JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 154, 164–183 (1991)

Real Wronskian Zeros of Polynomials with Nonreal Zeros

KARL DILCHER

Department of Mathematics, Statistics and Computing Science, Dalhousie University, Halifax, Nova Scotia, Canada B3H 3J5

Submitted by R. P. Boas

Received April 10, 1989

1. INTRODUCTION

The nonlinear differential operator W, defined by

$$Wf(z) = f(z)\,f''(z) - (f'(z))^2 = \begin{vmatrix} f(z) & f'(z) \\ f'(z) & f''(z) \end{vmatrix}$$

$$= (f(z))^2 \frac{d^2}{dz^2} \log f(z), \tag{1.1}$$

was studied in [4]; Wf was called "the Wronskian of f." While [4] contains a few general results, the main focus of that paper is on the case where f is a polynomial with real zeros only. In that case Wf has no real zeros, as we see below, except where f has multiple zeros. However, if the polynomial f has nonreal zeros, then Wf may have real zeros. In this regard, Craven, Csordas, and Smith [3] made the following conjecture:

Let $f(z)$ be a real polynomial of degree $n \geqslant 2$, and suppose that $f(z)$ has exactly $2d$ nonreal zeros. Let $Z_R(Wf(z))$ be the number of real zeros of $Wf(z)$. Then

$$Z_R(Wf(z)) \leqslant 2d. \tag{1.2}$$

The purpose of this paper is to prove some partial results towards the conjecture (1.2). The main results are as follows:

1. Any nontrivial real zero of $Wf(z)$ must lie on or inside the Jensen circle of some pair of complex zeros of $f(z)$ (Theorem 2.4).

2. If a pair of complex zeros of $f(z)$ is sufficiently isolated from other zeros of $f(z)$, then the associated closed Jensen disk contains exactly two real or complex zeros of $Wf(z)$ (Theorem 4.1). This proves the conjecture for polynomials whose zeros are sufficiently well spaced (Corollary 4.2).

0022-247X/91 $3.00

3. If there are real zeros of $f(z)$ sufficiently close to a pair of complex zeros of $f(z)$, then the Jensen disk associated with these complex zeros contains no real zeros of $Wf(z)$ (Theorems 5.1 and 5.3).

4. If in a fixed interval $f(z)$ has sufficiently many real zeros, depending on the number and location of the complex zeros of $f(z)$, then $Wf(z)$ has no real zeros at all (Corollaries 5.4–5.6).

5. If the zeros of $f(z)$ lie in two circles symmetric to and sufficiently distant from the real axis, then $Wf(z)$ has exactly two real zeros (Corollary 6.5).

6. If $f(z)$ has no multiple zeros, then outside the Jensen circles of $f(z)$ the complex zeros of $Wf(z)$ cannot lie too close to the real axis. A lower bound is given (Theorem 3.4).

2. Some General Properties

For easier reference, we give the following two lemmas from [4].

LEMMA 2.1. *Let f and g be polynomials, and c a constant. Then*

(a) $W(fg) = f^2 Wg + g^2 Wf$;

(b) $W(f^n) = nf^{2n-2} Wf$;

(c) $W(z-c) = -1$.

LEMMA 2.2. *If $f(z) = (z-z_1)^{m_1} \cdots (z-z_k)^{m_k}$, $m_j \in \mathbb{N}$, then*

$$Wf(z) = -(f(z))^2 \left\{ \frac{m_1}{(z-\alpha_1)^2} + \cdots + \frac{m_k}{(z-\alpha_k)^2} \right\}.$$

Parts (a) and (c) of Lemma 2.1 follow from (1.1) by direct computation; (b) is a direct consequence of (a). Lemma 2.2 is an easy consequence of Lemma 2.1(a) and (c).

For the remainder of this paper we adopt the following notations. Let $f(z)$ be a real polynomial of degree $n = m + 2d$ with m real zeros $\alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_m$ and d pairs of complex zeros $u_j \pm iv_j$, $1 \leqslant j \leqslant d$, numbered such that $u_1 \leqslant u_2 \leqslant \cdots \leqslant u_d$. If $u_j = u_{j+1}$, we assume that $v_j \leqslant v_{j+1}$.

With this notation we get the next lemma which is fundamental to what follows.

LEMMA 2.3.

$$Wf(z) = -(f(z))^2 \left\{ \sum_{j=1}^{m} \frac{1}{(z-\alpha_j)^2} + 2 \sum_{j=1}^{d} \frac{(z-u_j)^2 - v_j^2}{[(z-u_j)^2 + v_j^2]^2} \right\}. \tag{2.1}$$

This is immediate from Lemma 2.2 if we observe that

$$\frac{1}{(z-u_j-iv_j)^2}+\frac{1}{(z-u_j+iv_j)^2}=2\,\frac{(z-u_j)^2-v_j^2}{[(z-u_j)^2+v_j^2]^2}.$$

It is obvious from Lemma 2.3 that if $f(z)$ has only real zeros then $Wf(z)$ can have real zeros only where $f(z)$ has multiple zeros. The zeros of $Wf(z)$ induced by multiple zeros of $f(z)$ are called "trivial zeros." Note that trivial zeros occur only in even multiplicities.

To each pair $u_j \pm iv_j$ of complex conjugate zeros we construct a "Jensen circle" (see, e.g., [5, p. 25]), namely a circle whose diameter is the line segment joining the two zeros. Let J_j be the corresponding closed Jensen disk, i.e., the union of the Jensen circle and its interior.

THEOREM 2.4. *Any nontrivial real zero of $Wf(z)$ must lie in one of the closed Jensen disks of $f(z)$ or, in other words, in the set*

$$S := \bigcup_{j=1}^{d} (J_j \cap \mathbb{R}).$$

Proof. Using Lemma 2.3 we show that $Wf(z)<0$ if $f(z)\neq 0$, $z\in\mathbb{R}$, and $z\notin S$. First we note that

$$\sum_{j=1}^{m}\frac{1}{(z-\alpha_j)^2}>0 \qquad \text{for} \quad z\in\mathbb{R}. \tag{2.2}$$

Now we observe that $(z-u_j)^2-v_j^2>0$ if z is real and lies outside of J_j. Hence the second sum in (2.1) is positive if $z\notin S$. This, with (2.2), completes the proof.

It is a well-known fact from the geometry of polynomials that the point sets of the zeros of a polynomial and the zeros of its derivative have the same centroid (or center of mass) if we imagine a unit mass attached to each zero, counting multiplicities (see, e.g., [7, p. 9]). In analogy, we get the following result.

