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Note

## On generalized Kneser hypergraph colorings

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**Abstract**

In 2002, the second author presented a lower bound for the chromatic numbers of hypergraphs  $KG_s^r \mathcal{S}$ , “generalized  $r$ -uniform Kneser hypergraphs with intersection multiplicities  $s$ .” It generalized previous lower bounds by Kříž (1992/2000) for the case  $s = (1, \dots, 1)$  without intersection multiplicities, and by Sarkaria (1990) for  $\mathcal{S} = \binom{[n]}{k}$ . Here we discuss subtleties and difficulties that arise for intersection multiplicities  $s_i > 1$ :

- (1) In the presence of intersection multiplicities, there are two different versions of a “Kneser hypergraph,” depending on whether one admits hypergraph edges that are multisets rather than sets. We show that the chromatic numbers are substantially different for the two concepts of hypergraphs. The lower bounds of Sarkaria (1990) and Ziegler (2002) apply only to the multiset version.
- (2) The reductions to the case of prime  $r$  in the proofs by Sarkaria and by Ziegler work only if the intersection multiplicities are strictly smaller than the largest prime factor of  $r$ . Currently we have no valid proof for the lower bound result in the other cases.

We also show that all uniform hypergraphs without multiset edges can be represented as generalized Kneser hypergraphs.

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### 1. Introduction

The “generalized Kneser hypergraphs with intersection multiplicities,” as studied in [10], arose from the graphs (implicitly) studied by Kneser [4] in several subsequent generalization steps. They do form a large class of hypergraphs—indeed, we will show in Section 3 that every uniform hypergraph without multiplicities can be represented in this model.

When writing [10], the second author overlooked that the edges of the generalized Kneser hypergraph  $KG_s^r \mathcal{S}$  with intersection multiplicities  $s = (s_1, \dots, s_n)$  could be multisets if  $s_i > 1$ : Their edges can have repeated elements, as pointed out in [7]. Thus  $KG_s^r \mathcal{S}$  is not a hypergraph in the traditional sense of Berge [2], where the edges have to be sets. If one does not allow for repeated elements in the edges, then this yields a sub-hypergraph without multiplicities,

$$kg_s^r(\mathcal{S}) \subseteq KG_s^r(\mathcal{S}).$$

Both for  $kg_s^r(\mathcal{S})$  and for  $KG_s^r(\mathcal{S})$  we are faced with the problem to determine the chromatic number: How many colors are needed for the sets in  $\mathcal{S}$  if monochromatic hypergraph edges are forbidden? Clearly we have  $\chi(kg_s^r \mathcal{S}) \leq \chi(KG_s^r \mathcal{S})$ , but the two values can be far apart, as we will see below.

In this note, we discuss the chromatic numbers of generalized Kneser hypergraphs with intersection multiplicities, in view of some main topics and results from [10]. This includes errata and clarifications announced in [11]:

[10, Lemma 3.1] described an explicit coloring for the special case of  $\mathcal{S} = \binom{[n]}{k}$  and constant  $s = (s, \dots, s)$ . This coloring is valid for generalized Kneser hypergraphs *with* multiplicities, which yields

$$\chi\left(kg_s^r\left(\binom{[n]}{k}\right)\right) \leq \chi\left(KG_s^r\left(\binom{[n]}{k}\right)\right) \leq 1 + \left\lceil \frac{1}{\lfloor \frac{r-1}{s} \rfloor} \frac{ns - rk + 1}{s} \right\rceil.$$

[10, Theorem 5.1] states a lower bound

$$\left\lceil \frac{1}{r-1} cd_s^r \mathcal{S} \right\rceil \leq \chi(KG_s^r \mathcal{S})$$

for the generalized Kneser hypergraphs. This lower bound holds for generalized Kneser hypergraphs *with* multiplicities and is not valid for  $\chi(kg_s^r \mathcal{S})$  as we will see in Section 4. Moreover, Karsten Vogel (Magdeburg) has noticed that the reduction to the prime case in the proof of [10, Theorem 5.1] fails when the intersection multiplicities are not smaller than the prime factors of  $r$ . As we will analyze in Section 5, we get only the following theorem (with a combinatorial proof):

