let \( TM \) be the tangent bundle and let \( E_\ell = \bigwedge^\ell(TM) \), where \( \bigwedge^\ell \) denotes \( \ell \)-th-exterior power. Let

\[ A_\ell = \bigwedge^\ell (Df) : E_\ell \to E_\ell, \]

where \( e_1, \ldots, e_n \) from a basis in \( T_xM \). Let \( \{ e_1, e_2, \ldots, e_\ell \} \) and \( \{ w_1, w_2, \ldots, w_\ell \} \) be two bases in \( T_xM \). They span the same \( \ell \)-dimensional subspace iff

\[ w_1 \wedge w_2 \wedge \cdots \wedge w_\ell = ae_1 \wedge e_2 \wedge \cdots \wedge e_\ell, \quad a \in \mathbb{R}. \]

This gives an equivalence relation \( \sim \). Set

\[ D_\ell(x) := \{ e_1 \wedge e_2 \wedge \cdots \wedge e_\ell | \, e_1, e_2, \ldots, e_\ell \text{ are linearly independent in } T_xM \}. \]

Set \( G_\ell(x) := D_\ell(x) \setminus \sim \) and \( G_\ell := \bigcup_{x \in M} G_\ell(x) \). Then \( G_\ell \subset P(E_\ell) \) is a sub-bundle whose fiber over \( x \) is the projectivization of the set of decomposable \( \ell \)-vectors in \( E_\ell(x) \). \( G_\ell \) is invariant under the map \( P(A_\ell) : P(E_\ell) \to P(E_\ell), \ [u] \to [A_\ell u], \ u \in E_\ell \). Note that \( L_\ell \) is isomorphic to \( G_\ell \). Moreover, the isomorphism conjugates \( L_\ell : L_\ell \to L_\ell \).
to the restriction $P(A_\ell)|G_\ell : G_\ell \to G_\ell$ and sends $m_\ell \in E(L_\ell, L_\ell)$ to some $\tilde{m}_\ell \in E(G_\ell, P(A_\ell))$. We have, by Theorem 3.7, $\tilde{m}_\ell(L_\ell) = h_\nu(f)$ and thus by the variational principle (see Lemma 2.5 in the flow case) $h(L_\ell) = h(f)$.

Now we prove that the equality $h_{\mu_\ell}(F_\ell^\#) = h_\nu(f)$. Consider the bundle $L_{1, \ell}$ whose fiber over $x$ is $L_{1}(x) \times L_{2}(x) \times \cdots \times L_{u}(x)$ and let $L_{1, \ell}$ be the automorphism of $L_{1, \ell}$ given by

$$(x, V_1, \ldots, V_\ell) \to (f(x), Df(x)(V_1), \ldots, Df(x)(V_\ell)).$$

Denote by $p_{1, \ell} : L_{1, \ell} \to M$ the canonical bundle projection. Note by Lemma 3.6 that

$$h(L_i, L_i(x)) = h(P(A_i), G_i(x)) = 0, \quad i = 1, \ldots, \ell, \quad \forall x,$$

it is simple to show that $h(L_{1, \ell}, L_{1, \ell}(x)) = 0, \quad \forall x$. By Lemma 2.3 we have $h_\nu(f) = h_{\mu_{1, \ell}}(L_{1, \ell})$ for each $\mu_{1, \ell} \in E(L_{1, \ell}, L_{1, \ell})$ covering $\nu$, namely, $p_{(1, \ell)} \mu_{1, \ell} = \nu$. Next, let $H_{\ell}$ be the sub-bundle of $L_{1, \ell}$ formed by elements $(x, V_1, \ldots, V_\ell)$ such that $V_1 \subset V_2 \subset \cdots \subset V_\ell$. Then $L_{1, \ell}(H_{\ell}) = H_{\ell}$. For a given $\alpha = (u_1, u_2, \ldots, u_\ell) \in \mathcal{F}_\ell^\#$ define

$$V_1 = [u_1], \quad V_2 = [u_1, u_2], \quad \ldots, \quad V_\ell = [u_1, u_2, \ldots, u_\ell],$$

where $[u_1, \ldots, u_i]$ denotes the linear subspace in $T_x M$ generated by $u_1, \ldots, u_i$. Set

$$p : \mathcal{F}_\ell^\# \to H_{\ell}, \quad p(\alpha) := (x, V_1, V_2, \ldots, V_\ell).$$

Then $p$ is a finite-to-one surjective map satisfying $p \circ F_\ell^\# = L_{1, \ell} | H_{\ell} \circ p$. It follows by Lemma 2.3 that $h_{\mu_{1, \ell}}(F_\ell^\#) = h_{\mu_{1, \ell}}(L_{1, \ell}) = h_\nu(f)$. By the variational principle (see Lemma 2.5 in the flow case) $h(F_\ell^\#) = h(f)$. This completes Theorem 1.1(2).

