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Long Abelian ideals ☆

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Abstract

We study Abelian ideals of a Borel subalgebra consisting of long roots. It is shown that methods of Cellini and Papi can be extended to this situation. A uniform expression for the number of long Abelian ideals is given. We also show that there is a one-to-one correspondence between the long Abelian ideals and B-stable commutative subalgebras in the little adjoint representation of the Langlands dual Lie algebra.

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0. Introduction

Let $\mathfrak{b}$ be a Borel subalgebra of a simple Lie algebra $\mathfrak{g}$. Let $\mathfrak{Ab}$ denote the set of all Abelian ideals of $\mathfrak{b}$. It is easily seen that any $\mathfrak{a}\in\mathfrak{Ab}$ is actually contained in the nilpotent radical of $\mathfrak{b}$. Therefore $\mathfrak{a}$ is determined by the corresponding set of roots. More precisely, let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$ lying in $\mathfrak{b}$ and let Δ be the root system of the pair $(\mathfrak{g},\mathfrak{t})$. Choose the system of positive roots, Δ^+, so that the roots of $\mathfrak{b}$ are positive. Then $\mathfrak{a}=\bigoplus_{\gamma\in I}\mathfrak{g}_\gamma$, where I is a suitable subset of Δ^+ and $\mathfrak{g}_\gamma$ is the root space for $\gamma\in\Delta^+$.

A nice result of D. Peterson says that the cardinality of $\mathfrak{Ab}$ is $2^{\mathrm{rk}\,\mathfrak{g}}$. His approach uses a one-to-one correspondence between the Abelian ideals and the so-called 'minuscule' elements of the affine Weyl group $\widehat{W}$ (see Section 1 for precise definitions). An exposition of Peterson's results is found in [4]. Peterson's work

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appeared to be the point of departure for active recent investigations of Abelian ideals, and related problems of representation theory and combinatorics [1,2,4,5,8,9]. Our definition of minuscule elements follows Kostant's paper [4], so that $w \in \widehat{W}$ is minuscule in our sense if and only if w^{-1} is minuscule in the sense of Cellini–Papi [1,2]. An elegant proof of Peterson's theorem is given in [1]. Let $\mathscr{A}$ be the fundamental alcove of $\widehat{W}$. Cellini and Papi show that $w \in \widehat{W}$ is minuscule if and only if $w^{-1} \cdot \mathscr{A} \subset 2\mathscr{A}$. Since $2\mathscr{A}$ consists of $2^{\mathrm{rk}\,\mathfrak{g}}$ alcoves and $\widehat{W}$ acts simply transitively on the set of alcoves, Peterson's theorem follows.

In this paper, we first show that methods of [1] can be adapted to solve the following problems:

Suppose $\mathfrak{g}$ *has two root lengths.*

- *describe* (*enumerate*) *the Abelian ideals such that the corresponding set I consists only of long roots* (*such Abelian ideals are said to be long*);
- *give a characterization of the corresponding* (= *long*) *minuscule elements.*

Write $\mathfrak{Ab}_l$ (resp. $\mathscr{M}_l$) for the set of long Abelian ideal (resp. long minuscule elements). We give a uniform answer to both problems. Let $d > 1$ be the ratio of squares of root lengths and a the number of long simple roots. Then

$$\#(\mathfrak{Ab}_l) = d^a.$$

To obtain a characterization of the long minuscule elements, we consider a certain simplex $\mathscr{A}_s$ that lies between $\mathscr{A}$ and $2\mathscr{A}$. Then the answer is that $w \in \mathscr{M}_l$ if and only if $w^{-1} \cdot \mathscr{A} \subset \mathscr{A}_s$ if and only if any reduced decomposition of w contains no short simple reflections (we say that a reflection is short, if it corresponds to a short root). We also describe the rootlets of long Abelian ideals, see Section 2 for the precise definition.

