Magnetoelastic analysis of an arbitrary shape inclusion undergoing eigenfields and remote loadings

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A B S T R A C T

Within the framework of the linear theory of magnetoelasticity, the problem of an arbitrary shape inclusion in an entire plane under remotely uniform in-plane electromagnetic and anti-plane mechanical loadings is investigated in this paper. The inclusion is undergoing eigenfields and has different magneto-electro-mechanical moduli as that of the matrix. Using complex variable and Faber series method, the complex potential for each component can be expressed in series form with unknown coefficients. The continuity conditions of the interfaces are used to build up a set of linear equations to determine the unknown coefficients. Through solving these linear equations, one can obtain the complex potentials both inside the inclusion and in the matrix. Numerical results are provided and shown that different combinations of piezoelectric/piezomagnetic exhibit significant magneto-electro-mechanical coupling effect that is not present in single-phase piezoelectric or piezomagnetic material. Applying appropriate remote loadings and eigenfields can effectively reduce stress level and increase magneto-electric response around the inclusion.

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1. Introduction

Due to the capability of the conversion between mechanical, electric and magnetic energy, piezoelectric/piezomagnetic composite materials are widely used to design actuators, sensors and other electronic products. Few natural materials simultaneously possess the magneto-electro-mechanical coupling properties. However, the combination of two or more different components, magneto-electro-mechanical composite materials not only have original piezoelectric and piezomagnetic properties, but also exhibit a remarkable magneto-electro-mechanical coupling effect that is not present in single-phase piezoelectric or piezomagnetic materials. In some cases, the magneto-electric effect of magneto-electro-mechanical composite materials can be even obtained two orders higher than that in a single-phase magneto-electric material. Run et al. (1974) first observed that the magneto-electric coefficients of a BaTiO3–CoFe2O4 composite are two orders higher than the highest known magneto-electric coefficients of a Cr2O3 medium. Extensive studies of the properties of piezoelectric/piezomagnetic composites have been carried out by numerous investigators. Nan (1994) and Huang and Kuo (1997) proposed micromechanics models to estimate the effective properties of fibrous piezoelectric/piezomagnetic composite materials. Li (2000) studied the magneto-electro-mechanical multi-inclusion and inhomogeneity problems and their applications in composite materials. Wu and Huang (2000) presented the closed form solutions for the magneto-electric coupling coefficients in fibrous composites with piezoelectric and piezomagnetic phases. The magneto-electric-coupling effects result from product properties which are absent from each phases of the composite as described by the Nan et al. (2008) in their review. The important contributions to investigate the coupling effect of various magneto-electro-mechanical materials with defects such as dislocations, inclusions, inhomogeneities and cracks can be found in the following works. Li (2002) obtained explicit expressions for the magneto-electric Green’s functions for a transversely isotropic medium and used them to analyze the magneto-electric inclusion and inhomogeneity problems. Fang et al. (2005) discussed the interaction between a generalized screw dislocation and circular–arc interfacial rigid lines in magneto-electro-mechanical solids. Soh and Liu (2005) studied the problem of a partially debonded circular inhomogeneity in piezoelectric/piezomagnetic composites. Chue and Liu (2005) investigated the magneto-electro-elastic field of a bimaterial BaTiO3–CoFe2O4 composite wedge containing an interface impermeable crack under anti-plane deformation. Zheng et al. (2007) discussed the interaction between a generalized screw dislocation and circular–arc interfacial cracks along a circular inhomogeneity in magneto-electro-mechanical solids.