**Theorem 1.1.** *Let  $\mathcal{S} \subseteq 2^{[n]}$  be a set family, and let the intersection multiplicities  $s_i \geq 1$  be smaller than the largest prime factor of  $r \geq 2$ . Then*

$$\chi(KG_s^r \mathcal{S}) \geq \left\lceil \frac{1}{r-1} cd_s^r \mathcal{S} \right\rceil.$$

In particular, the conditions of this theorem are satisfied

- if there are no intersection multiplicities,  $s = (1, \dots, 1)$ . In this case  $kg_{(1, \dots, 1)}^r \mathcal{S} = KG_{(1, \dots, 1)}^r \mathcal{S}$ , and Theorem 1.1 reduces to the main result of Kříž [5,6], and
- in the case when  $r$  is prime, for arbitrary  $s_i < r$ .

We still believe that the theorem is valid for arbitrary  $s_i < r$ , as stated by [10, Theorem 5.1], but we have no proof for this generality—not even for the “complete uniform” case when  $\mathcal{S} = \binom{[n]}{k}$ ; indeed, we will see in Section 5 that the induction proof by Sarkaria [9] for this case is not valid, either.

[10, Example 7.2] analyzed some Kneser hypergraphs *without* multiplicities, including a computation of  $\chi(\text{kg}_{(2,\dots,2)}^4 \binom{[n]}{2})$ . Thus in Section 4 we discuss families of hypergraphs that include  $\text{KG}_{(2,\dots,2)}^4 \binom{[n]}{2}$ . We also collect evidence towards the conjecture that the upper bound  $\chi(\text{KG}_{(s,\dots,s)}^r \binom{[n]}{k}) \leq 1 + \lceil \frac{1}{\lfloor \frac{r-1}{s} \rfloor} \frac{ns-rk+1}{s} \rceil$  is tight in general.

## 2. Preliminaries

In this section we review the fundamental concepts for this study; compare [10, Section 2]. Let  $n \geq 1$  and denote  $[n] := \{1, \dots, n\}$ . By  $s$  we denote a vector of positive integers  $s = (s_1, \dots, s_n)$ . Throughout  $r \geq 2$  denotes an integer. We write  $s < r$  if  $s_i < r$  for all  $i$ .

**Definition 2.1** (*s-disjoint sets*). Subsets  $S_1, \dots, S_r$  of  $[n]$  are *s-disjoint* if each  $i \in [n]$  occurs in at most  $s_i$  of the sets  $S_k$ . Note that here equalities  $S_k = S_\ell$  are possible.

To illustrate this definition consider  $n = 3$  and  $s = (3, 2, 1)$ . The subsets  $\{1, 2\}$ ,  $\{1, 2\}$ , and  $\{2, 3\}$  are not *s-disjoint* because 2 occurs in all three sets. On the other hand,  $\{1, 2\}$ ,  $\{1, 2\}$ , and  $\{1, 3\}$  are *s-disjoint*, and so are  $\{1, 2\}$ ,  $\{1, 3\}$ , and  $\{2\}$ .

**Definition 2.2** (*s-disjoint r-colorability defect [10, p. 673]*). Let  $[n]^s$  denote the multiset in which the element  $i \in [n]$  occurs with multiplicity  $s_i$ . We denote the cardinality of  $[n]^s$  counting multiplicities by  $\bar{n}$ .