THEOREM 2.5. *The zeros of a polynomial of degree $n\geqslant 2$ and the zeros of its Wronskian $Wf(z)$ have the same centroid.*

Proof. First we note that the centroid of the set $\{z_1, z_2, ..., z_n\}$ of the zeros of

$$f(z)=(z-z_1)\cdot \cdots \cdot(z-z_n)=z^n+a_1z^{n-1}+\cdots+a_n$$

is

$$\frac{1}{n}(z_1 + z_2 + \cdots + z_n) = \frac{-1}{n} a_1.$$

It is easy to verify that

$$\begin{aligned} Wf(z) &= -nz^{2n-2} - (2n-2)\, a_1 z^{2n-3} - \cdots \\ &= -n\left(z^{2n-2} + \frac{2n-2}{n} a_1 z^{2n-3} + \cdots\right). \end{aligned}$$

Hence the centroid of the zeros of $Wf(z)$ is

$$\frac{-1}{2n-2}\left(\frac{2n-2}{n} a_1\right) = \frac{-1}{n} a_1;$$

this proves the theorem.

The following result gives a connection between the real zeros of $Wf(z)$ and the location of the complex zeros of $f(z)$.

THEOREM 2.6. *Let $f(z)$ be a polynomial having only real coefficients and at least one real zero. Then to each real nontrivial zero r of $Wf(z)$ there exists at least one pair of complex zeros $u \pm iv$ of $f(z)$ located in the intersection of the closed disk with radius $\sqrt{2h}\, R(r)$ centered at r and the angular region $|x - r| < |y|$ $(z = x + iy)$, where h is the number of pairs of complex zeros whose Jensen disks contain r in its interior and $R(r) := (\sum_{j=1}^{m} (r - \alpha_j)^{-2})^{-1/2}$, with the sum taken over all real zeros α_j of $f(z)$.*

Remarks. (1) If $f(z)$ has only one real zero α, or if α is a real zero with minimum distance to r, then clearly $R(r) \leqslant |r - \alpha|$, and the radius of the disk in Theorem 2.6 may be replaced by $\sqrt{2h}\, |r - \alpha|$.

(2) If nothing is known about the location of the complex zeros of $f(z)$, we may replace h by d, the number of pairs of complex zeros of $f(z)$, since $h \leqslant d$.

(3) Both the statement and the proof of Theorem 2.6 are analogous to a result by H. B. Mitchell [6] on the relationship between the complex zeros of a polynomial and the real zeros of its derivative.

(4) For an application of Theorem 2.6, see Corollary 5.6 below.

Proof of Theorem 2.6. If r is a nontrivial real zero of $Wf(z)$, then by (2.1) we have, with $R = R(r)$,

$$R^{-2} = \sum_{j=1}^{m} \frac{1}{(r - \alpha_j)^2} = 2 \sum_{j=1}^{d} \frac{v_j^2 - (r - u_j)^2}{[(r - u_j)^2 + v_j^2]^2}. \tag{2.3}$$

Let $u \pm iv$ be a pair of complex zeros of f corresponding to a maximum term in the sum on the right-hand side of (2.3). Let $h \leqslant d$ be the number of positive terms in this sum, and note that $h \geqslant 1$ since the sum itself is positive. Further note that a term is positive if and only if the corresponding Jensen disk contains r in its interior. Now we get from (2.3),

$$\frac{1}{R^2} \leqslant 2h \frac{v^2-(r-u)^2}{[(r-u)^2+v^2]^2} \leqslant 2h \frac{v^2}{(v^2)^2} = \frac{2h}{v^2}, \tag{2.4}$$

or

$$v^2 \leqslant 2hR^2. \tag{2.5}$$

From the left-hand inequality in (2.4) we get

$$[(r-u)^2+v^2]^2 \leqslant 2hR^2[v^2-(r-u)^2],$$

which is equivalent to

$$[(r-u)^2+v^2+hR^2]^2 \leqslant 4hR^2v^2+h^2R^4.$$

Now with (2.5) we get

$$[(r-u)^2+v^2+hR^2]^2 \leqslant 9h^2R^4,$$

or

$$(u-r)^2+v^2 \leqslant 2hR^2.$$

This proves the theorem if we note that r lies in the interior of the Jensen disk belonging to $u \pm iv$ if and only if $|u-r|<|v|$.

Finally in this section, we state a general result that is in fact a special case of Theorem (8.1) in [5]. It is relevant to the topic of this paper, although it will not be needed later on.

THEOREM 2.7. *If the zeros of $f(z)$ lie inside or on the unit circle then the zeros of $Wf(z)$ lie inside or on the circle of radius $\sqrt{2}$ centered at the origin. This result is best possible.*

3. ZERO-FREE REGIONS

To each real zero α_j of $f(z)$ define the double angular set

$$S_j := \{z = x+iy : |x-\alpha_j| \leqslant |y|\},$$

and to each pair of complex zeros $u_j \pm iv_j$ define the set

$$T_j := \{z : |z-u_j| \leqslant v_j\} \cup \{z = x+iy : (x-u_j)^2 + v_j^2 \leqslant y^2\}.$$

Note that T_j consists of the hyperbola $(x-u_j)^2 + v_j^2 = y^2$ and its interior, and of the closed disk centered at $z = u_j$ and whose boundary touches the hyperbola at $u_j \pm iv_j$. Let S_f be the union of all S_j and all T_j.

THEOREM 3.1. *With $f(z)$ and S_f defined as above, we have*

$$\operatorname{Re} \frac{Wf(z)}{(f(z))^2} < 0 \qquad \text{for} \quad z \notin S_f.$$

Proof. By Lemma 2.3 it suffices to show that

$$\operatorname{Re}(z-\alpha_j)^{-2} > 0 \qquad \text{for} \quad 1 \leqslant j \leqslant m, \tag{3.1}$$

and

$$\operatorname{Re} \frac{(z-u_j)^2 - v_j^2}{[(z-u_j)^2 + v_j^2]^2} > 0 \qquad \text{for} \quad 1 \leqslant j \leqslant d. \tag{3.2}$$

With $\alpha_j = \alpha$ and $z = x + iy$, we have

$$\operatorname{Re} \frac{1}{(z-\alpha)^2} = \frac{(x-\alpha)^2 - y^2}{[(x-\alpha)^2 + y^2]^2}$$

which is positive if $z \notin S_j$; this proves (3.1).