Eigenstrains (or stress-free strains) are all kinds of inelastic strains such as thermal expansion strains, phase transformation strains, initial strains, plastic strains and misfit strains. The determination of the stress field for an infinite homogeneous elastic body
containing an inclusion undergoing uniform eigenstrains is particularly significant since these eigenstrains may be replaced with the equivalent body force occurred in many physical problems. Current examples include passivated interconnector lines in large scale integrated circuits, and strained semiconductor laser devices, where residual stresses induced by thermal or lattice mismatch between buried active components and surrounding barrier materials crucially affect the electronic performance of devices and sometimes are identified as a major cause of degradation (Ru, 2003). Since the pioneering works of Eshelby (1957), he showed that the elastic field in an ellipsoidal inclusion is uniform when a uniform eigenstrain is distributed in the ellipsoidal inclusion, several researchers investigated the remarkable characteristics of particular non-ellipsoidal inclusions embedded in an infinite medium (Franciosi, 2005). By using the Green’s function method, Mura et al. (1994) and Mura (1997) claimed that an m-point polygonal inclusion subjected to uniform eigenstrain would produce uniform stress field inside the inclusion, if m is an odd number. However, the exact solutions obtained by Rodin (1996), Nozaki and Taya (1997) and Lubarda and Markenscoff (1998) showed that polygonal and polyhedral inclusions undergoing uniform eigenstrains produce non-uniform the stress fields inside the inclusions. The complex variable and conformal mapping method is another powerful method for two-dimensional non-ellipsoidal inclusion problems although the solutions obtained by conformal mapping are not exact because the mapping functions include infinite terms. Based on the assumption that the inclusion and matrix have the same elastic constants, Ru (1999, 2000) developed a method for solving arbitrary shape inclusion problems in elastic media and piezoelectric media. Wang and Shen (2003) used the same method to obtain the analytical solutions for arbitrary shape inclusion problems in magnetoelectroelastic composites.

Based on the Faber series method, Gao and Noda (2004) investigated the anti-plane piezoelectric problem of an arbitrarily shaped inclusion with different piezoelectric constants from the matrix. Using the complex variable method, Wang and Gao (2011) studied the stress field inside an arbitrary-shape elastic inclusion in a three-phase composite subjected to uniform stresses at infinity. The results showed that a uniform hydrostatic stress field within the inclusion in the three-phase composite can be achieved by properly designing the inclusion shape, the elastic constants of the composite phases, and the thickness of the interphase layer. Stagni (2001) used complex variable technique to solve the plane-elasticity problem of a multilayered circular hollow-cored inclusion with eigenstrains under arbitrary uniform remote loadings. It was found that eigenstrains may be effective in reducing stress levels inside or around the fiber.

In this paper, we used the Faber-series method to solve the anti-plane magnetoelectroelastic problem of an arbitrarily shaped inclusion with different material constants from the matrix. The inclusion undergoes uniform eigenfields and the matrix is subjected to uniform remote loadings. Numerical results examined the magnetoelectroelastic fields for various constituents of piezoelectric/piezomagnetic composite undergoing eigenfields. Moreover, we try to select appropriate remote loadings and eigenfields for reducing stress levels and increasing magnetoelectric responses around the inclusion. It can protect the vulnerable piezoelectric/piezomagnetic device from mechanical damaging and increase the performance of the piezoelectric/piezomagnetic sensor.

2. Basic equations of magnetoelectroelasticity

In a fixed rectangular coordinates system $(x_1, x_2, x_3)$, the basic equations for linear magnetoelectroelastic materials can be written as

\begin{align}
\tau_{ij} &= c_{ijkl}\gamma_{kl} - e_{ijl}E_k - q_{ijk}H_k \\
A_i &= e_{ijkl}\gamma_{kl} + e_{ijk}E_k + di_kH_k \\
B_i &= q_{ijkl}\gamma_{kl} + d_{ijk}E_k + \mu_{ik}H_k \\
\gamma_{ij} &= 0, \quad D_{ii} = 0, \quad B_{ij} = 0 \\
\gamma_{ij} &= \frac{1}{2}(u_{ij} + u_{ji}), \quad E_i = -\varphi_j, \quad H_i = -\varphi_j
\end{align}

where a comma in the subscripts stands for partial differentiation; repeated indices denote summation; $c_{ijkl}, e_{ijkl}, q_{ijkl}$ and $d_{ijk}$ are the corresponding elastic, piezoelectric, piezomagnetic and magneto-electric constants, respectively. $e_{ij}, \mu_{ik}$ and $\mu_{ik}$ are the dielectric permittivity and the magnetic permeabilities, respectively. $u_{ij}, \tau_{ij}, \gamma_{ij}, \varphi, D_{ij}, E_i, \phi, B_i$ and $H_i$ are displacement, stress, strain, electric potential, electric displacement, electric field, magnetic potential, magnetic induction and magnetic field, respectively. The material constants have the following symmetries:

\begin{align*}
c_{ijkl} &= c_{kijl} = c_{ijlk} = c_{lijk} \\
e_{ijkl} &= e_{ijlk} = e_{klij} = e_{klij} \\
q_{ijkl} &= q_{iklj} = q_{lijk} = q_{lijk} \\
d_{ijkl} &= d_{ijk} = d_{ik} = d_{kl} \\
\mu_{ik} &= \mu_{ki}
\end{align*}