The *s-disjoint r-colorability defect*  $\text{cd}_s^r \mathcal{S}$  of a set  $\mathcal{S} \subseteq 2^{[n]}$  is the minimal number of elements one has to remove from the multiset  $[n]^s$  such that the remaining multiset can be covered by  $r$  subsets of  $[n]$  that do not contain any element from  $\mathcal{S}$ :

$$\text{cd}_s^r \mathcal{S} = \bar{n} - \max \left\{ \sum_{j=1}^r |R_j| \mid \begin{array}{l} R_1, \dots, R_r \subseteq [n] \text{ s-disjoint subsets} \\ \text{and } S \not\subseteq R_j \text{ for all } S \in \mathcal{S} \text{ and all } j \end{array} \right\}.$$

Here the sets  $R_\ell$  need not be distinct, and they may be empty. Note for further reference that  $\text{cd}_s^r \emptyset > 0$  if  $s_i > r$  for some  $i$ .

As example we consider  $n = 3$ ,  $s = (3, 2, 1)$ , and  $\mathcal{S} = \{S_1\}$  where  $S_1 = \{2, 3\}$ . We are therefore never allowed to pick  $\{1, 2, 3\}$  or  $\{2, 3\}$  as one of the sets  $R_j$ . For  $r = 1$ ,  $R_1 = \{1, 2\}$  is a possible choice of largest cardinality, thus  $\text{cd}_{(3,2,1)}^1 \mathcal{S} = 6 - 2 = 4$ . For  $r = 2$ , we may pick  $R_1 = \{1, 2\}$  and  $R_2 = \{1, 3\}$  as examples that maximize  $|R_1| + |R_2|$ , hence  $\text{cd}_{(3,2,1)}^2 \mathcal{S} = 2$ . For  $r = 3$ , the value  $|R_1| + |R_2| + |R_3|$  is maximized by  $R_1 = \{1, 2\}$ ,  $R_2 = \{1, 2\}$ , and  $R_3 = \{1, 3\}$ , so  $\text{cd}_{(3,2,1)}^3 \mathcal{S} = 0$ .

**Definition 2.3** (*r-uniform hypergraphs with/without multiplicities*). An *r-multisubset*  $X$  of  $[n]$  is an unordered collection of  $r$  elements  $x_1, \dots, x_r$  of  $[n]$  that need not be distinct. We denote it by  $X = \{\{x_1, \dots, x_r\}\}$ .

An  $r$ -uniform hypergraph in the sense of Berge [2] (without multiplicities) is a pair  $(V, E)$  that consists of a finite set of vertices  $V$  and a set  $E$  of edges, which are  $r$ -subsets of  $V$ .

An  $r$ -uniform hypergraph with multiplicities is a pair  $(V, E)$  that consists of a finite set of vertices  $V$  and a set  $E$  of edges, which are  $r$ -multisubsets of  $V$ .

A hypergraph (with multiplicities) is loop-free if every edge has at least two distinct elements (vertices). In the following, all hypergraphs are supposed to be loop-free.

According to this definition, hypergraphs do not have multiple edges; this makes sense for our purposes, since multiple edges are irrelevant for coloring.

There are two loop-free  $r$ -uniform hypergraph analogs of the complete graph  $K_n$  on  $n$  vertices, which we denote by  $K_n^r$  and  $k_n^r$ . The vertex set of  $K_n^r$  is  $[n]$ , and the edges are all the  $r$ -multisubsets of  $[n]$  that contain at least two distinct vertices. Thus  $K_n^r$  has  $\binom{n}{r} - n = \binom{n+r-1}{r} - n$  edges. The analogous complete  $r$ -uniform hypergraph  $k_n^r$  without multiplicities has  $\binom{n}{r}$  edges.

**Definition 2.4** ( $r$ -uniform  $s$ -disjoint Kneser hypergraphs). For any finite set  $\mathcal{S} = \{S_1, \dots, S_m\}$  of non-empty subsets of  $[n]$ , the  $r$ -uniform  $s$ -disjoint Kneser hypergraph  $\text{KG}_s^r \mathcal{S}$  with multiplicities has the vertex set  $\mathcal{S}$  and the edge set

$$E(\text{KG}_s^r \mathcal{S}) := \{\text{all } s\text{-disjoint } r\text{-multisets whose elements are sets } S_i \in \mathcal{S}\}.$$

If  $s_i < r$  for all  $i \in [n]$  then  $\text{KG}_s^r \mathcal{S}$  is loop-free.