With $u_j = u$ and $v_j = v$, we get

$$\frac{(z-u)^2 - v^2}{[(z-u)^2 - v^2]^2} = \frac{[(z-u)^2 - v^2][(\bar{z}-u)^2 + v^2]}{|(z-u)^2 + v^2|^4},$$

and it is easy to verify that

$$\operatorname{Re} \frac{(z-u)^2 - v^2}{[(z-u)^2 + v^2]^2} = \frac{(|z-u|^4 - v^4)((x-u)^2 - y^2 + v^2) + 8(x-u)^2 y^2 v^2}{|(z-u)^2 + v^2|^4}. \tag{3.3}$$

By the definition of T_j we see now immediately that the right-hand side of this last equation is positive whenever $z \notin T_j$. This proves (3.2), and the proof of the theorem is complete.

The first corollary follows immediately from Theorem 3.1. It can be considered an extension of Theorem 2.4.

COROLLARY 3.2. *All zeros of $Wf(z)$ lie in the set S_f.*

COROLLARY 3.3. (a) *Let α_j and α_{j+1} be two consecutive real zeros of $f(z)$ such that the open interval (α_j, α_{j+1}) has an empty intersection with all Jensen disks of $f(z)$. Then the interior of the square that has the line segment $[\alpha_j, \alpha_{j+1}]$ as a diagonal is free of zeros of $Wf(z)$.*

(b) *Part* (a) *is still true if α_j is replaced by $u_k + v_k$ or if α_{j+1} is replaced by $u_{k'} - v_{k'}$, where $u_k \pm iv_k$ and $u_{k'} \pm iv_{k'}$, $v_k > 0$, $v_{k'} > 0$ are complex zeros of $f(z)$.*

The next theorem gives a quantitative version of Theorem 2.4. It shows that the zeros outside of the Jensen disks cannot lie too close to the real axis.

THEOREM 3.4. *Let $f(z)$ be a polynomial with real coefficients, and d the minimum distance between adjacent points in the set containing all real zeros α_j, $j = 1, ..., m$ and the points $z = u_j$, where $u_j \pm iv_j$, $j = 1, ..., d$, are the complex zeros of $f(z)$. Then the zeros of $Wf(z)$ are either in a Jensen disk or outside the strip $|\mathrm{Im}\, z| < d\sqrt{3/2}/2\pi$.*

Proof. First we suppose that z lies in the intersection D_j of the angular region $|x - \alpha_j| \leqslant y$ and the closed disk of radius $d\sqrt{3}/2\pi$ around the real zero α_j of $f(z)$. By Lemma 2.3, $Wf(z)$ is not zero if we can show that

$$|z-\alpha_j|^{-2} > \sum_{i \neq j} |z-\alpha_i|^{-2} + \sum_{i=1}^{d} (|z-u_i-iv_i|^{-2} + |z-u_i+iv_i|^2). \tag{3.4}$$

We note that $|z-\alpha_i|^{-2} \leqslant |x-\alpha_i|^{-2}$ and $|z-u_i \pm iv_i|^{-2} \leqslant |x-u_i|^{-2}$; hence (3.4) hold when

$$|z-\alpha_j|^{-2} > \sum_{i \neq j} |x-\alpha_i|^{-2} + 2\sum_{i=1}^{d} |x-u_i|^{-2}$$

which in turn is true if

$$|z-\alpha_j|^{-2} > 2\sum_{\beta} |x-\beta|^{-2}, \tag{3.5}$$

where the sum extends over $\beta = u_1, ..., u_d, \alpha_1, ..., \alpha_k$, but $\beta \neq \alpha_j$. Since the distance between two consecutive β is at least d, we have

$$\sum_{\beta} |x-\beta|^{-2} < \sum_{k=1}^{\infty} (|\tilde{x}-kd|^{-2} + |\tilde{x}+kd|^{-2}),$$

where $\tilde{x} := x - \alpha_j$. Without loss of generality we may assume $\tilde{x} \leqslant 0$. Then

$$|\tilde{x}kd|^{-2} \leqslant (kd)^{-2}, \qquad k = 1, 2, ...$$

and also, since $|\tilde{x}| = |x - \alpha_j| \leqslant d/2$,

$$|\tilde{x} + kd|^{-2} \leqslant \left(-\frac{d}{2} + kd\right)^{-2}, \qquad k = 1, 2, ...$$

Hence

$$\sum_{\beta} |x - \beta|^{-2} < d^{-2} \left(\sum_{k=1}^{\infty} k^{-2} + 4 \sum_{k=1}^{\infty} (2k-1)^{-2} \right)$$
$$= d^{-2} \left(\frac{\pi^2}{6} + 4 \frac{\pi^2}{8} \right) = \frac{2}{3} \pi^2 d^{-2}.$$

On the other hand,

$$|z - \alpha_j|^{-2} \geqslant \tfrac{4}{3} \pi^2 d^{-2}$$

since $z \in D_j$; hence (3.5) and therefore (3.4) hold.

Next we suppose that z lies in the set D_j, where α_j is replaced by some u_j. If $v_j \geqslant d\sqrt{3}/2\pi$, then the Jensen disk belonging to the complex zeros $u_j \pm iv_j$ covers all of D_j, and nothing remains to prove. Hence we assume that $v_j < d\sqrt{3}/2\pi$. By Corollary 3.2 we may restrict our attention to those $z = x + iy$ satisfying

$$(x - u_j)^2 + v_j^2 \leqslant y^2. \tag{3.6}$$

To simplify notation,set $u = u_j$ and $v = v_j$. By Corollary 2.3, $Wf(z)$ is not zero if we can show that

$$2 \frac{|(z-u)^2 - v^2|}{|(z-u)^2 + v^2|^2} > \sum_{i=1}^{m} |z - \alpha_i|^{-2} + \sum_{\substack{i=1 \\ i \neq j}}^{d} (|z - u_i - iv_i|^{-2} + |z - u_i + iv_i|^{-2}). \tag{3.7}$$

To estimate the left-hand side of (3.7), we first note that

$$|(z-u)^2 - v^2| = |z-u|^4 - v^2\{(z-u)^2 + (\bar{z}-u)^2\} + v^4$$
$$= |z-u|^4 + 2v^2[y^2 - (x-u)^2] + v^4$$
$$> |z-u|^4$$

since z lies in the angular region $|x-u|<y$. Hence

$$|(z-u)^2-v^2|>|z-u|^2. \tag{3.8}$$

Furthermore, taking (3.6) into account, we get

$$\begin{aligned}|(z-u)^2+v^2|^2 &= [(x-u)^2+v^2-y^2]^2+4y^2(x-u)^2\\ &\leqslant [(x-u)^2-y^2]^2+4y^2(x-u)^2\\ &= |z-u|^4.\end{aligned}$$

Hence, with (3.8) we have

$$\frac{|(z-u)^2-v^2|}{|(z-u)^2+v^2|^2}>\frac{1}{|z-u|^2}.$$

So (3.7) is certainly true if (3.5) holds, with α_j replaced by u. But (3.5) has already been verified.