4. Proof of Theorem 1.2

Let $\phi : X \times R \to X$ be a continuous flow on a compact metric space and let $\nu \in E(X, \phi)$. Define

$$Q_\nu(X, \phi) := \left\{ x \in M \mid \lim_{t \to \pm \infty} \frac{1}{t} \int_0^t \xi(\phi_s(x)) \, ds = \int \xi \, d\nu, \quad \forall \xi \in C^0(X) \right\}.$$ 

A point in $Q_\nu(X, \phi)$ is called a quasi-regular point for $\nu$. By the Birkhoff ergodic theorem $\nu(Q_\nu(X, \phi)) = 1$ and $\phi_t(Q_\nu(X, \phi)) = Q_\nu(X, \phi), t \in \mathbb{R}$.

**Lemma 4.8.** Let $\tilde{\nu} : X \times R \to X, \psi : Y \times R \to Y$ be two continuous flows on compact metric spaces and let $p : X \to Y$ be a continuous surjective map with $p \circ \tilde{\nu} = \psi \circ p, t \in \mathbb{R}$. Let $\nu \in E(Y, \psi)$ and let $A = \{ \mu \in E(X, \tilde{\nu}) \mid p_* \mu = \nu \}$ and let $Q = \bigcup_{\mu \in A} Q_\mu(X, \phi)$. Let $\rho = p|Q : Q \to Q_\nu(Y, \psi), \quad \rho^{-1} Q_\nu(Y, \psi) \subset Q$. Suppose that $\text{Card}(\rho^{-1}(a)) = N$ for some $a \in Q_\nu(Y, \psi)$, some positive integer $N$, then $\text{Card} A \leq N$.

**Proof.** It is known (see [6, Lemma 2.1]) that $A \neq \emptyset$. Now we show that $\text{Card} A \leq N$.

Suppose on the contrary that $\text{Card} A > N$. Take $\mu_n \in A, n = 1, 2, \ldots, N + 1$ and suppose that $\mu_i \neq \mu_j, i \neq j$. Let $A = Q_\nu(Y, \psi) \cap \bigcap_{n=1}^{N+1} p Q_{\mu_n}(X, \phi)$. Then $\nu(A) = 1$ and $\psi_t(A) = A, t \in \mathbb{R}$. Fix $a \in A$. Since $\text{Card} \rho^{-1}(a) = N$ then there exists $b \in \rho^{-1}(a)$ so that $b \in Q_{\mu_i}(X, \tilde{\nu}) \cap Q_{\mu_j}(X, \tilde{\nu})$ for some $i \neq j$. Thus

$$\int_0^t \xi \, d\mu_i = \lim_{t \to \pm \infty} \frac{1}{t} \int_0^t \xi(\tilde{\nu}_s(b)) \, ds = \int \xi \, d\mu_j, \quad \forall \xi \in C^0(X).$$

This implies that $\mu_i = \mu_j$, a contradiction to the choice of $\mu_n$. Therefore, $\text{Card} A \leq N$.

In light of Lemma 4.8, the bounds of the number of covering probabilities rely on the cardinality of the quasi-regular points that are pre-images of a quasi-regular point of $\nu$ under the natural bundle projection. Liao qualitative functions (see below for definition) will turn out to be crucial while dealing with these quasi-regular points. These functions are a generalization of the function $\omega : TM \to R, \omega(u) := \frac{d}{dt} \|\Phi_t(u)\|_{t=0}$. 

Now we recall Liao qualitative functions, which were first introduced in [5].