The  $r$ -uniform  $s$ -disjoint Kneser hypergraph  $\text{kg}_s^r \mathcal{S}$  without multiplicities has the same vertex set  $\mathcal{S}$ , but all of its edges are sets rather than multisets:

$$E(\text{kg}_s^r \mathcal{S}) := \{\text{all } s\text{-disjoint } r\text{-subsets of } \mathcal{S}\}.$$

The generalized Kneser hypergraphs  $\text{kg}_s^r \mathcal{S}$  are loop-free for any  $s$ .

We use  $\text{KG}_s^r \mathcal{S}$  as a shorthand for  $\text{KG}_{(s, \dots, s)}^r \mathcal{S}$  in the case of constant intersection multiplicity  $s = (s, \dots, s)$ , and similarly we write  $\text{kg}_s^r \mathcal{S}$  and  $\text{cd}_s^r \mathcal{S}$ .

The previously defined complete  $r$ -uniform hypergraphs  $K_n^r$  and  $k_n^r$  are examples of  $r$ -uniform  $s$ -disjoint Kneser hypergraphs. We have  $K_n^r = \text{KG}_{r-1}^r \binom{[n]}{1}$ ,  $k_n^r = \text{kg}_{r-1}^r \binom{[n]}{1}$ , and in this particular situation  $\text{KG}_1^r \binom{[n]}{1} = \text{kg}_{r-1}^r \binom{[n]}{1}$ .

We can obtain  $\text{kg}_s^r \mathcal{S}$  from  $\text{KG}_s^r \mathcal{S}$  by discarding edges. In this sense,  $\text{kg}_s^r \mathcal{S}$  is a subhypergraph of  $\text{KG}_s^r \mathcal{S}$ . In the special case that  $s_i \equiv 1$  we have  $\text{KG}_s^r \mathcal{S} = \text{kg}_s^r \mathcal{S}$  since pairwise disjoint non-empty sets are distinct. In particular, for  $r = 2$  and  $s_i \equiv 1$  both definitions specialize to the generalized Kneser graph of  $\mathcal{S} \subseteq 2^{[n]}$ .

**Definition 2.5** (Hypergraph colorings [3]). A coloring of an  $r$ -uniform hypergraph  $H$  (multiplicity-free or not) with  $m$  colors is a map  $c : V(H) \rightarrow [m]$  that assigns to each vertex of  $H$  a color such that no edge is monochromatic, that is, for each  $e \in E(H)$  we have  $|\{c(x) \mid x \in e\}| \geq 2$ . Any coloring  $c$  of  $H$  by  $m$  colors induces a homomorphism  $H \rightarrow K_m^r$  of hypergraphs. The chromatic number  $\chi(H)$  is the smallest number  $m$  such that there is a coloring of  $H$  with  $m$  colors.

### 3. How general are generalized Kneser hypergraphs?

Matoušek and Ziegler [8, p. 76] observed that every (finite, simple) graph can be represented as a Kneser graph: For any  $G = (V, E)$  there is a set system  $\mathcal{S} = \{S_v \mid v \in V\} \subseteq 2^{[m]}$ , for some  $m$ ,

such that  $S_v \cap S_w = \emptyset$  if and only if  $\{v, w\} \in E$ . Thus it is natural to ask which hypergraphs (with or without multiplicities) can be represented as generalized Kneser hypergraphs. The following proposition collects our answers to this question.

For this, call a hypergraph  $H([n], E)$  *up-monotone* if for  $e, e' \in \binom{[n]}{r}$  with  $e \in E$  we also have  $e' \in E$  whenever the support of  $e'$  contains that of  $e$ . Every  $r$ -uniform hypergraph without multiplicities is up-monotone, as is every generalized Kneser hypergraph  $\text{KG}_{r-1}^r \mathcal{S}$ .