2.1. Anti-plane deformation

Fig. 1 shows an arbitrary inclusion embedded in an infinite magnetoelectroelastic matrix subjected to anti-plane mechanical and in-plane electric and magnetic loadings. The arbitrary inclusion undergoes uniform anti-plane eigenstrains $(\gamma_{11x}, \gamma_{22x})$ and uniform in-plane eigenmagnetic field $(H_x, H_y)$. The matrix and the inclusion are assumed to be transversely isotropic and poled in the $x_3$ direction. For current problem, the out-of-plane displacement $u_3$, the in-plane electric potential $\varphi$ and in-plane magnetic potential $\phi$ are functions of $x_1$ and $x_2$ only, that

$u_1 = u_2 = 0$, $u_3 = u_3(x_1, x_2)$, $\varphi = \varphi(x_1, x_2)$, $\phi = \phi(x_1, x_2)$

The basic Eqs. (1)–(5) can be simplified to

\begin{align}
c_{44}\nabla^2 u_3 + e_{15}\nabla^2 \varphi + q_{15}\nabla^2 \phi &= 0
\end{align}
\[ e_{15} \nabla^2 u_3 - e_{11} \nabla^2 \phi + d_{11} \nabla^2 \phi = 0, \quad (7) \]
\[ q_{15} \nabla^2 u_3 - d_{11} \nabla^2 \phi + \mu_{11} \nabla^2 \phi = 0, \quad (8) \]
where \( \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \) is the two-dimensional Laplace operator.

According to Zheng et al. (2007), the generalized displacement, stress and strain vectors can be expressed in terms of an analytical function vector \( f(z) \) as
\[
\begin{align*}
\{ u_3 \} &= \text{Re} \left\{ f_0(z) \right\} = \text{Re} \left[ f(z) \right] \\
\{ \phi \} &= \left\{ f_0(z) \right\} = \left\{ f_0(z) \right\} = \left\{ f(z) \right\} \\
\{ \psi \} &= \left\{ f_0(z) \right\} = \left\{ f_0(z) \right\} = \left\{ f(z) \right\}
\end{align*}
\]
\[
\begin{bmatrix}
\left( \frac{\alpha_{31} - i \alpha_{32}}{D_1 - i D_2} \right) \\
\left( \frac{\alpha_{31} - i \alpha_{32}}{B_1 - i B_2} \right)
\end{bmatrix} = \left[ \begin{array}{c}
\cos \eta - i \sin \eta \\
\sin \eta - i \cos \eta
\end{array} \right] = \left[ \begin{array}{c}
\cos \eta - i \sin \eta \\
\sin \eta - i \cos \eta
\end{array} \right] = f(z)
\]
\[
\left( \begin{array}{c}
\frac{\gamma_{31} - i \gamma_{32}}{-E_1 + i E_2} \\
\left( -H_1 + i H_2 \right)
\end{array} \right) = f(z)
\]
where \( \text{Re} \) denotes the real part and \( \text{Im} \) denotes the imaginary part of a complex function.

2.2. Conformal mapping

We introduce a mapping function as in England (1971)
\[
Z = \Omega(z) = R \left( c + \sum_{n=1}^{\infty} m_n z^n \right)
\]
(13)