Second, we show that the theory of long Abelian ideals is closely related to describing of commutative subalgebras in the isotropy representations of some symmetric spaces. Let θ be the highest root and θ_s the short dominant root of Δ^+. Write $\mathbb{V}(\lambda)$ for the $\mathfrak{g}$-module with highest weight λ. Assume that $|\theta|^2/|\theta_s|^2 = 2$, i.e., we exclude the case of $\mathbf{G}_2$. Then $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{V}(\theta_s)$ has a structure of a $\mathbb{Z}_2$-graded Lie algebra, so that it makes sense to speak about commutative subalgebras of $\mathbb{V}(\theta_s)$. In Section 3, we prove that there is a natural one-to-one correspondence between the $\mathfrak{b}$-stable commutative subalgebras of $\mathbb{V}(\theta_s)$ and the long Abelian $\mathfrak{b}^\vee$-ideals in $\mathfrak{g}^\vee$. Here $\mathfrak{g}^\vee$ is the Langlands dual Lie algebra for $\mathfrak{g}$.

1. Notation and other preliminaries

1.1. Main notation. Δ is the root system of $(\mathfrak{g}, \mathfrak{t})$ and W is the usual Weyl group. For $\alpha \in \Delta$, $\mathfrak{g}_\alpha$ is the corresponding root space in $\mathfrak{g}$.

Δ^+ is the set of positive roots and $\rho = \frac{1}{2}\sum_{\alpha \in \Delta^+} \alpha$.

$\Pi = \{\alpha_1, \dots, \alpha_p\}$ is the set of simple roots in Δ^+.

$\varphi_1, \dots, \varphi_p$ are the fundamental weights corresponding to Π.

We set $V := \mathfrak{t}_{\mathbb{R}} = \oplus_{i=1}^{p} \mathbb{R}\alpha_i$ and denote by $(\,,\,)$ a W-invariant inner product on V. As usual, $\mu^\vee = 2\mu/(\mu,\mu)$ is the coroot for $\mu \in \Delta$. Letting $\widehat{V} = V \oplus \mathbb{R}\delta \oplus \mathbb{R}\lambda$, we extend the inner product $(\,,\,)$ on $\widehat{V}$ so that $(\delta, V) = (\lambda, V) = (\delta,\delta) = (\lambda,\lambda) = 0$ and $(\delta,\lambda) = 1$.

$\widehat{\Delta} = \{\Delta + k\delta \mid k \in \mathbb{Z}\}$ is the set of affine real roots and $\widehat{W}$ is the affine Weyl group.

Then $\widehat{\Delta}^+ = \Delta^+ \cup \{\Delta + k\delta \mid k \geqslant 1\}$ is the set of positive affine roots and $\widehat{\Pi} = \Pi \cup \{\alpha_0\}$ is the corresponding set of affine simple roots. Here $\alpha_0 = \delta - \theta$, where θ is the highest root in Δ^+. The inner product $(\,,\,)$ on $\widehat{V}$ is $\widehat{W}$-invariant.

For $\alpha_i (0 \leqslant i \leqslant p)$, we let s_i denote the corresponding simple reflection in $\widehat{W}$. The length function on $\widehat{W}$ with respect to $s_0, s_1, \dots, s_p$ is denoted by ℓ. We consider two different actions of $\widehat{W}$: the linear action on $\widehat{V}$ and the affine-linear action on V. To distinguish these two, we use dot '$\cdot$' for denoting the second action. For any $w \in \widehat{W}$, we set

$$\widehat{N}(w) = \{\alpha \in \widehat{\Delta}^+ \mid w(\alpha) \in -\widehat{\Delta}^+\}.$$

Our convention concerning $\widehat{N}(w)$ is the same as in [4,8] but is opposite to that in [1,2].