Finally, we note that the union of the sets D_j and the complement of S_f cover the part of the strip $|\mathrm{Im}\, z|<d\sqrt{3/2}/2\pi$ that lies outside of any Jensen disk. This completes the proof.

Remarks. (1) Theorem 3.4 can be extended, without change, to polynomials $f(z)$ with real zeros of multiplicity two. Indeed, we may simply consider a real double zero as a pair $u \pm iv$ of complex zeros with $v=0$. The proof is still valid in this case.

(2) An analogue of Theorem 3.4 for the case where $f(z)$ has only real zeros was proved in [4].

4. The Number of Zeros on a Jensen Disk

By Theorem 2.4, the real zeros of $Wf(z)$ lie on the closed Jensen disks associated with the complex zeros of $f(z)$. If it could be shown that each such Jensen disk contains at most two zeros of $Wf(z)$, this would prove the conjecture (1.2). The following theorem is a partial result in this direction, for the case where the zeros of $f(z)$ are sufficiently isolated.

Theorem 4.1. *Let $u \pm iv$ be a pair of simple complex zeros of $f(z)$ such that its closed Jensen disk J has no points in common with the Jensen disks of any other zeros of $f(z)$, and that no real zero of $f(z)$ and no center of any other Jensen disk lies in the interval $[u-v\sqrt{2}, u+v\sqrt{2}]$. Then $Wf(z)$ has exactly two zeros in J.*

Proof. Without loss of generality we may assume that the zeros $u \pm iv$ lie on the imaginary axis, i.e., $u = 0$.

For $\varepsilon > 0$, let C_ε be the circle of radius $\sqrt{v^2 + \varepsilon}$ centered at the origin. We show that

$$F(z) := -Wf(z)/(f(z))^2$$

goes around the origin twice in the negative direction as z traverses C_ε once in the positive direction, if ε is sufficiently small. Then by the argument principle we have

$$Z_F - P_F = -2,$$

where Z_F and P_F are the numbers of zeros, resp. poles of $F(z)$ in the interior of C_ε. But $F(z)$ has exactly four poles (counting multiplicities) in the interior of C_ε, namely the zeros $z = \pm iv$ of $f(z)$. Hence $Z_F = 2$, which was to be shown.

Next we note that the hypotheses ensure that if ε is sufficiently small, the contour C_ε has no point in common with any of the sets S_j associated with the real zeros of $f(z)$, nor with any of the sets T_j associated with the complex zeros of $f(z)$ distinct from $\pm iv$. Hence the terms on the right-hand side of (2.1) associated with these zeros all have positive real parts for $z \in C_\varepsilon$, by (3.1) and (3.2). With (2.1) we can therefore write

$$F(z) = 2\,\frac{z^2 - v^2}{[z^2 + v^2]^2} + G(z),$$

where $\operatorname{Re} G(z) \geqslant 0$ for $z \in C_\varepsilon$ if ε is sufficiently small. It is also clear that $\operatorname{Im} G(z) = 0$ if z is real, and that $G(z)$ remains bounded for $z \in C_\varepsilon$. It is easy to verify that

$$\operatorname{Re} F(z) = 2\,\frac{(|z|^4 - v^4)(x^2 - y^2 + v^2) + 8x^2y^2v^2}{|z^2 + v^2|^4} + \operatorname{Re} G(z) \qquad (4.1)$$

and

$$\operatorname{Im} F(z) = 4xy\,\frac{2v^2(x^2 - y^2 + v^2) - (|z|^4 - v^4)}{|z^2 + v^2|^4} + \operatorname{Im} G(z). \qquad (4.2)$$

We subdivide the upper semicircle of C_ε into five sections. Let $z_1 := \sqrt{v^2 + \varepsilon}$ and $z_6 := -\sqrt{v^2 + \varepsilon}$. Let z_2 and z_5 be the two points in the first, resp. second, quadrant where C_ε intersects the hyperbola $x^2 - y^2 + v^2 = 0$; their real parts satisfy $x^2 = \varepsilon/2$, which can be seen by adding the equations $x^2 - y^2 + v^2 = 0$ and $x^2 + y^2 = v^2 + \varepsilon$. Finally, let z_3 and z_4 be the two points on C_ε in the first, resp. second, quadrant whose real parts satisfy $x^2=\varepsilon^2/8v^2$. For $j=1, ..., 5$, let C_j be the sections of C_ε joining z_j with z_{j+1}.

To estimate the denominator in (4.1) and (4.2), we use the relationship $y^2=v^2+\varepsilon-x^2$ to get

$$\begin{aligned}|z^2+v^2|^2&=(x^2-y^2+v^2)^2+4x^2y^2\\&=(2x^2-\varepsilon)^2+4x^2(v^2+\varepsilon-x^2)\\&=4x^2v^2+\varepsilon^2;\end{aligned}$$

hence

$$|z^2+v^2|^4=(4x^2v^2+\varepsilon^2)^2. \tag{4.3}$$

We also note that

$$x^2-y^2+v^2=2x^2-\varepsilon \tag{4.4}$$

and

$$|z|^4-v^4=\varepsilon(2v^2+\varepsilon); \tag{4.5}$$

all, of course, under the condition $|z|^2=v^2+\varepsilon$.

Now we let z traverse the upper half of C_ε in the positive direction, starting at $z=z_1$, where $\operatorname{Im} F(z_1)=0$ and $\operatorname{Re} F(z_1)>\operatorname{Re} G(z_1)\geqslant 0$.

(i) For $z\in C_1$, it is clear from (4.4) and (4.1) that $\operatorname{Re} F(z)>0$.