For \( \alpha = (u_1, u_2, \ldots, u_\ell) \in \mathcal{F}_\ell \) denote \( \chi_t(\alpha) = (u_1(t), u_2(t), \ldots, u_\ell(t)) \) and define \( \zeta_{\alpha k}(t) := \|u_k(t)\| \). It is clear that \( \zeta_{\alpha 1}(t) = \|d\phi_t(u_1)\| \). The limit
\[
\lim_{t \to +\infty} \frac{1}{t} \log \zeta_{\alpha k}(t).
\]

whenever it exists, coincides with the Lyapunov exponent
\[
\lim_{t \to +\infty} \frac{1}{t} \log \|\Phi_t(u_k)\|,
\]
for \( k = 1 \) and is not necessarily equal to the Lyapunov exponent for \( k = 2, \ldots, \ell \). Liao qualitative functions \( \omega_k : \mathcal{F}_\ell \to \mathbb{R} \) are defined by
\[
\omega_k(\alpha) := \frac{d\zeta_{\alpha k}(t)}{dt} \bigg|_{t=0}, \quad k = 1, \ldots, \ell.
\]

All these functions are continuous, see [5]. For \( \alpha \in \mathcal{F}_\ell^# \), it holds clearly that \( \omega_k(\chi_t(\alpha)) = \frac{d\zeta_{\alpha k}(t)}{dt} \) and \( \omega_k(\chi^#_t(\alpha)) = \frac{1}{\zeta_{\alpha k}(t)} \frac{d\zeta_{\alpha k}(t)}{dt} \) and thus
\[
\log \|d\phi_t(u_1)\| = \int_0^t \omega_k(\chi^#_s(\alpha)) \, ds.
\]

(4.1)

We point out that (4.1) is a quite natural formula while \( k = 1 \) by the following argument:
\[
\zeta_{\alpha 1}(t) = \|d\phi_t(u_1)\|,
\]
\[
\omega_1(\alpha) := \frac{d\|d\phi_s(u_1)\|}{ds} \bigg|_{s=0}, \quad \omega_1(\chi^#_t(\alpha)) = \frac{1}{\zeta_{\alpha 1}(t)} \frac{d\zeta_{\alpha 1}(t)}{dt},
\]
\[
\log \|d\phi_t(u_1)\| = \int_0^t \frac{d\|d\phi_s(u_1)\|}{ds} \, ds.
\]

where \( u_1 \) is the first vector in \( \alpha = (u_1, \ldots, u_\ell) \), \( \|u_1\| = 1 \).

**Proof of Theorem 1.2.** We prove (1) with \( \ell = n \) and leave other cases to readers.

**Step 1. The upper bounds.** Let \( O_\nu(M, \phi) \) denote the Oseledec basin for \( \nu \), it consists of all points \( x \in M \) for which there is a splitting \( T_x M = E^1_x \oplus \cdots \oplus E^n_x \) with the invariant property \( \Phi_t(E^i_x) = E^i_{\phi_t(x)} \) and satisfying
\[
\lim_{t \to \pm \infty} \frac{1}{t} \log \|\Phi_t|_{E^i_x}\| = \lambda_i, \quad i = 1, \ldots, n,
\]
where \( \lambda_1 < \cdots < \lambda_n \) are Lyapunov exponents for \( \nu \). From the Oseledec theorem [8], the angle \( \angle(E^i_x, E^j_x) \) of two sub-bundles \( E^i_x, E^j_x \) is measurable and positive with respect to \( \nu \)-a.a. \( x \in M \), hence by Birkhoff ergodic theorem
\[
\lim_{t \to \pm \infty} \frac{1}{t} \int_0^t \angle(E^i_{\phi_s x}, E^j_{\phi_s x}) \, ds = \int_0^\infty \angle(E^i_x, E^j_x) \, dv(y) > 0
\]
\( \mu \)-a.a. \( x \in M \). Denote
\[
G_{ij}(\nu) = \{ x \in M \mid (4.2) \text{ holds} \}, \quad i, j = 1, 2, \ldots, n.
\]

Let
\[
H(\nu) = Q_\nu(M, \phi) \cap O_\nu(M, \phi) \bigcap_{i,j=1, \ldots, n; i \neq j} G_{ij}(\nu).
\]

Then \( \phi_t H(\nu) = H(\nu) \) and \( \nu(H(\nu)) = 1 \). Denote by \( \rho \) the restriction map \( p_n \big|_{\bigcup_{\mu \in \mathcal{A}} Q_\mu(X^\phi \times X^\phi)} \).

**Claim.** Card \( \rho^{-1}(x) = 2^n n! \) for any given \( x \in H(\nu) \).

For the proof of claim, please see Appendix A.

From the claim and Lemma 4.8, Card \( \mathcal{A} \leq 2^n n! \) when \( \ell = n \).
One shows similarly that
\[ \operatorname{Card} \mathcal{A} \leq 2^\ell A^\ell_n \]  
(4.3)
when \( 1 \leq \ell < n \).