Taking \( z \) on the unit circle and substituting \( z = \cos \eta + i \sin \eta \) into Eq. (13), it becomes
\[
Z = \Omega(z) = R \left( \cos \eta + \sum_{n=1}^{\infty} m_n \cos \eta + i \left( \rho \sin \eta - \sum_{n=1}^{\infty} m_n \sin \eta \right) \right)
\]
and shows that the mapping function maps the outside region \( \Omega_b \) of an arbitrary inclusion in \( z \)-plane onto the outside region of a unit circle \( S_b \) in \( z \)-plane. Where \( m_n \) are the constants corresponding to the power of \( z \) in the mapping function for a given shape of inclusion and \( R \) is a constant for size of inclusion (see Fig. 2). The technique of conformal mapping provides a powerful method for the stress analysis of two-dimensional elastic bodies containing either holes or rigid inclusions of any shapes. The truncated mapping function constants for some rounded shape inclusions are listed in Table 1 of Ukadgaonker and Rao (2000) and the specific conformal mapping functions for a regular \( N \)-polygonal section and a regular \( N \)-pointed star-shaped inclusion are derived by Chen and Chiang (1997). Because of non-existence of a conformal mapping which maps, simultaneously, the exterior and interior of the inclusion onto a plane with a simple interface, the conformal mapping technique cannot be applied directly to inclusions of arbitrary shape. However, the conformal mapping technique in conjunction with the Faber polynomials and Laurent series can be employed to determine the non-uniform stress field within a two-dimensional inclusion of arbitrary shape (e.g., Gao and Noda, 2004; Tsukrov and Novak, 2004; Wang and Sudak, 2006; Zou et al., 2010). The normal and tangential components of the interfacial stress, electric displacement, magnetic induction, strain, electric field and magnetic field related to the complex function vector \( f(z) \) are expressed as
\[
\begin{bmatrix}
\frac{\gamma_{31} - i \gamma_{32}}{-E_1 + i E_2} \\
\left( -H_1 + i H_2 \right)
\end{bmatrix} = f(z)
\]
\[
\left( \begin{array}{c}
\frac{\gamma_{31} - i \gamma_{32}}{-E_1 + i E_2} \\
\left( -H_1 + i H_2 \right)
\end{array} \right) = f(z)
\]
(14)
(15)
where \( \alpha \) is defined as the angle between the normal direction and the positive \( x_1 \)-axis in \( z \)-plane. In the mapped plane, the angle \( \alpha \) can be found by the following relation as England (1971)
\[
\alpha = \arg \Omega(z) + \eta, \quad \zeta = \rho e^{i\eta}
\]
(16)

3. General solution

3.1. Series forms of complex potentials \( f_a(z) \) and \( f_b(z) \)

The complex potential \( f_a(z) \) which is analytical in the region \( \Omega_a \), can be extended into Faber series and expressed as follow
\[
f_a(z) = \sum_{k=1}^{\infty} a_k P_k(z) \quad z \in \Omega_b
\]
(17)
where \( a_k \) are unknown coefficient vectors to be determined and \( P_k(z) \) is the \( k \)-th degree Faber polynomial which has the following form
\[
P_k(z) = z^k + \sum_{n=1}^{\infty} \beta_{k,n} z^{-n}(z)
\]
(18)
and the coefficients \( \beta_{k,n} \) can be determined by the following recurrence relations
\[
\begin{align*}
\beta_{1,n} &= m_n \\
\beta_{k+1,n} &= \beta_{k,n} + \sum_{i=1}^{n} m_n \beta_{k,i} \sum_{i=1}^{k} \beta_{k,i} \beta_{k,n+i} (k, n = 1, 2, 3 \ldots)
\end{align*}
\]
(19)

Fig. 2. The conformal mapping from \( z \)-plane to \( \zeta \)-plane.

Table 1

<table>
<thead>
<tr>
<th>Inclusion/matrix</th>
<th>Inclusion shape</th>
</tr>
</thead>
<tbody>
<tr>
<td>BaTiO_3/CoFe_2O_4</td>
<td>Triangle</td>
</tr>
<tr>
<td>5.15</td>
<td>5.40</td>
</tr>
<tr>
<td>CoFe_2O_4/BaTiO_3</td>
<td>5.75</td>
</tr>
</tbody>
</table>
Substituting Eq. (18) into Eq. (17) yields

\[ f_\theta(\zeta) = \sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \left( \sum_{k=1}^{\infty} b_{nk} z^n \right) \left( 1 - \frac{z}{\zeta} \right) \]

(20)

When the matrix is subjected to remote anti-plane mechanical and in-plane electric and magnetic loadings the complex potential \( f_\theta(\zeta) \) which is analytical in the region \( |\zeta| > 1 \) except at infinity, can be represented by the Laurent series

\[ f_\theta(\zeta) = c^e R^e + \sum_{n=1}^{\infty} b_n z^n \]

(21)

where \( b_n \) are unknown coefficient vectors to be determined and \( c^e \) is a constant related to the load conditions at infinity. There are two cases of combined electric and mechanical loadings, for each case the constant \( c^e \) is given as follows.