1.2. Abelian ideals. Let $\mathfrak{a} \subset \mathfrak{b}$ be an Abelian ideal. It is easily seen that $\mathfrak{a} \subset [\mathfrak{b},\mathfrak{b}]$. Therefore $\mathfrak{a} = \oplus_{\alpha \in I} \mathfrak{g}_\alpha$ for a subset $I \subset \Delta^+$, which is called the *set of roots of* $\mathfrak{a}$. In what follows, an Abelian ideal will be identified with the respective set of roots. That is, I is said to be an Abelian ideal, too. We also say that $\mathfrak{a}$ is a *geometric* Abelian ideal, while I is a *combinatorial* Abelian ideal. In the combinatorial context, the definition of an Abelian ideal can be stated as follows.

$I \subset \Delta^+$ is an Abelian ideal, if the following two conditions are satisfied:

(a) for any $\mu, \nu \in I$, we have $\mu + \nu \notin \Delta$;
(b) if $\gamma \in I, \nu \in \Delta^+$, and $\gamma + \nu \in \Delta$, then $\gamma + \nu \in I$.

Following D. Peterson, an element $w \in \widehat{W}$ is said to be *minuscule*, if $\widehat{N}(w)$ is of the form $\{\delta - \gamma \mid \gamma \in I\}$, where I is a subset of Δ^+. It was shown by Peterson that such an I is a combinatorial Abelian ideal and, conversely, each Abelian ideal occurs in this way, see [1,4, Proposition 2.8]. Hence one obtains a one-to-one correspondence between the Abelian ideals of $\mathfrak{b}$ and the minuscule elements of $\widehat{W}$. The set of minuscule elements in $\widehat{W}$ is denoted by $\mathscr{M}$. If $w \in \mathscr{M}$, then we write I_w(resp. $\mathfrak{a}_w$) for the corresponding combinatorial (resp. geometric) Abelian ideal. That is,

$$I_w = \{\gamma \in \Delta^+ \mid \delta - \gamma \in \widehat{N}(w)\} \text{ and } \mathfrak{a}_w = \bigoplus_{\alpha \in I_w} \mathfrak{g}_\alpha.$$

Conversely, given $I \in \mathfrak{Ab}$, we write $w\langle I\rangle$ for the respective minuscule element. Notice that

$$\dim \mathfrak{a}_w = \#(I_w) = l(w).$$

Throughout the paper, I or I_w stands for a combinatorial Abelian ideal.

2. Long minuscule elements and long Abelian ideals

From now on, we assume that Δ^+ has two root lengths. To distinguish different objects related to long and short roots, we use the subscripts 'l' and 's'. For instance, Π_l is the set of long simple roots and Δ_l^+ is the set of long positive roots. Accordingly, each simple reflection s_i is either short or long. Since θ is long, the reflection s_0 is regarded as long. Write θ_s for the unique short dominant root in Δ^+.

We say that $I \in \mathfrak{Ab}$ *is long*, if $I \subset \Delta_l^+$. Write $\mathfrak{Ab}_l$ for the set of all long Abelian ideals. The corresponding subset of $\mathscr{M}$ is denoted by $\mathscr{M}_l$. Notice that the analogous notion of a short Abelian ideal does not make sense, for any non-empty Abelian ideal contains the highest root θ, which is long.

2.1. Proposition. *Suppose $w \in \mathscr{M}$. Then $w \in \mathscr{M}_l$ if and only if any reduced decomposition of w does not contain short simple reflections.*

Proof. '$\Rightarrow$' Suppose $w = w'' s_i w'$, where α_i is a short simple root. As was noticed in [8, Section 2], every right substring in a reduced decomposition of w is again a minuscule element and the corresponding (combinatorial) ideal is a subset of I_w. In particular, we have $s_i w', w' \in \mathscr{M}$. Furthermore,

$$I_{s_i w'} = \{\gamma\} \cup I_{w'},$$

where $w'(\delta - \gamma) = \alpha_i$. It follows that γ is a short root lying in $I_{s_i w'} \subset I_w$.