When \( \ell = n \), there is exactly one \( m \in E(L_n, \varphi) \) to cover \( v \). Now let \( 1 \leq \ell < n \) and consider a map \( \operatorname{id}_\ell : F^\#_\ell \to L_\ell(M), (u_1, \ldots, u_\ell) \to [u_1, \ldots, u_\ell] \). We have then
\[ \operatorname{id}_\ell \circ E(F^\#_\ell, \chi^\#) = E(L_\ell, \varphi), \quad p_\ell \circ E(L_\ell, \varphi) = E(M, \phi) = p_\ell \circ E(F^\#_\ell, \chi^\#). \]

So by (4.3) the number of \( \mu \in E(L_\ell(M), \varphi) \) covering \( v \) is less than or equal to \( 2^n A_n^\ell \).

**Step 2. The lower bounds.** To show the lower bounds we quote the following Liao’s reordering lemma.

**Lemma 4.9.** ([6, Theorem 4.1], [12, Theorem 2.1]) Let \( 1 \leq \ell \leq n \) and let \( \mu \in E(F^\#_\ell, \chi^\#) \) cover \( v \in E(M, \phi) \), \( p_\ell \circ \mu = v \). For a permutation
\[ \gamma : \{1, 2, \ldots, \ell\} \to \{\gamma(1), \gamma(2), \ldots, \gamma(\ell)\}, \]
there exists \( \mu_\gamma \in E(F^\#_\ell, \chi^\#) \) such that \( p_\ell \circ \mu_\gamma = p_\ell \circ (\mu_\gamma) \) and
\[ \int \omega_{\gamma(k)} d\mu_\gamma = \int \omega_k d\mu, \quad k = 1, 2, \ldots, \ell. \]
Moreover, for each permutation \( \gamma : \{1, 2, \ldots, n\} \to \{\gamma(1), \gamma(2), \ldots, \gamma(n)\}, \)
\[ \left\{ \int \omega_{\gamma(k)} d\mu_\gamma, k = 1, \ldots, n \right\} = \{\lambda_1, \ldots, \lambda_n\}, \]
where \( \lambda_1 < \cdots < \lambda_n \) are all Lyapunov exponents of \( v \), and \( v \) is given in the assumption of Theorem 1.2.

From Lemma 4.9, covering probabilities corresponding to different permutations are different. Note there are \( n! \) many permutations, so \( \operatorname{Card} \mathcal{A} \geq n! \) while \( \ell = n \). If \( 1 \leq \ell < n \), one shows similarly that \( \operatorname{Card} \mathcal{A} \geq A_n^\ell \). This completes Theorem 1.2. \( \square \)

**Remark 4.1.** Let us make an explanation of Theorem 1.2 in the case when \( \dim M = 2 \).

Let \( f : M \to M \) be a \( C^1 \) diffeomorphism preserving an ergodic hyperbolic probability \( v \) with Lyapunov exponents \( \lambda_1 < 0 < \lambda_2 \). Define \( O_\nu(M, f) \) to be all points \( x \in M \) for which there is a splitting \( T_x M = E^1_x \oplus E^2_x \) with invariant property \( d f | E^i_x = E^i f(x) \) and satisfying
\[ \lim_{n \to +\infty} \frac{1}{n} \log \| d f | E^i_x \| = \lambda_i, \quad i = 1, 2. \]
Let
\[ H(v) = O_v(M, f) \cap Q\nu(M, f). \]
Then \( f(H(v)) = H(v) \) and \( v(H(v)) = 1 \). Since \( \dim E^1_x = 1 \), there are exactly two unit vectors \( u^1, u^2 \) in each \( E^1_x, \quad i = 1, 2 \).

Recall that \( \mathcal{A} = \{ \mu \in E(F^\#_2, F^\#_2) \mid p_\ell \circ \mu = v \} \) and \( Q = \bigcup_{\mu \in \mathcal{A}} Q\mu(F^\#_2, F^\#_2) \). Denote by \( \rho \) the restriction map \( p_2|_{\bigcup_{\mu \in \mathcal{A}} Q\mu(F^\#_2, F^\#_2)} \).

Fix \( x \in H(v) \) and let \( \alpha = (u_1, u_2) \) denote arbitrarily a frame in \( \rho^{-1}(x) \). From the proof in Appendix A, \( u_1 \) belongs to \( E^1_x \) or \( E^2_x \).