(ii) $z\in C_2$. First we note that with (4.3), (4.4), and (4.5) we get

$$I:=4xy\,\frac{2v^2(x^2-y^2+v^2)-(|z|^4-v^4)}{|z^2+v^2|^4}=4xy\,\frac{4v^2(x^2-\varepsilon)-\varepsilon^2}{(4x^2v^2+\varepsilon^2)^2}.$$

On C_2 we have $\varepsilon^2/8v^2\leqslant x^2\leqslant\varepsilon/2$; hence

$$(4x^2v^2+\varepsilon^2)^2\leqslant(4x^2v^2+8x^2v^2)^2=144x^4v^4.$$

Also $4v^2(x^2-\varepsilon)-\varepsilon^2\leqslant 4v^2(x^2-\varepsilon)$, and

$$y=\sqrt{v^2+\varepsilon-x^2}\geqslant\sqrt{v^2+\varepsilon/2}>v,$$

so that

$$I<\frac{x^2-\varepsilon}{9x^3v}$$

(note that $x^2 - \varepsilon < 0$). It is easy to verify that the right-hand side is increasing with x if $x^2 < 3\varepsilon$; hence

$$I < \frac{\varepsilon/2 - \varepsilon}{9(\varepsilon/2)^{3/2}\, v} = -\frac{\sqrt{2}}{9v\varepsilon}$$

for $\varepsilon^2/8v^2 \leqslant x^2 \leqslant \varepsilon/2$. Hence by (4.2), $\operatorname{Im} F(z) < 0$ for $z \in C_2$ if ε is sufficiently small.

(iii) On C_3, we have $x^2 \leqslant \varepsilon^2/8v^2$. Now with (4.5) and (4.4),

$$\begin{aligned}
(|z|^4 - v^4)&(x^2 - y^2 + v^2) + 8x^2y^2v^2 \\
&= \varepsilon(2v^2 + \varepsilon)(2x^2 - \varepsilon) + 8x^2v^2(v^2 + \varepsilon - x^2) \\
&\leqslant \varepsilon(2v^2 + \varepsilon)\left(\frac{\varepsilon^2}{4v^2} - \varepsilon\right) + \varepsilon^2\left(v^2 + \varepsilon - \frac{\varepsilon^2}{8v^2}\right) \\
&= -\varepsilon^2\left(v^2 - \frac{\varepsilon}{2} - \frac{\varepsilon^2}{8v^2}\right).
\end{aligned}$$

With (4.3) we get

$$|z^2 + v^2|^4 \leqslant (\tfrac{1}{2}\varepsilon^2 + \varepsilon^2)^2 = \tfrac{9}{4}\varepsilon^4,$$

and therefore, with (4.1),

$$\operatorname{Re} F(z) \leqslant -\frac{8}{9}\left(v^2 - \frac{\varepsilon}{2} - \frac{\varepsilon^2}{8v^2}\right)\frac{1}{\varepsilon^2} + \operatorname{Re} G(z)$$

for $z \in C_3$. Hence $\operatorname{Re} F(z) < 0$ for $z \in C_3$ if ε is sufficiently small.

(iv) On C_4 we have the same situation as on C_2, with the only difference being that $x < 0$. Hence $\operatorname{Im} F(z) > 0$ for $z \in C_4$ if ε is sufficiently small.

(v) On C_5 we clearly have $\operatorname{Re} F(z) > 0$ again, as on C_1. For $z = -\sqrt{v^2 + \varepsilon}$ we have $\operatorname{Im} F(z) = 0$ and $\operatorname{Re} F(z) > \operatorname{Re} G(z) \geqslant 0$.

Finally, as z traverses the lower semicircle of C_ε, $F(z)$ goes around the origin once more: this can be seen in essentially the same way as in (i)–(v). The proof is now complete.

As an immediate consequence of Theorem 4.1, and with Theorem 2.4, we get the following result which is relevant to the conjecture (1.2).

COROLLARY 4.2. *Let $f(z)$ have d pairs of simple nonreal zeros $u_j \pm iv_j$, $j = 1, ..., d$, such that the Jensen disk J_j of each pair has no point in common* *with any other Jensen disk, and that no real zero of* $f(z)$ *and no center of any other Jensen disk lies in the interval* $[u_j - v_j\sqrt{2}, u_j + v_j\sqrt{2}]$. *Then* $Wf(z)$ *has at most* $2d$ *real zeros.*

5. Polynomials without Real Wronskian Zeros

In this section we see that the presence of real zeros of $f(z)$ in proximity of a pair of complex zeros $u \pm iv$ of $f(z)$ can prevent the existence of real zeros of $Wf(z)$ in the interval $[u-v, u+v]$. As consequences, we find conditions on f under which $Wf(z)$ has no real zeros at all.

Theorem 5.1. *Let* $u \pm iv$ *be a pair of complex zeros of* $f(z)$. *If there is a real zero* α *of* $f(z)$ *such that* $|u-\alpha| < c_1 v$, *where*

$$c_1 := \sqrt{3/2}\,[(16\sqrt{2}+13)^{1/3} - (16\sqrt{2}-13)^{1/3} \cong 0.4947092,$$

then $Wf(z)$ *has no real zero in the interval* $I := [u-v, u+v] - (S \cap \mathbb{R})$; *here* S *is the union of the closed Jensen disks belonging to all the other complex zeros of* $f(z)$.

Proof. Without loss of generality we may assume that $u=0$. From the proof of Theorem 2.4 we know that all the terms of the right-hand side of (2.1), except for the one belonging to $\pm iv$, are positive if $z \in I$. Hence we are done if we can show that

$$\frac{1}{(z-\alpha)^2} + 2\frac{z^2-v^2}{(z^2+v^2)^2} > 0 \qquad \text{for} \quad z \in [-v, v].$$

For a given v we want to find those α for which the numerator $N(z)$ of

$$\frac{1}{(z-\alpha)^2} + 2\frac{z^2-v^2}{(z^2+v^2)^2} = \frac{(z^2+v^2)^2 + 2(z-\alpha)^2\,(z^2-v^2)}{(z-\alpha)^2\,(z^2+v^2)^2}$$

is positive for all z. First we note that

$$\begin{aligned} N(z) &= 3z^4 - 4\alpha z^3 + 2\alpha^2 z^2 + 4\alpha v^2 z - 2\alpha^2 v^2 + v^4 \\ &= 3v^4\left[\left(\frac{z}{v}\right)^4 - \frac{4}{3}\frac{\alpha}{v}\left(\frac{z}{v}\right)^3 + \frac{2}{3}\left(\frac{\alpha}{v}\right)^2\left(\frac{z}{v}\right)^2\right. \\ &\qquad \left. + \frac{4}{3}\frac{\alpha}{v}\frac{z}{v} - \frac{2}{3}\left(\frac{\alpha}{v}\right)^2 + \frac{1}{3}\right]. \end{aligned}$$