'$\Leftarrow$' Argue by induction on the length of w. If $w' \in \mathscr{M}_l, w \in \mathscr{M}$, and $w = s_i w'$ for a long reflection s_i, then $I_w = \{\gamma\} \cup I_{w'}$, where $w'(\delta - \gamma) = \alpha_i$. Since γ is long, we conclude that $w \in \mathscr{M}_l$. $\square$

Consider the collection of affine hyperplanes in V

$$H_{\mu,k} = \{x \in V \mid (\mu, x) = k\},$$

where $\mu \in \Delta^+$ and $k \in \mathbb{Z}$. The connected components of $V \setminus \bigcup_{\mu,k} H_{\mu,k}$ are called *alcoves*. It is well-known that $\widehat{W}$, as group of affine transformations of V, acts simply transitively on the set of alcoves. Recall that the fundamental alcove of $\widehat{W}$ is

$$\mathscr{A} = \{x \in V \mid (\alpha, x) > 0 \ \forall \alpha \in \Pi \,\&\, (\alpha, \theta) < 1\}.$$

Set

$$\mathscr{A}_s = \{x \in V \mid (\alpha, x) > 0 \quad \forall \alpha \in \Pi \& (\alpha, \theta_s) < 1\}.$$

It is clear that $\mathscr{A}_s$ is a union of (finitely many) alcoves and it contains $\mathscr{A}$.

2.2. Proposition. *Let $w \in \widehat{W}$ be a minuscule element. We have $I_w \in \mathfrak{Ab}_l$ if and only if $w^{-1} \cdot \mathscr{A} \subset \mathscr{A}_s$.*

Proof. We use the following result proved in [1, 1.1]:

$$-\mu + k\delta \in N(w) \Leftrightarrow H_{\mu,k} \text{ separates } \mathscr{A} \text{ and } w^{-1} \cdot \mathscr{A} \quad (k > 0),$$

$$\mu + k\delta \in N(w) \Leftrightarrow H_{\mu,-k} \text{ separates } \mathscr{A} \text{ and } w^{-1} \cdot \mathscr{A} \quad (k \geqslant 0).$$

We also know that $w^{-1} \cdot \mathscr{A} \subset 2\mathscr{A}$, since $w \in \mathscr{M}$.

(a) If $w^{-1} \cdot \mathscr{A} \not\subset \mathscr{A}_s$, then the hyperplane $H_{\theta_s,1}$ separates $\mathscr{A}$ and $w^{-1} \cdot \mathscr{A}$. Hence $\delta - \theta_s \in N(w)$, i.e., $\theta_s \in I_w$.

(b) Conversely, if $w^{-1} \cdot \mathscr{A} \subset \mathscr{A}_s$, then any hyperplane $H_{\mu,k}$ separating $\mathscr{A}$ and $w^{-1} \cdot \mathscr{A}$ must meet $\mathscr{A}_s$. If $x \in \mathscr{A}_s \cap H_{\mu,k}$ and $\mu \in \Delta_s^+$, then $0 < (x, \mu) \leqslant (x, \theta_s) < 1$, which is impossible. Hence the hyperplanes $H_{\mu,1}$, with μ short, do not separate $\mathscr{A}$ and $w^{-1} \cdot \mathscr{A}$. Thus, $I_w \cap \Delta_s^+ = \emptyset$. □

It follows that the number of long Abelian ideals is equal to the number of alcoves that fit in $\mathscr{A}_s$. In other words,

$$\#(\mathfrak{Ab}_l) = \text{vol } (\mathscr{A}_s)/\text{vol } (\mathscr{A}). \tag{2.3}$$

2.4. Theorem.

$$\frac{\text{vol } (\mathscr{A}_s)}{\text{vol } (\mathscr{A})} = \left(\frac{|\theta|^2}{|\theta_s|^2}\right)^{\#\Pi_l}.$$