**Case 1:** \( u_1 \in E^1_x \). In this case it follows that
\[ \lambda_2 = \lim_{t \to +\infty} \frac{1}{t} \log \| \Phi_t(u_2) \| > \lim_{t \to +\infty} \frac{1}{t} \log \| \Phi_t(u_2) \| = \lambda_1. \]
**Case 2:** \( u_1 \in E^2_x \). In this case it follows that
\[ \lambda_2 = \lim_{t \to +\infty} \frac{1}{t} \log \| \Phi_t(u_2) \| > \lim_{t \to -\infty} \frac{1}{t} \log \| \Phi_t(u_2) \| = \lambda_1. \]
The set \( \rho^{-1}(x) \) consists of exactly \( 2^2 \times 2! = 8 \) elements, they are all orthonormal frames expressed by

\[
(u^i, v) \in \mathcal{F}_2^\#(x),
\]
while \( v \) has two choices for each \( u^i, 1 \leq i, j \leq 2 \). For each \( \alpha = (u^i, v) \), denote by \( \mu \) its individual ergodic probability, namely the probability satisfies

\[
\lim_{n \to \pm \infty} \frac{1}{n} \sum_{i=1}^{n-1} \xi ((F_2^\#)(\alpha)) = \int \xi \, d\mu, \quad \forall \xi \in C^0(\mathcal{F}_2^\#).
\]

Then \( \mu \in \mathcal{A} \), and from Lemma 4.8 and its proof, \( \mathcal{A} \) consists only of probabilities that are individual of the frames \( (u^i, v) \). So the cardinality of \( \mu \in \mathcal{A} \) is no greater than \( 8 = 2^2 \times 2! \).

Observe that the two individual probabilities \( \mu \) of \((u^{11}, v)\) and \(\tilde{\mu}\) of \((u^{22}, v')\) are different. To show this, let us consider a continuous function \( \omega_1 : \mathcal{F}_2^\# \to \mathbb{R} \), \( \alpha = (u_1, u_2) \to \omega_1(\alpha) = \log \|df(u_1)\| \). Now that the following two limits are different:

\[
\int \omega_1 \, d\mu = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \omega_1(F_2^\#(u^{11}, v)) = \lim_{n \to \infty} \frac{1}{n} \log \|df^n(u^{11})\| = \lambda_1,
\]

\[
\int \omega_1 \, d\tilde{\mu} = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \omega_1(F_2^\#(u^{22}, v')) = \lim_{n \to \infty} \frac{1}{n} \log \|df^n(u^{22})\| = \lambda_2,
\]

so \( \mu \neq \tilde{\mu} \). Thus the cardinality of \( \mathcal{A} \) is greater than or equal to 2. Finally,

\[ A_1^2 = 2 \leq \text{Card} \mathcal{A} \leq 8 = 2^2 \times 2! \.

Appendix A

**Proof of claim in the proof of Theorem 1.2.** One can prove the claim in a similar way as in Claim 1 in the proof of [14, Theorem B]. We give here a different and more natural proof, from which the interesting relation (see (A.7), (A.10) and (A.11)) between the usual Lyapunov exponent \( \lim_{t \to \pm \infty} \frac{1}{t} \log \|\Phi_t(u_k)\| \) and the number \( \lim_{t \to \pm \infty} \frac{1}{t} \log \zeta_{\alpha k}(t) \) defined in Liao theory, where \( u_k \) is the \( k \)th vector of \( \alpha \), becomes clear.

Take an \( n \)-frame \( \alpha = (u_1, \ldots, u_n) \) to represent a common point in \( \rho^{-1}(x) \), now we figure out how many choices each \( u_i \) has. Since \( \alpha \) is in the set \( \bigcup_{\mu \in A} Q_\mu(\mathcal{F}^\#, \chi^\#) \) and \( \omega_\alpha \) is continuous, it holds from (4.1)

\[
\lim_{t \to \pm \infty} \frac{1}{t} \log \zeta_{\alpha k}(t) = \lim_{t \to +\infty} \frac{1}{t} \log \zeta_{\alpha k}(t) = \lim_{t \to +\infty} \frac{1}{t} \int s(\chi_k(\alpha)) \, ds = \lambda_{i_k}.
\]

The first vector \( u_1 \) in \( \alpha \) must be taken from some subspace \( E_{i_1}^{x} \). Indeed, by definition, \( \|\Phi_t(u_1)\| = \zeta_{\alpha_1}(t) \) and by (A.1), \( \lim_{t \to +\infty} \frac{1}{t} \log \|\Phi_t(u_1)\| \) coincides with \( \lim_{t \to +\infty} \frac{1}{t} \log \|\Phi_t(u_1)\| \). The common limit coincides with a Lyapunov exponent \( \lambda_{i_1} \) for some \( i_1 \in \{1, \ldots, n\} \). This implies by the Oseledec theorem that \( u_i \in E_{i_1}^{x} \). Observe that \( i_1 \) is taken from \( \{1, \ldots, n\} \), and the one-dimensional sub-bundle \( E_x^{i_1} \) has exactly 2 unit vectors, so the maximal choice that \( u_1 \) has is \( 2n \).