Now let $\tilde{z} := z/v$ and $\tilde{\alpha} := \alpha/v$, and denote the term in square brackets by

$M(\tilde{z})$. Then we must find those $\tilde{\alpha}$ for which $M(\tilde{z})>0$ for all $\tilde{z}$. It is easy to see that we can write

$$\begin{aligned}M(\tilde{z}) &= (\tilde{z}-\tfrac{1}{3}\tilde{\alpha})^4+(\tfrac{4}{27}\tilde{\alpha}^3+\tfrac{4}{3}\tilde{\alpha})\tilde{z}-\tfrac{1}{81}\tilde{\alpha}^4-\tfrac{2}{3}\tilde{\alpha}^2+\tfrac{1}{3}\\ &= \tfrac{1}{81}\{(3\tilde{z}-\tilde{\alpha})^4+12(\tilde{\alpha}^3+9\tilde{\alpha})\tilde{z}-\tilde{\alpha}^4-54\tilde{\alpha}^2+27\}.\end{aligned}$$

Let $K(\tilde{z}):=81\,M(\tilde{z})$. We differentiate $K(\tilde{z})$ with respect to $\tilde{z}$, and find that $K(\tilde{z})$ is minimal when $\tilde{z}=\tilde{z}_0$, where

$$3\tilde{z}_0=\tilde{\alpha}-(\tilde{\alpha}^3+9\tilde{\alpha})^{1/3};$$

we get

$$K(\tilde{z}_0)=3[(3-\tilde{\alpha}^2)^2-((\tilde{\alpha}^2+9)^2\,\tilde{\alpha}^4)^{1/3}].$$

This increases as $\tilde{\alpha}^2$ increases; it is zero when

$$2\tilde{\alpha}^6+9\tilde{\alpha}^4+108\tilde{\alpha}^2-27=0,$$

or when

$$t^3+21t-26=0, \qquad \text{where} \quad \tilde{\alpha}^2=\tfrac{3}{2}(t-1).$$

Hence by Cardano's formula we see that $K(\tilde{z})>0$ whenever

$$|\tilde{\alpha}|<\sqrt{3/2}\,[(16\sqrt{2}+13)^{1/3}-(16\sqrt{2}-13)^{1/3}-1]^{1/2},$$

which completes the proof.

Corollary 5.2. *Let $u_j \pm iv_j$, $j=1, ..., d$ be the complex zeros of $f(z)$. If there are $m \geqslant d$ real zeros $\alpha_1, ..., \alpha_m$, and they can be numbered in such a way that $|u_j-\alpha_j|<c_1 v_j$ for $j=1, ..., d$ (with c_1 as in Theorem 5.1), then $Wf(z)$ has no real zeros.*

Proof. For $u_j \pm iv_j$ and α_j, $j=1, ..., d$ we have (5.1). For $\alpha_{d+1}, ..., \alpha_m$, (2.2) holds. The result now follow from (2.1) and Theorem 2.4.

The next result is an extension of Theorem 5.1 to the case where a larger number of real zeros of $f(z)$ are located at a greater distance from $z=u$.

Theorem 5.3. *Let $u \pm iv$ be a pair of complex zeros of $f(z)$. If there are $k>1$ real zeros $\alpha_1, ..., \alpha_k$ such that $|u-\alpha_j|<c_k v$ for $j=1, ..., k$, where $c_k:=\frac{1}{2}(\sqrt{9k^2-8k}-k)^{1/2}$, then $Wf(z)$ has no real zero in the interval I (I as in Theorem 5.1).*

Proof. We may assume that $u=0$. We are done if we can show that

$$2\frac{z^2-v^2}{(z^2+v^2)^2}+\sum_{j=1}^{k}\frac{1}{(z-\alpha)^2}>0 \tag{5.1}$$

for $z\in[-v, v]$. This inequality was dealt with by Conrey and Rubel [2] in a somewhat different setting. For convenience, we repeat their argument which yields the bounds c_k. Let $F=F(z, \alpha_1, ..., \alpha_k)$ denote the left-hand side of (5.1). To find the minimum M of F, we first observe that the terms $(z-\alpha_j)^{-2}$ are minimal when the α_j have the greatest distance from z. By symmetry, a minimum value for F occurs in $z\geqslant 0$; here we must take $\alpha_j=-c_k v$. Therefore we must find the minimum of

$$\begin{aligned}M(z) &:= 2\frac{z^2-v^2}{(z^2+v^2)^2}+\frac{k}{(z+c_k v)^2}\\ &=\frac{2(z^2-v^2)(z+c_k v)^2+k(z^2+v^2)^2}{(z^2+v^2)^2\,(z+c_k v)^2}.\end{aligned}$$

If we set $\tilde{z}:=z/v$, we see that $M(z)>0$ for $0\leqslant z\leqslant v$ if

$$g(\tilde{z}):=2(\tilde{z}^2-1)(\tilde{z}+c_k)^2+k(\tilde{z}^2+1)^2>0$$

for $0\leqslant\tilde{z}\leqslant 1$. We have

$$\begin{aligned}g(\tilde{z}) &= (k+2)\tilde{z}^4+4c_k\tilde{z}^3+2(k-1+c_k^2)\tilde{z}^2-4c_k\tilde{z}+(k-2c_k^2)\\ &\geqslant 2(k-1+c_k^2)\tilde{z}^2-4c_k\tilde{z}+(k-2c_k^2)\end{aligned}$$

for $\tilde{z}\geqslant 0$ and $c_k\geqslant 0$. The last expression is positive for all $\tilde{z}$ if

$$0>16c_k^2-8(k-1+c_k^2)(k-2c_k^2)=8(2c_k^4+kc_k^2-k(k-1))$$

which holds if

$$c_k^2<[-k+\sqrt{k^2+8k(k-1)}]/4.$$

This proves Theorem 5.3.

The following corollary is derived from Theorems 5.3 and 5.1 just as Corollary 5.2 follows from Theorem 5.1 alone.