A hypergraph  $H = ([n], E)$  is *convex* if every integral weight vector  $(a_1, \dots, a_n)$  in the convex hull of multiplicity vectors of edges of  $H$  (thus  $0 \leq a_i < r$ ) is the multiplicity vector of an edge of  $H$ .

**Proposition 3.1.**

- (1) *There are  $r$ -uniform hypergraphs without multiplicities that cannot be represented as a Kneser hypergraph  $\text{KG}_1^r \mathcal{S}$ .*
- (2) *An  $r$ -uniform hypergraph  $H = ([n], E)$  with multiplicities can be represented as  $\text{KG}_{r-1}^r \mathcal{S}$  if and only if it is up-monotone.*  
*In particular, every  $r$ -uniform hypergraph without multiplicities can be represented as a Kneser hypergraph  $\text{KG}_{r-1}^r \mathcal{S}$ .*
- (3) *If an  $r$ -uniform hypergraph is representable by a generalized Kneser hypergraph with intersection multiplicities then it is convex. (The converse is not true.)*  
*In particular, there are  $r$ -uniform hypergraphs with multiplicities that cannot be represented as a Kneser hypergraph  $\text{KG}_s^r \mathcal{S}$ .*

**Proof.** (1) Consider  $([4], \{124, 134, 234\})$ . If  $\text{KG}_1^3 \{S_1, \dots, S_4\}$  has  $\{S_1, S_2, S_4\}$ ,  $\{S_1, S_3, S_4\}$  and  $\{S_2, S_3, S_4\}$  as edges, then each of the triples of sets is pairwise disjoint, so in particular  $S_1, S_2, S_3$  are pairwise disjoint. Thus also  $\{S_1, S_2, S_3\}$  is an edge in the Kneser hypergraph.

(2) The following construction generalizes the construction for graphs in [8]. Let  $H = ([n], E)$  be up-monotone, and let  $\bar{H} = ([n], \bar{E})$  be the complementary hypergraph of  $H$ , i.e. the hypergraph that has the same vertices as  $H$  and all edges of  $K_n^r$  that are not edges of  $H$ . Define the set system  $\mathcal{S} = \{S_i \mid i \in [n]\}$  by

$$S_i := \{i\} \cup \{\bar{e} \in \bar{E} \mid i \in \bar{e}\}.$$

The  $S_i$  are clearly distinct. If  $e = \{i_1, \dots, i_r\}$  is an edge of  $H$ , then

$$S_{i_1} \cap \dots \cap S_{i_r} = \{i_1\} \cap \dots \cap \{i_r\} \cap \{\bar{e} \in \bar{E} \mid i_1, \dots, i_r \in \bar{e}\},$$

where the first part is empty since  $H$  does not have loops (so the  $i_k$  cannot all be equal) and the last set is empty since  $H$  is up-monotone. Thus  $S_{i_1} \cap \dots \cap S_{i_r}$  is an edge of the Kneser hypergraph. Conversely, if  $S_{i_1} \cap \dots \cap S_{i_r} = \emptyset$ , then in particular it does not contain the element  $e = \{i_1, \dots, i_r\}$ , so  $e \in E$ .

(3) The intersection multiplicities  $s_i$  define the hypergraph  $\text{KG}_s^r \mathcal{S}$  as a subgraph of  $K_m^r$ , for  $m = |\mathcal{S}|$ , by linear conditions on the multiplicity vectors of the edges.