Now we assert that there exists \( i_2 \in \{1, \ldots, n\} \setminus \{i_1\} \) so that \( u_2 \in E_{i_1}^{x} \oplus E_{i_2}^{x} \) and

\[
\lim_{t \to +\infty} \frac{1}{t} \log \zeta_{\alpha 2}(t) = \lambda_{i_2}.
\]

Observe from the Oseledec theorem that the filtration

\[
E^1 \subset E^1 \oplus E^2 \subset \cdots \subset E^1 \oplus \cdots \oplus E^n = TM
\]

is \( d\Phi_t \) invariant. There is a minimal index \( j \) such that \( u_2 \in E_j^{x} \oplus \cdots \oplus E_i^{x} \). There are three possibilities, \( j = i_1, j < i_1 \), and \( j > i_1 \).
The first case: $j = i_1$. We see that $i_1 > 1$ in this case, because the 1-dimensional space $E_i^1$ cannot contain two orthonormal vectors $u_1$ and $u_2$. We take $i_2 := i_1 - 1$. From the choice of $j$, which equals to $i_1$ in this case, we see that $u_2 \in E_i^1 \oplus \cdots \oplus E_x^i$, hence
\[
\lim_{t \to +\infty} \frac{1}{t} \log \| \Phi_t(u_2) \| = \lambda_{i_1}.
\]

Note from definition that $\| \Phi_t(u_2) \| \geq \eta_{i_2}(t)$, hence we have by (A.1)
\[
\lambda_{i_1} = \lim_{t \to +\infty} \frac{1}{t} \log \| \Phi_t(u_2) \| \geq \lim_{t \to +\infty} \frac{1}{t} \log \eta_{i_2}(t) = \lim_{t \to +\infty} \frac{1}{t} \log \eta_{i_2}(t) \geq \lim_{t \to -\infty} \frac{1}{t} \log \| \Phi_t(u_2) \|.
\]

Let us write
\[
u_2 := v_1 + \cdots + v_{i_2} + v_{i_1}, \quad v_1 \in E_x^1, \ldots, v_{i_2} \in E_x^{i_2}, \quad v_{i_1} \in E_x^{i_1}.
\]

Denote by $\text{proj}_j : F_n \to TM$ the projection that sends an $n$-frame to its $i$th vector. Since $\langle \text{proj}_1 \chi_i^j(\alpha), \text{proj}_2 \chi_i^j(\alpha) \rangle = 0$ and $\Phi_t(v_{i_1}) \in E_x^{i_1}$, $\Phi_t(u_2)$ and $\Phi_t(v_1 + \cdots + v_{i_2})$ have the same projection in the direction determined by $\text{proj}_2 \chi_i^j(\alpha)$. Since $\chi_i^j(\alpha)$ is an orthonormal frame on $T_{\phi_t(x)}M$ and $\Phi_t(v_1 + \cdots + v_{i_2})$ is orthogonal to the linear subspace in $T_{\phi_t(x)}M$ generated by $\text{proj}_3 \chi_i^j(\alpha), \ldots, \text{proj}_n \chi_i^j(\alpha)$, we can represent $\Phi_t(v_1 + \cdots + v_{i_2})$ as
\[
\frac{\Phi_t(v_1 + \cdots + v_{i_2})}{\| \Phi_t(v_1 + \cdots + v_{i_2}) \|} = a_1(t) \text{proj}_1 \chi_i^j(\alpha) + a_2(t) \text{proj}_2 \chi_i^j(\alpha)
\]
with $|a_i(t)| \leq 1$. Since $\text{orb}(x, \phi) \subset H(v)$ we have by (4.2) that
\[
\pi \geq \limsup_{t \to +\infty} \left( \langle E_x^{i_1}, E_x^{i_2} \rangle \right) > 0.
\]