**Case 1:** A uniform far-field anti-plane strain \( \tau_{31}^e, \tau_{32}^e \), in-plane electric field \( E_1^e, E_2^e \) and in-plane magnetic field \( H_1^m, H_2^m \)

\[ c^e = \left[ \begin{array}{c} \tau_{31}^e - i \tau_{32}^e \\ -E_1^e + i E_2^e \\ -H_1^m + i H_2^m \end{array} \right] \]

(22)

**Case 2:** A uniform far-field anti-plane shear \( \tau_{31}^e, \tau_{32}^e \), in-plane electric displacement \( D_1^e, D_2^e \) and in-plane magnetic induction \( B_1^m, B_2^m \)

\[ c^e = \left[ \begin{array}{ccc} c_{44}^e & e_{15}^e & q_{15}^e \\ e_{15}^e & -c_{11}^e & -d_{11}^e \\ q_{15}^e & -d_{11}^e & -\mu_{11}^e \end{array} \right]^{-1} \left( \begin{array}{c} \tau_{31}^e - i \tau_{32}^e \\ -E_1^e + i E_2^e \\ -H_1^m + i H_2^m \end{array} \right) \]

(23)

### 3.2. Boundary conditions

In this work, we consider the inclusion is undergoing uniform eigenfields (eigenstrain, eigenelectric field and eigenmagnetic field). The additional displacement, electric potential and magnetic potential in the inclusion induced by the uniform eigenfields are express as

\[ U_a = \left\{ \begin{array}{c} u_j^e \\ \phi^e \end{array} \right\} = \left\{ \begin{array}{c} 2(\gamma_{31} x_1 + \gamma_{32} x_2) \\ -(E_{1x} + E_{2x}) \end{array} \right\} = \Pi z + \overline{\Pi z} \]

(24)

where

\[ \Pi = \left[ \begin{array}{cc} \gamma_{31} - i \gamma_{32} \\ \frac{1}{2}(-E_1 + i E_2) \\ \frac{1}{2}(-H_1^m + i H_2^m) \end{array} \right] \]

(25)

We assume that the interface is perfect bonding, i.e., tractions, normal electric displacement, normal magnetic induction, displacement, electric potential and magnetic potential are continuous across the interface \( L \). The continuity conditions can be expressed as

\[ \text{Re}[f_a(\sigma)] + U_a = \text{Re}[f_\theta(\sigma)] \quad \sigma \in L_a \]

(26)

\[ \text{Im}[C_a f_a(\sigma)] = \text{Im}[C_\theta f_\theta(\sigma)] \quad \sigma \in L_a \]

(27)

where

\[ C_a = \left[ \begin{array}{ccc} c_{44}^e & e_{15}^e & q_{15}^e \\ e_{15}^e & -c_{11}^e & -d_{11}^e \\ q_{15}^e & -d_{11}^e & -\mu_{11}^e \end{array} \right], \quad C_\theta = \left[ \begin{array}{ccc} c_{44}^e & e_{15}^e & q_{15}^e \\ e_{15}^e & -c_{11}^e & -d_{11}^e \\ q_{15}^e & -d_{11}^e & -\mu_{11}^e \end{array} \right] \]

### 3.3. Determinations of \( f_a(z) \) and \( f_\theta(z) \)

Substituting the series expressions of \( f_a(\zeta) \) and \( f_\theta(\zeta) \) as in Eqs. (20) and (21) into the boundary conditions (26) and (27) gives

\[ \sum_{n=1}^{\infty} a_n \sigma^n + \sum_{n=1}^{\infty} \left( \sum_{k=1}^{\infty} b_{nk} \sigma^n \right) \left( 1 - \frac{\sigma}{\zeta} \right) = \left( R^e + \sum_{n=1}^{\infty} (b_n \sigma^n) \right) \]

\[ + 2 \left[ \text{PIR} \left( \sigma + \sum_{n=1}^{\infty} m_n \sigma^n \right) \sigma \right] \]

\[ = \left( R^e + \sum_{n=1}^{\infty} (b_n \sigma^n) \right) \sigma \in L_a \]