Proof. Write $\theta = \sum_{i=1}^p m_i\alpha_i$ and $\theta_s = \sum_{i=1}^p c_i\alpha_i$. Then

$$\frac{\text{vol } (\mathscr{A}_s)}{\text{vol } (\mathscr{A})} = \prod_{i=1}^p \frac{(\varphi_i, \theta)}{(\varphi_i, \theta_s)} = \prod_{i=1}^p \frac{m_i}{c_i}.$$

Here the first equality follows from elementary-geometric considerations, since the simplices $\mathscr{A}_s$ and $\mathscr{A}$ generate the same cone in V.

Consider the dual root system $\Delta^\vee$. We have $\{\alpha^\vee \mid \alpha\in\Pi\}$ is a set of simple roots, $\theta_s^\vee$ is the highest root in $\Delta^\vee$, and

$$\theta_s^\vee = \frac{2}{(\theta_s,\theta_s)}\sum_{i=1}^{p} c_i\alpha_i = \sum_{\alpha_i\in\Pi_s} c_i\alpha_i^\vee + \sum_{\alpha_i\in\Pi_l} c_i\frac{|\theta|^2}{|\theta_s|^2}\alpha_i^\vee.$$

On the other hand, the collection of the coefficients of the highest root in $\Delta^\vee$ is the same as in Δ. Hence

$$\prod_{i=1}^{p} m_i = \prod_{i=1}^{p} c_i\cdot\left(\frac{|\theta|^2}{|\theta_s|^2}\right)^{\#\Pi_l},$$

and we are done. □

2.5. Corollary. *The number of long Abelian ideals equals*:
2^{p-1} *for* $\mathfrak{so}_{2p+1}$; 2 *for* $\mathfrak{sp}_{2p}$; 4 *for* $\mathbf{F}_4$; 3 *for* $\mathbf{G}_2$.

In [8], we introduced the notion of the *rootlet* of an Abelian ideal. Let I_w, $w\in\mathcal{M}$, be a non-trivial Abelian $\mathfrak{b}$-ideal. Then $\tau(I_w) := w(\alpha_0)+\delta$ is a long positive root [8, 2.4], which is called the *rootlet* of I_w. It is natural to inquire as to what the rootlets of long Abelian ideals are.

2.6. Proposition. *Suppose I is a nontrivial Abelian ideal. Then $I\in\mathfrak{Ab}_l$ if and only if $\theta-\tau(I)$ is a certain sum of only long simple roots, i.e., θ and $\tau(I)$ have equal coefficients of all short simple roots.*

Proof. '⇒' By Proposition 2.1, a reduced decomposition of $w = w\langle I\rangle$ contains only reflections s_0 and $s_i, i>0$, for $\alpha_i\in\Pi_l$. We argue by induction on $\ell(w)$.

(a) If $\ell(w)=1$, then $w=s_0$ and $s_0(\alpha_0)+\delta=\theta$.

(b) Suppose $w\in\mathcal{M}_l$ and $w(\alpha_0)+\delta=\gamma\in\Delta_l^+$. If $s_iw\in\mathcal{M}$, then $s_i(\gamma-\delta)+\delta=s_i(\gamma)$. Consider two possibilities for s_i. If s_i corresponds to $\alpha_i\in\Pi_l$, then γ and $s_i(\gamma)$ have equal coefficients of the short simple roots. If $s_i=s_0$, then

$$s_0(\gamma)=\begin{cases}\gamma & \text{if } (\gamma,\theta)=0,\\ \delta-(\theta-\gamma) & \text{if } (\gamma,\theta)\neq 0.\end{cases}$$

As the last case is impossible in the minuscule situation (for, we would obtain a root which is not in Δ^+), the induction step is complete.