Remember that $\langle \text{proj}_1 \chi_i^j(\alpha), \text{proj}_2 \chi_i^j(\alpha) \rangle = \frac{\pi}{2}$. Then
\[
\limsup_{t \to +\infty} \left( \langle E_x^{i_2}, \text{proj}_2 \chi_i^j(\alpha) \rangle > 0.
\]

This together with the following equality
\[
\lim_{t \to +\infty} \frac{1}{t} \log \| \Phi_t(v_1 + \cdots + v_{i_2}) \| = \lim_{t \to +\infty} \frac{1}{t} \log \| \Phi_t(v_{i_2}) \|
\]
implies that $\limsup_{t \to +\infty} |a_2(t)| > 0$. Observe that $\text{proj}_2 \chi_i^j(\alpha)$ and $a_2(t) \| \Phi_t(v_1 + \cdots + v_{i_2}) \| \text{proj}_2 \chi_i^j(\alpha)$ denote the same projection of $\Phi_t(v_1 + \cdots + v_{i_2})$ to the direction determined by $\text{proj}_2 \chi_i^j(\alpha)$, so $\eta_{i_2}(t) = |a_2(t)| \| \Phi_t(v_1 + \cdots + v_{i_2}) \|$. Thus
\[
\limsup_{t \to +\infty} \frac{1}{t} \log \eta_{i_2}(t) = \lim_{t \to +\infty} \frac{1}{t} \log \| \Phi_t(v_1 + \cdots + v_{i_2}) \| = \lambda_{i_2}.
\]

This together with (A.2) gives
\[
\lambda_{i_1} = \lim_{t \to +\infty} \frac{1}{t} \log \| \Phi_t(u_2) \| > \lim_{t \to +\infty} \frac{1}{t} \log \eta_{i_2}(t) = \lim_{t \to +\infty} \frac{1}{t} \log \eta_{i_2}(t) = \lambda_{i_2}.
\]

Now we show that $u_2 \in E_x^{i_2} \oplus E_x^{i_1}$. Otherwise, by (A.3) there is $k < i_2$ such that
\[
u_2 := v_k + \cdots + v_{i_2} + v_{i_1}, \quad 0 \neq v_k \in E_x^k, \ldots, v_{i_2} \in E_x^{i_2}, \quad v_{i_1} \in E_x^{i_1}.
\]

One can deduce by using a similar argument as above while $t \to -\infty$ that,
\[
\lim_{t \to -\infty} \frac{1}{t} \log \eta_{i_2}(t) = \lim_{t \to -\infty} \frac{1}{t} \log \| \Phi_t(u_2) \| = \lambda_{i_2},
\]
which contradicts to (A.5). Therefore, $u_2 \in E_x^{i_1} \oplus E_x^{i_2}$ and
\[
\lambda_{i_1} = \lim_{t \to +\infty} \frac{1}{t} \log \| \Phi_t(u_2) \| > \lim_{t \to +\infty} \frac{1}{t} \log \eta_{i_2}(t) = \lim_{t \to +\infty} \frac{1}{t} \log \eta_{i_2}(t) = \lim_{t \to -\infty} \frac{1}{t} \log \| \Phi_t(u_2) \| = \lambda_{i_2}.
\]
This completes the first case.

The second case: \( j < i_1 \). In this case we set \( i_2 = j \). From (A.1) and from the choice of \( j \) it follows that

\[
\lambda_{i_2} = \lim_{i \to \infty} \frac{1}{i} \log \left\| \Phi_t(u_2) \right\| = \lim_{i \to \infty} \frac{1}{i} \log \zeta_{a_2}(t) = \lim_{i \to \infty} \frac{1}{i} \log \zeta_{a_2}(t) \geq \lim_{i \to \infty} \frac{1}{i} \log \left\| \Phi_t(u_2) \right\|. \quad (A.8)
\]

Since \( \chi^\#(\alpha) \) is an orthonormal frame on \( T_{\phi^{(\alpha)}}M \) and by definition \( \Phi_t(u_2) \) have no projection on the subspace generated by \( \text{proj}_3 \chi^\#(\alpha) \), ..., \( \text{proj}_n \chi^\#(\alpha) \), then we can represent \( \frac{\Phi_t(u_2)}{\left\| \Phi_t(u_2) \right\|} = b_1(t) \text{proj}_1 \chi^\#(\alpha) + b_2(t) \text{proj}_2 \chi^\#(\alpha) \)

with \( |b_1(t)| \leq 1 \). We observe that \( \langle \text{proj}_1 \chi^\#(\alpha), \text{proj}_2 \chi^\#(\alpha) \rangle = \frac{\pi}{2} \) and by (4.2) that

\[
\frac{\pi}{2} \geq \limsup_{t \to \infty} \langle E^{i_1}_{\phi^{(\alpha)}}, E^{i_2}_{\phi^{(\alpha)}} \rangle > 0.
\]