COROLLARY 5.4. *Let $u_j\pm iv_j$, $j=1, ..., d$ be the complex zeros of $f(z)$, and let a sequence $m_1, ..., m_d$ of positive integers with $m_1+\cdots+m_d\leqslant m$ be given. If to each pair of complex zeros $u_j\pm iv_j$ one can associate m_j real zeros of $f(z)$ which lie at a distance of less than $c_{m_j}v_j$ from u_j and such that none of the real zeros are associated with two pairs of complex zeros, then $Wf(z)$ has no real zeros.*

We conclude this section with two more general (but generally weaker) criteria for the nonexistence of real zeros of $Wf(z)$.

COROLLARY 5.5. *Let $u_j \pm iv_j$, $j=1, ..., d$ be the complex zeros of $f(z)$, and let $v := \min(|v_j| \; j=1, ..., d\}$. Suppose that $f(z)$ has $m \geqslant d$ real zeros, and denote $k := [m/d]$ (the greatest integer less than or equal to m/d). Let D be the smallest number such that all the zeros of $f(z)$ satisfy $a \leqslant \mathrm{Re}(z) \leqslant a+D$ for some a. If*

$$D < vc_k, \tag{5.2}$$

with c_k as in Theorems 5.1 and 5.3, then $Wf(z)$ has no real zeros.

Proof. We use Corollary 5.4 with $m_1 = \cdots = m_d = k$; then $m_1 + \cdots + m_d = d[m/d] \leqslant m$. By definition of D, the distance between any u_i and any real zero of $f(z)$ is at most D. On the other hand, we have $v \leqslant v_j$ for $j=1, ..., d$. Hence (5.2) guarantees that the hypotheses of Corollary 5.4 are satisfied. It follows that $Wf(z)$ cannot have any real zeros.

A result very similar to Corollary 5.5 and stronger in certain cases can be derived as a direct consequence of Theorem 2.6.

COROLLARY 5.6. *Let $u_j \pm iv_j$ and v be as in Corollary 5.5. Let $\tilde{D}$ be the smallest number such that all m real zeros and all d Jensen circles of $f(z)$ lie in the region $b \leqslant \mathrm{Re}(z) \leqslant b + \tilde{D}$ for some b. If*

$$\tilde{D} < v\sqrt{m/2d}, \tag{5.3}$$

then $Wf(z)$ has no real zeros.

Proof. By the definition of $\tilde{D}$ and by Theorem 2.4, the distance between a possible real zero r of $Wf(z)$ and any real zero α_j of $f(z)$ is at most $\tilde{D}$. If $Wf(z)$ had a real zero r, we would get

$$R(r)^{-2} = \sum_{j=1}^{m} (r-\alpha_j)^{-2} \geqslant \sum_{j=1}^{m} \tilde{D}^{-2} = m\tilde{D}^{-2},$$

or

$$R(r) \leqslant \tilde{D}/\sqrt{m}.$$

Hence by Theorem 2.6 and Remark (2) following it, at least one complex zero of $f(z)$ would be no further than $\sqrt{2d}\, R(r) \leqslant \tilde{D}\sqrt{2d/m}$ from the real axis. But this is a contradiction to (5.3). Hence such a real zero r of $Wf(z)$ cannot exist.

Remarks. (1) If $f(z)$ has a real zero located "to the left" and one "to the right" of all Jensen circles of $f(z)$, then $\tilde{D} = D$. In this case it is obvious that Corollary 5.6 is stronger than Corollary 5.5 since

$$\tfrac{1}{2}(\sqrt{9k^2-8k}-k)^{1/2} < \tfrac{1}{2}\sqrt{2k} \leqslant \sqrt{m/2d}.$$

However, for large $k = [m/d]$, the left and the right term of the above inequalities are asymptotically equal.

(2) Since $\tilde{D} \geqslant 2\,|v_j|$ for all nonreal zeros $u_j \pm iv_j$ of $f(z)$, (5.3) cannot hold unless $\sqrt{m/2d} > 2$. Hence Corollary 5.6 is vacuous unless the number of real zeros of $f(z)$ is more than four times the number of complex zeros. Corollary 5.5, on the other hand, requires only half that number, namely $m \geqslant d$.

6. An Analogue of Walsh's Two-Circle Theorem

The polynomial $f(z) := (z-a)^m (z-b)^n$, a, $b \in \mathbb{C}$, has critical points at $z = a$ and $z = b$ (of order $m-1$ and $n-1$, respectively) and a simple critical point at $z_0 := (mb+na)/(m+n)$, i.e., located on the straight line segment between a and b. The Two-Circle Theorem of Walsh (see, e.g., [5, p. 89] or [7, p. 13]) now states that if $f(z)$ has m zeros in the closed interior of a circle C_1 with center a and n zeros in the closed interior of a circle C_2 with center b, then each of the $m+n-1$ critical points of $f(z)$ lies either in the closed interiors of C_1 or C_2, or inside or on a circle around z_0 with radius a simple function of m, n, and the radii of C_1 and C_2.

In this section we see that the Wronskian zeros have very similar properties. We begin with an example which is generalized by the following theorem.

Example 6.1. Let $f(z) := (z-a)^m (z-b)^n$, a, $b \in \mathbb{C}$. Then Lemma 2.1 gives

$$Wf(z) = -(z-a)^{2m-2}(z-b)^{2n-2}\,[m(z-b)^2 + n(z-a)^2].$$

The nontrivial zeros of $Wf(z)$ are the zeros of the term in brackets, i.e., the roots of

$$z^2 - 2\frac{mb+na}{m+n}z + \frac{mb^2+na^2}{m+n} = 0,$$

namely

$$z_0 = \frac{1}{m+n}\,[mb+na \pm (a-b)\sqrt{mn}]. \tag{6.1}$$

Note that for all choices of m and n, the zeros z_0 lie on the circle $|z-(a+b)/2| = |(b-a)/2|$, i.e., the circle that has the straight line segment connecting a and b as a diameter.

The following result is the Wronskian analogue to Walsh's Two-Circle Theorem.

THEOREM 6.2. *Suppose that the polynomial f of degree $m+n$ has m zeros in the closed disk D_1 with center $a \in \mathbb{C}$ and radius r and n zeros in the closed disk D_2 with center $b \in \mathbb{C}$ and radius s. Then each zero of Wf lies in the closed disk D'_1 with center a and radius $2r$, or in D'_2 with center b and radius $2s$, or in the disks D_3 or D_4 with centers c, d given by the right-hand side of* (6.1) *and common radius*

$$R = \frac{r\sqrt{n} + s\sqrt{m}}{\sqrt{m+n}}.$$

Before proving this result, we derive several corollaries. The first one is an obvious special case.