'⇐' Assume that a reduced decomposition of w contains a short reflection. Then the first occurrence of short reflections will certainly reduce some short simple root coefficients of the current rootlet. It is conceivable that a consequent occurrence of short reflections would restore the previous coefficients of the short simple roots. However, it is not possible in view of [8, Corollary 3.3]. Thus, the reduced decompositions of w cannot contain short simple reflections. □

In [8], we studied the poset structure of the fibres of the mapping $\tau : \mathfrak{Ab}\backslash\{\emptyset\} \to \Delta_l^+$ that takes a non-trivial Abelian ideal to its rootlet. By the previous proposition, we have that the set of non-trivial long Abelian ideals are the union of fibres of this mapping.

2.7. Examples. We use the notation and numbering of roots as in [10, Tables].

1. $\mathfrak{g} = \mathfrak{sp}_{2p}$. Here $\theta = 2\varepsilon_1 = 2\varphi_1$ and $\theta_s = \varepsilon_1 + \varepsilon_2 = \varphi_2$. Normalize the bilinear form $(\,,)$ so that $(\varepsilon_i, \varepsilon_i) = 1$. Then $\mathscr{A}$ has the vertices $\varepsilon_1, \varepsilon_1 + \varepsilon_2, \ldots, \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_p$, whereas $\mathscr{A}_s$ has the vertices $2\varepsilon_1, \varepsilon_1 + \varepsilon_2, \ldots, \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_p$. The unique non-trivial long Abelian ideal is $\{\theta\}$.

2. $\mathfrak{g} = \mathfrak{so}_{2p+1}$. Here $\theta = \varepsilon_1 + \varepsilon_2 = \varphi_2$ and $\theta_s = \varepsilon_1 = \varphi_1$. The unique maximal long Abelian ideal consists of the roots $\{\varepsilon_i + \varepsilon_j \mid 1 \leqslant i < j \leqslant p\}$. The only generator is $\varepsilon_{p-1} + \varepsilon_p$. The corresponding rootlet is also $\varepsilon_{p-1} + \varepsilon_p = \alpha_{p-1} + 2\alpha_p$.

3. $\mathfrak{g} = \mathbf{F}_4$. Here $\theta = 2\alpha_1 + 4\alpha_2 + 3\alpha_3 + 2\alpha_4 = \varphi_4$ and $\theta_s = 2\alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4 = \varphi_1$. The non-trivial long Abelian ideals appear in the first three rows of Table 1 in [8]. The unique maximal long Abelian ideal is $\{\theta, \theta - \alpha_4, \theta - \alpha_4 - \alpha_3\}$.

4. $\mathfrak{g} = \mathbf{G}_2$. The non-trivial long Abelian ideals are $\{3\alpha_1 + 2\alpha_2\}, \{3\alpha_1 + 2\alpha_2, 3\alpha_1 + \alpha_2\}$.

3. Long Abelian ideals and little adjoint representations

In this section, we elaborate on a relationship between long Abelian ideals and commutative subalgebras in the isotropy representations of some symmetric spaces.

We still assume that $\mathfrak{g}$ has two root lengths. Then the $\mathfrak{g}$-representation with highest weight θ_s is said to be *little adjoint*. The corresponding $\mathfrak{g}$-module is denoted by $\mathbb{V}(\theta_s)$. The properties of the $\mathfrak{g}$-module $\mathbb{V}(\theta_s)$ are similar to that of $\mathfrak{g} = \mathbb{V}(\theta)$. For instance, the set of non-zero weights of $\mathbb{V}(\theta_s)$ is Δ_s; the dimension of the zero-weight space is $\#(\Pi_s)$ and each weight space corresponding to $\mu \in \Delta_s^+$ is one-dimensional, see [7, 2.8].

From now on, we stick to the case, where $\frac{|\theta|^2}{|\theta_s|^2} = 2$. Then $\mathbb{V}(\theta_s)$ is the isotropy representation of a symmetric space; more precisely,

$$\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{V}(\theta_s)$$

has a natural structure of $\mathbb{Z}_2$-graded simple Lie algebra. The following table shows all possibilities for θ_s and $\tilde{\mathfrak{g}}$.