For an example consider  $([3], \{113, 223\})$ . If  $\text{KG}_s^3 \{S_1, S_2, S_3\}$ , with  $S_1, S_2, S_3 \subseteq [m]$ , does not have  $\{S_1, S_2, S_3\}$  as an edge, then there is some  $i \in [m]$  such that  $S_1, S_2, S_3$  contain  $i$  more than  $s_i$  times. However, that cannot be if both  $\{\{S_1, S_1, S_3\}\}$  and  $\{\{S_2, S_2, S_3\}\}$  are edges, so  $S_1, S_1, S_3$  and  $S_2, S_2, S_3$  contain  $i$  at most  $s_i$  times.

An example of a convex uniform hypergraph with multiplicities that cannot be represented as a generalized Kneser hypergraph is  $([3], \{112, 223\})$ . □

For the purpose of coloring, any hypergraph  $H = (V, E)$  with multiplicities can be replaced by an up-monotone uniform hypergraph with multiplicities, on the same ground set, and with the same chromatic number: For this replace each edge  $e$  by all multisets of cardinality  $r$  which contain the support of  $e$ , for some large enough  $r$ . By Proposition 3.1(2), the resulting  $r$ -uniform hypergraph with multiplicities  $H'$  can be represented as a generalized Kneser graph, which yields topological lower bounds for  $\chi(H)$  in terms of the colorability defect of  $H'$ . In particular, this applies to (non-uniform) hypergraphs in the sense of Berge.

#### 4. Two counterexamples

The purpose of this section will be to show that the lower bound  $\chi(\text{KG}_s^r \mathcal{S}) \geq \lceil \frac{1}{r-1} \text{cd}_s^r \mathcal{S} \rceil$  of Theorem 1.1 is not valid for  $\text{kg}_s^r \mathcal{S}$ . Indeed, the proof for [10, Theorem 5.1] is valid only if multiplicities are included: The argument at [10, p. 679] yields  $p$  subsets  $S_1, \dots, S_p$  of  $[n]$  that are  $s$ -disjoint, but they need not be pairwise different.

**Example 4.1.** For  $n \geq 5$  and  $r \geq 4$  with  $n \geq r - 1$ , let  $\mathcal{S} := \{12, 13, \dots, 1n, 23, 45\} \subset \binom{[n]}{2}$ . Then all edges of  $\text{kg}_{r-2}^r \mathcal{S}$  are of the form  $\{1i_1, \dots, 1i_{r-2}, 23, 45\}$ , so they contain both 23 and 45. Thus  $\chi(\text{kg}_{r-2}^r \mathcal{S}) = 2$ . A straightforward argument (cf. [7, p. 83] for details) shows that  $\text{cd}_{r-2}^r \mathcal{S} = 3r - 10$ . For  $r > 8$  this yields  $\text{cd}_{r-2}^r \mathcal{S} > (r - 1)\chi(\text{kg}_{r-2}^r \mathcal{S})$ .

The next example shows that for Kneser hypergraphs without multiplicities, the colorability defect lower bound does not even hold in the special case of  $\mathcal{S} = \binom{[n]}{k}$ . (Sarkaria [9] speaks of “ $p$ -tuples of  $S$ -subsets,” so his treatment clearly concerns the Kneser hypergraphs  $\text{KG}_s^r \binom{[n]}{k}$  with multisets as hypergraph edges.)

**Example 4.2.** For  $n, r \geq 4$  the hypergraph  $\text{kg}_{r-1}^r \binom{[n]}{2}$  has a greedy  $(n - 2)$ -coloring, by  $c : S \mapsto \min\{\min S, n - 2\}$ . (The hypergraph is non-empty if  $r \leq \binom{n}{2}$ . Its chromatic number will be computed in Example 6.2.)

On the other hand,  $\text{cd}_{r-1}^r \binom{[n]}{2} = \max\{n(r - 1) - r, 0\} = (n - 1)(r - 1) - 1$  by [10, Lemma 3.2]. Thus

$$(r - 1)\chi\left(\text{kg}_{r-1}^r \binom{[n]}{2}\right) < \text{cd}_{r-1}^r \binom{[n]}{2}.$$

#### 5. The induction to non-prime cases

For the case of Theorem 1.1 when  $p$  is a prime Ziegler [10] has given a combinatorial proof; an alternative topological proof was given by Lange [7, Section 4.4]. The special case when  $\mathcal{S} = \binom{[n]}{k}$  is due to Sarkaria [9].