(28)

\[ C_a \left( \sum_{n=1}^{\infty} a_n^e \sigma^n - \sum_{n=1}^{\infty} \left( \sum_{k=1}^{\infty} b_{nk} \sigma^n \right) \right) = \left( R^e + \sum_{n=1}^{\infty} (b_n \sigma^n) \right) \sigma \in L_a \]

(29)

Equaling the coefficients of \( \sigma^n \ (n \geq 1) \), we have

\[ a_n + \sum_{n=1}^{\infty} b_{nk} \sigma^n = \bar{b}_n + (R^e - 2R^T \Pi) \delta_{1n} - 2R^T \Pi m_n \]

(30)

\[ C_a \left( \sum_{n=1}^{\infty} a_n^e \sigma^n - \sum_{n=1}^{\infty} \left( \sum_{k=1}^{\infty} b_{nk} \sigma^n \right) \right) = \left( R^e + \sum_{n=1}^{\infty} (b_n \sigma^n) \right) \]

(31)

where \( \delta_{1n} \) is 1 for \( n = 1 \) or \( \delta_{1n} = 0 \) for \( n \neq 1 \).

Decoupling of Eqs. (30) and (31) yields

\[ (1 - C_a^e C_a^e) a_n + (1 - C_a^e C_a^e) \sum_{n=1}^{\infty} b_{nk} \sigma^n = (2R^e - 2R^T \Pi) \delta_{1n} - 2R^T \Pi m_n \]

(32)

Taking the conjugate of Eq. (32), we have

\[ (1 - C_a^e C_a^e) a_n + (1 - C_a^e C_a^e) \sum_{n=1}^{\infty} b_{nk} \sigma^n = (2R^e - 2R^T \Pi) \delta_{1n} - 2R^T \Pi m_n \]

(33)

Truncating the above infinite system of linear algebraic equations at a large \( n \) terms, Eqs. (32) and (33) constitute a system of \( 2n \) linear equations to determine the \( 2n \) unknown coefficient vectors \( a_n \) and \( \bar{b}_n \). After \( a_n \) is found the other unknown coefficient vectors \( b_{nk} \) can be easily found by the relations of Eqs. (30) and (31).

### 3.4. Special case of elliptical inclusion

For an elliptical inclusion, the mapping function in Eq. (13) has an exact form as

\[ z = \Omega(\zeta) = R(\zeta + m_1 \zeta^{-1}) \]

(34)

where

\[ R = \frac{a + b}{2}, \quad m_1 = \frac{a - b}{a + b} \]

(35)

\( a \) and \( b \) are the semi-major and semi-minor axes of the elliptical inhomogeneity. In this case, one can obtain from Eq. (19) that

\[ \beta_{kk} = m_1, \quad \beta_{kn} = 0 \ (k \neq n) \]

(36)

Solving Eqs. (32) and (33) produce

\[ a_n = 2R((1 + C_a^e C_a^e)^2 - (1 - C_a^e C_a^e) m_1^{-2} - \Pi - \Pi m_1 - (1 - C_a^e C_a^e) m_1 (\zeta^{-1} - \Pi - \Pi m_1)) \]


\(a_k = 0, \quad k \geq 2\) \hspace{1cm} (37)

Now, we examine that the inclusion has the same magnetoelastic moduli as that of the matrix, but undergoing eigenfields. By letting \(C_k = C = C_\text{m}\) in Eq. (37), one has

\[a_1 = -R[(\Pi + \Pi_\text{m})_1], \quad a_k = 0, \quad k \geq 2\] \hspace{1cm} (38)

The exact internal magnetoelastic field is given by

\[
\begin{align*}
&\{T_{\Sigma} - I\} \hat{T}_{\Sigma} \\
&\{D_1 - iD_2\} = -C(\Pi + \Pi_\text{m})_1 \\
&\{B_1 - iB_2\}
\end{align*}
\] \hspace{1cm} (39)

These results are in agreement with those of Wang and Shen (2003).