COROLLARY 6.3. *If in Theorem* 6.2 *we have $m = n$, then*

$$c = \frac{a+b}{2} + i\frac{a-b}{2}, \qquad d = \frac{a+b}{2} - i\frac{a-b}{2},$$

and

$$R = \tfrac{1}{2}\sqrt{2}\,(r+s).$$

The next corollary and its proof are analogous to the statement and proof of Corollary (19, 1) in [5].

COROLLARY 6.4. *If the closed disks D_1, D_2, D_3, and D_4 of Theorem* 6.2 *are pairwise disjoint then they contain* $2m-2$, $2n-2$, 1, *and* 1 *zeros of $Wf(z)$, respectively.*

Proof. We let the m zeros of $f(z)$ in D_1 approach a single point in D_1 (say, a) along regular paths entirely in D_1, and similarly we let the n zeros in D_2 approach a point in D_2 (say, b). Now by Example 6.1, a and b are zeros of $Wf(z)$ of multiplicities $2m-2$ and $2n-2$, respectively, and the remaining two zeros are located at c and d, given by (6.1). By Theorem 6.2 no zero of $Wf(z)$ can leave the disks $D_1, \ldots, D_4$ during this process since the disks are pairwise disjoint. Hence the number of zeros in $D_1, \ldots, D_4$ was also originally $2m-2$, $2n-2$, 1, and 1.

We apply now the preceding corollaries to the main topic of this paper.

COROLLARY 6.5. *Let f be a polynomial of degree $2n$ with real coefficients and such that n zeros lie on a closed disk centered at $u+iv$ $(u, v \in \mathbb{R})$ with radius $r < |v|/2$. Then $Wf(z)$ has exactly two real zeros. They are located in the intervals $[u-|v|-r\sqrt{2}, u-|v|+r\sqrt{2}]$, resp. $[u+|v|-r\sqrt{2}, u+|v|+r\sqrt{2}]$.*

Proof. By Corollary 6.3 with $a := u+i|v|$, $b := u-i|v|$, and $r=s$ we have $c=u-|v|$, $d=u+|v|$ and $R=r\sqrt{2}$. It is now easy to verify that $D_1, ..., D_4$ are pairwise disjoint. Since $f(z)$ has only real coefficients then so does $Wf(z)$, and therefore the single zeros in the disks D_3 and D_4 must be real and consequently lie in the intervals of the statement.

The Two-Circle Theorem mentioned earlier in this section is usually proved by way of the Coincidence Lemma of J. L. Walsh (see, e.g., [5, p. 62]). Similarly, we use here a generalization of the Coincidence Lemma due to Boese [1]; the following is a special case of the much more general result in [1].

LEMMA 6.6. *Let $a_k \in \mathbb{C}$ and $A_k > 0$, $1 \leqslant k \leqslant n$, be such that $A_1 + \cdots + A_n = 1$ and $|a_k| \leqslant r$ for $1 \leqslant k \leqslant n$. Then among the $a = a(z)$ satisfying*

$$\sum_{k=1}^{n} A_k(z-a_k)^{-2} = (z-a)^{-2}$$

for $|z| > 2r$ there is one for which $|a(z)| \leqslant r$.

Proof of Theorem 6.2. Let $a_1, ..., a_m$ be the zeros of f located in D_1, and $b_1, ..., b_n$ be those in D_2. Consider

$$F(z) := \sum_{j=1}^{m} (z-a_j)^{-2} + \sum_{j=1}^{n} (z-b_j)^{-2};$$

note that by Lemma 2.2 we have $F(z) = -Wf(z)/(f(z))^2$, so that it suffices to consider the zeros of $F(z)$. If z lies outside the union of D_1' and D_2' then by Lemma 6.6 (with $A_j = 1/m$, resp. $1/n$) there is an $a' = a'(z) \in D_1$ and $b' = b'(z) \in D_2$ such that

$$F(z) = \frac{m}{(z-a')^2} + \frac{n}{(z-b')^2} = \frac{m(z-b')^2 + n(z-a')^2}{(z-a')^2\,(z-b')^2}.$$

In Example 6.1 we already saw that $F(z) = 0$ only when z is as in (6.1), with a and b replaced by a' and b', or

$$z = \frac{n \pm i\sqrt{mn}}{m+n}\,a' + \frac{m \mp i\sqrt{mn}}{m+n}\,b',$$

where it is understood that a' and b' vary over the disks D_1 and D_2, respectively. Then by Lemma (17, 2a) of [5], z lies in one of the disks with centers given by (6.1) and radius

$$R=\left|\frac{n\pm i\sqrt{mn}}{m+n}\right| r+\left|\frac{m+\sqrt{mn}}{m+n}\right| s=\frac{r\sqrt{n}+s\sqrt{m}}{\sqrt{m+n}};$$

this proves the theorem.

Remark. The Wronskian zeros of polynomials with real coefficients and three or four (or in special cases more) distinct zeros are still relatively easy to determine explicitly. Hence Lemma 6.6 can be used to prove results analogous to Theorem 6.2 and Corollary 6.5, for polynomials whose zeros lie in three, four, or more circular disks. In this way the conjecture (1.2) can be verified for further classes of polynomials.

Acknowledgment

I thank K. B. Stolarsky for his encouragement and for useful discussions concerning this work.

References

1. G. Boese, Verallgemeinerung eines Lemmas von J. L. Walsh nebst einigen Anwendungen, *Z. Angew. Math. Mech.* **59** (1979), 625–635.
2. J. B. Conrey and L. A. Rubel, On the location of the zeros of the derivative of a polynomial, *Proc. Amer. Math. Soc.* **86** (1982), 37–41; Erratum, *Proc. Amer. Math. Soc.* **104** (1988), 1002.
3. T. Craven, G. Csordas, and W. Smith, The zeros of derivatives of entire functions and the Pólya-Wiman Conjecture, *Ann. of Math.* **125** (1987), 405–431.
4. K. Dilcher and K. B. Stolarsky, Zeros of the Wronskian of a polynomial, *J. Math. Anal. Appl.*, (to appear).
5. M. Marden, "Geometry of Polynomials," American Mathematical Society, Providence, RI, 1966.
6. H. B. Mitchell, On the imaginary roots of a polynomial and the real roots of its derivative, *Trans. Amer. Math. Soc.* **19** (1918), 43–52.
7. J. L. Walsh, "The Location of Critical Points of Analytic and Harmonic Functions," American Mathematical Society, Providence, RI, 1950.