So \( \limsup_{t \to \infty} \langle E^{i_2}_{\phi^{(\alpha)}}, \text{proj}_2 \chi^\#(\alpha) \rangle > 0 \). This implies that \( \limsup_{t \to \infty} |b_2(t)| > 0 \). Note that \( \zeta_{a_2}(t) = |b_2(t)| \times \left\| \Phi_t(u_2) \right\| \), we have that

\[
\lim_{i \to \infty} \frac{1}{i} \log \zeta_{a_2}(t) = \lim_{i \to \infty} \frac{1}{i} \log \left\| \Phi_t(u_2) \right\| = \lambda_{i_2}.
\]

This together with (A.8) gives

\[
\lambda_{i_2} = \lim_{i \to \infty} \frac{1}{i} \log \left\| \Phi_t(u_2) \right\| = \lim_{i \to \infty} \frac{1}{i} \log \zeta_{a_2}(t) = \lim_{i \to \infty} \frac{1}{i} \log \zeta_{a_2}(t). \quad (A.9)
\]

We now show that \( u_2 \in E^{i_2}_x \oplus E^{i_1}_x \). Otherwise, there exists \( k < i_2 \) such that \( u_2 = v_k + \cdots + v_{i_2}, v_j \in E^j_x, j = k, \ldots, i_2 \), \( v_k \neq 0 \). It is not difficult to show that

\[
\lim_{i \to -\infty} \frac{1}{i} \log \zeta_{a_2}(t) = \lim_{i \to -\infty} \frac{1}{i} \log \left\| \Phi_t(u_2) \right\| = \lambda_k < \lambda_{i_2},
\]

which contradicts Eq. (A.9). Therefore, \( u_2 \in E^{i_2}_x \oplus E^{i_1}_x \), and by (A.9)

\[
\lambda_{i_2} = \lim_{i \to \infty} \frac{1}{i} \log \left\| \Phi_t(u_2) \right\| = \lim_{i \to \infty} \frac{1}{i} \log \zeta_{a_2}(t) = \lim_{i \to \infty} \frac{1}{i} \log \zeta_{a_2}(t) = \lim_{i \to \infty} \frac{1}{i} \log \zeta_{a_2}(t) = \lambda_{i_2}. \quad (A.10)
\]

This completes the second case.

The third case: \( j > i_1 \). We take \( i_2 = j \) in this case. A similar argument shows that \( u_2 \in E^{i_1}_x \oplus E^{i_2}_x \) and

\[
\lambda_{i_2} = \lim_{i \to \infty} \frac{1}{i} \log \left\| \Phi_t(u_2) \right\| = \lim_{i \to \infty} \frac{1}{i} \log \zeta_{a_2}(t) = \lim_{i \to \infty} \frac{1}{i} \log \zeta_{a_2}(t) > \lim_{i \to -\infty} \frac{1}{i} \log \left\| \Phi_t(u_2) \right\| = \lambda_{i_1}. \quad (A.11)
\]

From cases 1–3, the above assertion follows. From the assertion, \( u_2 \) has two choices in each 2-dimensional space \( E^{i_1}_x \oplus E^{i_2}_x \), while \( i_2 \) is chosen from \( \{1, \ldots, n\} \setminus \{i_1\} \). So the maximal choices of \( u_2 \) are \( 2(n-1) \).

By induction, there exists \( i_j \in \{1, \ldots, n\} \setminus \{i_1, \ldots, i_{j-1}\} \) such that

\[
u_j \in E^{i_{j-1}}_x \oplus E^{i_{j}}_x
\]

and

\[
\lim_{i \to \infty} \frac{1}{i} \log \zeta_{ai_j} = \lim_{i \to \infty} \frac{1}{i} \log \zeta_{ai_j}(t) = \lambda_{i_j}, \quad j = 1, \ldots, n.
\]

This implies that the maximal choices of \( u_j \) are \( 2(n-j), j = 1, \ldots, n \). Therefore the maximal choices of \( \alpha \) are \( 2^n n! \), namely, the cardinality of \( \rho^{-1}(x) \) is no greater than \( 2^n n! \). So the claim follows. \( \square \)

Acknowledgements

The authors thank very much IMPA/TWAS and IME-USP and the referee.
References