Then, we inspect that the inclusion and the matrix have different magnetoelastic constants, and the subject is to out-of-plane shear stress and in-plane electric field at infinity. By letting \(\Pi = 0\) in Eq. (37), one has

\[a_1 = 2R[(I + C_\text{m})C_\Sigma^2 - (I - C_\Sigma)C_\Sigma C_\text{m}i_1 C_\Sigma]C^{\infty} - (I - C_\Sigma)C_\text{m}i_1 C^{\infty}\] \hspace{1cm} (40)

Thus the exact internal magnetoelastic field is given by

\[
\begin{align*}
&\{T_{\Sigma} - I\} \hat{T}_{\Sigma} \\
&\{D_1 - iD_2\} = -C_\Sigma \frac{A_1}{R} \\
&\{B_1 - iB_2\}
\end{align*}
\] \hspace{1cm} (41)

These results are in agreement with those of Gao and Noda (2004) who deal with piezoelectric materials. From Eq. (37), we observe another phenomenon that when piezoelectric/piezomagnetic composite with elliptical inclusion undergoing uniform eigenfields and the matrix subjected remote uniform loadings the magnetoelastic fields occurred inside the inclusion are uniform. The similar phenomenon had been discussed by Zou et al. (2010) and Wang and Gao (2011) for elastic problems. This uniformity has the important consequence that by composing appropriate uniform eigenfields and remote uniform loading may reduce stress levels of piezoelectric/piezomagnetic composite. From Eq. (37), letting

\[C^{\infty} = \Pi + \Pi_\text{m}\] \hspace{1cm} (42)

It is effective in reducing all magnetoelastic fields inside or around the elliptical inclusion. For a non-elliptical inclusion, it is difficult to give the exact solutions. In practical applications, the numerical computation can be used to solve the linear algebraic equations Eqs. (32) and (33) for the arbitrary-shape (non-elliptical) inclusion. The results are shown in following discussion.

4. Numerical results and discussion

In this section, the linear algebraic equations derived in the preceding section will be used to analyze the following examples associated with piezoelectric/piezomagnetic composite subjected to magnetoelastic/mechanical loading. Numerical results are obtained for the composite BaTiO\(_3\)–CoFe\(_2\)O\(_4\), the constituents of which are: piezoelectric BaTiO\(_3\) and piezomagnetic CoFe\(_2\)O\(_4\), whose material constants are given as Zheng et al. (2007).

Piezoelectric BaTiO\(_3\):

\[c_{44} = 43 \times 10^9 \text{ N/m}^2, \quad e_{15} = 11.6 \text{ C/m}^2, \quad q_{15} = 0, \quad \varepsilon_{11} = 11.2 \times 10^{-9} \text{ C/N/m}^2, \quad d_{11} = 0, \quad \mu_{11} = 5.0 \times 10^{-6} \text{ Ns}^2/\text{C}^2\]

Piezomagnetic CoFe\(_2\)O\(_4\):

\[c_{44} = 45.3 \times 10^9 \text{ N/m}^2, \quad e_{15} = 0, \quad q_{15} = 550 \text{ N/Am}, \quad \varepsilon_{11} = 0.08 \times 10^{-9} \text{ C/N/m}^2, \quad d_{11} = 0, \quad \mu_{11} = -590 \times 10^{-6} \text{ Ns}^2/\text{C}^2\]

4.1. Uniform eigenfields

We first consider the influence of different constituents of piezoelectric/piezomagnetic composite on the magnetoelastic fields of an arbitrary shaped inhomogeneity with eigenfield \(\gamma_{11} = -10^{-10}, \gamma_{12} = 0, \quad E_1 = E_2 = 0, \quad H_1 = H_2 = 0\). The specific conformal mapping function for a regular \(N\)-polygonal section inclusion is derived by Chen and Chiang (1997) as follows

\[z = \Omega(i) = \frac{R}{N} \left[ i^{\frac{2}{N}} \frac{2}{N^2} i^{\frac{1}{N} - 1} + \frac{N - 2}{N^2(N - 1)} i^{\frac{1}{N} - 2N} + \cdots + i^{\frac{1}{N} - (N - 1)N} + \frac{1}{N} \right]\] \hspace{1cm} (43)

For comparing with the existent numerical result, we truncate the mapping function to \(\Omega(i) = \xi + \frac{1}{N} i^{\frac{1}{N} - N(\theta - 1)} + \frac{1}{N} i^{\frac{1}{N} - (N - 1)N} + \frac{1}{N} i^{\frac{1}{N} - (N - 2)N} + \cdots + \frac{1}{N} i^{\frac{1}{N} - \theta N} + \frac{1}{N} \).