In this section, we show that the reductions of the situation with general  $r$  to the case of prime  $r$  by Sarkaria [9] and by Ziegler [10] are both incomplete. We also argue that argument given in [10] suffices to establish the result in the generality of Theorem 1.1.

Sarkaria’s proof [9, (3.2)] starts with the assumption that  $\text{KG}_{j-1}^p \binom{[N]}{S}$  has an  $M$ -coloring, with  $(p - 1)M < N(j - 1) - p(S - 1) = \text{cd}_{j-1}^p \binom{[N]}{S}$ , for some non-prime  $p = p_1 p_2$ . Then one constructs a coloring of  $\text{KG}_{j-1}^{p_1} \binom{[N']}{N'}$  with

$$N' := M(p_2 - 1) + p_2(S - 1) + 1,$$

and tries to get a monochromatic  $(j - 1)$ -disjoint  $p_1$ -tuple of  $N'$ -subsets of  $[N]$ . The argument fails if  $N'$  is larger than  $N$ , so there will not be any  $N'$ -subsets of  $[N]$ . Concrete parameters where this happens are  $p = 4, p_1 = p_2 = 2, j = 4, S = 2, M = N - 2$ , which yields  $N' = M + 3 = N + 1$ . (The problem does not occur for  $j \leq 3$ , so in particular the proof specializes correctly to the case  $j = 2$  treated by Alon, Frankl and Lovász [1].)

Ziegler’s reduction to the prime case in [10, pp. 679–680] is an extension of Kříž’ proof [6], which in turn generalizes the argument of Alon, Frankl and Lovász. Let  $r = r'r''$  with  $r' \leq r''$ . The goal is to derive a contradiction if we assume that  $cd_s^r \mathcal{S} > (r - 1)\chi(\text{KG}_s^r \mathcal{S})$ . A crucial ingredient is the set

$$\mathcal{T} := \{N \subseteq [n] \mid cd_1^{r'} \mathcal{S}|_N > (r' - 1)\chi(\text{KG}_s^{r'} \mathcal{S})\},$$

where  $\mathcal{S}|_N$  denotes the elements of  $\mathcal{S}$  that are subsets of  $N$ . One then wants to argue that

$$(r'' - 1)\chi(\text{KG}_s^{r''} \mathcal{T}) \geq cd_s^{r''} \mathcal{T}.$$

But this can be concluded by induction only if  $s < r''$ . Moreover, it definitely fails if  $s_i > r''$  for some  $i$  and  $\mathcal{T} = \emptyset$ : In this situation  $\chi(\text{KG}_s^{r''} \emptyset) = 0$  since there are no vertices to color, but  $cd_s^{r''} \emptyset > 0$  since at least  $s_i - r''$  elements have to be removed from  $[n]^s$  to cover the remaining elements with  $r''$  subsets of  $[n]$ . The case  $\mathcal{T} = \emptyset$  can occur, as we have seen above for the special case of  $\mathcal{S} = \binom{[N]}{S}$ .

Thus, [10, Theorem 5.1] can currently only be established in the generality given above as Theorem 1.1. To establish this, one uses the induction given at [10, pp. 679–680], factoring non-prime  $r = r'r''$  so that  $r''$  is the largest prime number that divides  $r$ .