1. Figs. 3–5 shows that the piezoelectric/piezomagnetic composite subjected to an eigenstrain will exhibit the magnetoelectroelastic effect. These results verify that two different combinations of piezoelectric/piezomagnetic composite materials not only have the original piezoelectric and piezomagnetic properties, but also showed significant magnetoelastic coupling effect that does not present in single-phase piezoelectric or piezomagnetic materials.

2. All the interfacial normal shear, electric displacement and magnetic induction are disturbed vigorously across the vertices of triangular inclusion.

Fig. 3. The normal shear \(\tau_{\Sigma}\) along the triangular boundary.
The specific conformal mapping function for a regular $N$-pointed star-shaped inclusion ($N \geq 5$) is derived by Chen and Chiang (1997).

$$z = \Omega(\zeta) = R \left\{ \zeta + \frac{6}{N(N-1)} z^{-1-N} + \frac{N-18}{N(N-2)(N-1)} z^{-2-N} + \frac{2(N^2-3N-48)}{N^3(N-1)(N-4)} z^{-3-N} + \cdots \right. $$

$$+ \left. \sum_{k=0}^{p} \left(-1\right)^k C_k^N \frac{1}{1-p} \right\},$$

(44)

Figs. 9 and 10 shows the interfacial normal shear for a 5-pointed star-shaped inclusion, in which, the mapping function is truncated at $p = 10$ and $p = 50$, respectively. Form these figures we can observe that the edge shape and the distribution of stress in Fig. 10 are smoother than those in Fig. 9. Since the vertexes of the star-shaped inclusion in Fig. 10 are sharper than those in Fig. 9, the singularities in Fig. 10 are larger. Truncated the mapping function at $p = 50$ achieve a good accuracy for the calculation of a 5-pointed star-shaped inclusion, the truncated selection also appear in Wang and Sudak (2006). Fig. 11 shows the interfacial normal shear for a 6-pointed star-shaped inclusion with the truncated mapping terms $p = 50$.

By integrating over the inclusion, the average shear stresses in the inclusions are calculated and shown in Table 1. Following the approaches provided by Chen and Chiang (1997) and Chen et al. (2002) and Tsukrov and Novak (2004), the derived solutions can be used to predict effectively magnetoelectroelastic properties of arbitrary shaped inclusions. The effective properties of solids with inhomogeneities are important to predict the overall mechanical behavior of fiber and particle reinforced composites.

4.2. Uniform remote loadings

In following discussion we focus on investigating the interaction between eigenfields, arbitrary shaped inclusions and remote loadings. The proposed solution provides a numerical approach for adjusting appropriate remote loadings and eigenfields to reduce stress levels and increase magnetoelectric response. Supposing the matrix is piezoelectric materials BaTiO$_3$, and the inclusion is made of piezomagnetic materials CoFe$_2$O$_4$ undergoing an eigenstrain ($\gamma_{31} = -10^{-10}$) which may be caused by thermal expansion.
Fig. 8. The normal magnetic induction $B_n$ along the triangular interface.

Fig. 9. The normal shear $\tau_{3n}$ along the boundary of 5-pointed star-shaped inclusion ($p = 10$).

Fig. 10. The normal shear $\tau_{3n}$ along the boundary of 5-pointed star-shaped inclusion ($p = 50$).

Fig. 11. The normal shear $\tau_{3n}$ along the boundary of 6-pointed star-shaped inclusion ($p = 50$).

Fig. 12. The normal electric displacement $D_n$ along the triangular interface.

Fig. 13. The normal electric displacement $D_n$ along the triangular interface.
or other initial strains. Figs. 12–14 respectively show the interfacial ROC under the contract number NSC 100-2221-E-252-004-MY2.

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References