### 6. More examples

In [10, Section 7] the second author had raised the question whether the upper bound of [10, Lemma 3.1]

$$\chi(\text{KG}_s^r) \binom{[n]}{k} \leq 1 + \left\lceil \frac{1}{\lfloor \frac{r-1}{s} \rfloor} \frac{ns - rk + 1}{s} \right\rceil \tag{*}$$

is always tight, for  $n \geq k \geq 2, r > s \geq 2, rk \leq sn$ . In [10, Example 7.2] he had claimed that (\*) is not sharp for  $\text{KG}_2^4 \binom{[n]}{2}$ . However, this is not true: The analysis given there referred to the corresponding Kneser hypergraph without multiplicities, that is, it established that

$$\chi\left(\text{kg}_2^4 \binom{[n]}{2}\right) = n - \left\lfloor \sqrt{2n + \frac{1}{4} - \frac{1}{2}} \right\rfloor.$$

Thus the tightness question is open for now. By Theorem 1.1, (\*) is tight if  $s$  is smaller than the largest prime factor of  $r$ , and divides  $r - 1$  (cf. [10, Corollary 7.1]). The following example yields more cases where (\*) is tight, including the case of  $\text{KG}_2^4 \binom{[n]}{2}$ .

**Example 6.1.** Assume that  $k = 2$  and  $\lfloor (r - 1)/s \rfloor = 1$ , i.e.  $r/2 \leq s < r - 1$ . Then

$$\chi\left(\text{KG}_s^r \binom{[n]}{2}\right) = 1 + \left\lceil \frac{ns - 2r + 1}{s} \right\rceil = 1 + n - \left\lfloor \frac{2r - 1}{s} \right\rfloor.$$

Indeed, the vertices of  $H = \text{KG}_s^r \binom{[n]}{2}$  are the edges of a complete graph  $K_n$ . By  $s \geq \lceil r/2 \rceil$ , an edge of  $H$  cannot contain two disjoint edges from  $K_n$ . Thus the edges of  $H$  are supported

only on stars or on triangles—the latter is permitted if  $s \geq 2r/3$ . Thus the possible color classes  $C \subset E(K_n)$  are stars, or they are triangles—the latter is permitted if  $s < 2r/3$ . In either case the greedy colorings that provide the upper bound are optimal:  $n - 1$  colors are needed for  $2r/3 \leq s \leq r - 1$ , while  $n - 2$  colors are needed for  $r/2 \leq s < 2r/3$ .

**Example 6.2.** The Kneser hypergraphs without multiplicities  $kg_{r-1}^r \binom{[n]}{2}$  have chromatic numbers

$$\chi \left( kg_{r-1}^r \binom{[n]}{2} \right) = \begin{cases} \lceil \frac{1}{r-1} \binom{[n]}{2} \rceil, & n < r, \\ n - \lfloor \frac{r}{2} \rfloor, & 2 \leq r \leq n. \end{cases}$$

Indeed, any edge of this hypergraph forms an  $r$ -set of edges in  $K_n$  that is not a star. Thus for a color class we can use any star, or any set of at most  $r - 1$  edges. An optimal coloring in case of  $2 \leq r \leq n$  uses  $n - r$  stars, and then  $\lceil \frac{1}{r-1} \binom{[n]}{2} \rceil = \lceil \frac{r}{2} \rceil$  edge sets of size at most  $r - 1$  to cover the remaining uncolored subgraph  $K_r$ . If  $n < r$ , an optimal coloring uses  $\lceil \frac{1}{r-1} \binom{[n]}{2} \rceil$  sets of size at most  $r - 1$ .

In summary, we see that

$$n - \lfloor \frac{r}{2} \rfloor = \chi \left( kg_{r-1}^r \binom{[n]}{2} \right) \ll \chi \left( KG_{r-1}^r \binom{[n]}{2} \right) = n - 1$$

and

$$n - \lfloor \sqrt{2n + \frac{1}{4}} - \frac{1}{2} \rfloor = \chi \left( kg_2^4 \binom{[n]}{2} \right) \ll \chi \left( KG_2^4 \binom{[n]}{2} \right) = n - 1$$

for sufficiently large  $n$  and  $r$ . This shows a huge difference between the chromatic numbers of generalized Kneser hypergraphs with and without multiplicities.

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