$\mathfrak{g}$	$\mathfrak{so}_{2p+1}$	$\mathfrak{sp}_{2p}$	$\mathbf{F}_4$
θ_s	φ_1	φ_2	φ_1
$\dim \mathbb{V}(\theta_s)$	$2p+1$	$2p^2 - p - 1$	26
$\tilde{\mathfrak{g}}$	$\mathfrak{so}_{2p+2}$	$\mathfrak{sl}_{2p}$	$\mathbf{E}_6$

Hence, given a subspace $\mathfrak{a} \subset \mathbb{V}(\theta_s)$, it makes sense to say that it can be a commutative subalgebra (of $\tilde{\mathfrak{g}}$).

The problem of describing Abelian $\mathfrak{b}$-ideals can be generalized to the setting of $\mathbb{Z}_2$-graded Lie algebras as follows:

Suppose $\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{V}$ *is a semisimple* $\mathbb{Z}_2$*-graded Lie algebra, i.e.,* $\mathbb{V}$ *is an* $\mathfrak{h}$*-module and* $[\mathbb{V}, \mathbb{V}] \subset \mathfrak{h}$. Let $\mathfrak{b}(\mathfrak{h})$ *be a Borel subalgebra of* $\mathfrak{h}$. *Describe (enumerate) the* $\mathfrak{b}(\mathfrak{h})$*-stable commutative subalgebras in* $\mathbb{V}$.

The interest of commutative subalgebras in $\mathbb{V}$ is explained by the fact that if $\dim \mathfrak{a} = k$, then the corresponding decomposable k-vector in $\wedge^k \mathbb{V}$ is an eigenvector a Casimir element of $\mathfrak{h}$ whose eigenvalue is maximal possible (and equal $k/2$); the converse is also true, see [6, Section 4]. Originally, the connection between decomposable vectors in the exterior algebra and commutative subalgebras of $\mathfrak{g}$ was studied by Kostant in [3].

I do not think that an answer to the above problem can be given in a uniform way (see also examples below). But, for $\mathbb{Z}_2$-gradings connected with little adjoint representations, there is a reasonably nice description.

Let $\mathfrak{g}^\vee$ denote the Langlands dual Lie algebra for $\mathfrak{g}$, i.e., the root system of $\mathfrak{g}^\vee$ is $\Delta^\vee$. We have $(\Delta^+)^\vee$ is a set of positive roots in $\Delta^\vee$. Write $\mathfrak{b}^\vee$ for the respective Borel subalgebra of $\mathfrak{g}^\vee$. Notice that $(\Delta_l^+)^\vee = (\Delta^\vee)_s^+$ and $\mathbb{V}(\theta^\vee)$ is the little adjoint module for $\mathfrak{g}^\vee$. Given a set S of short roots in $\Delta^\vee$, we write $\mathfrak{a}(S)$ for the respective subspace of $\mathbb{V}(\theta^\vee)$.

3.1. Lemma. *If* $\mathfrak{a}$ *is a commutative* $\mathfrak{b}^\vee$*-stable subalgebra of* $\mathbb{V}(\theta^\vee)$, then $\mathfrak{a} = \mathfrak{a}(S)$ *for some* $S \subset (\Delta_l^+)^\vee$.

Proof. It follows from [6, Proposition 4.9] that $\mathfrak{a}$ has no zero weight and $\mathfrak{a} = \mathfrak{a}(S)$ for some S lying in an open halfspace of V. However, if $-\mu^\vee \in S$ for some $\mu \in (\Delta_l^+)^\vee$, then the $\mathfrak{b}^\vee$-invariance of $\mathfrak{a}$ implies that it has a non-zero component in the zero-weight space of $\mathbb{V}(\theta^\vee)$. $\square$

The following result asserts that there is a bijection between the long Abelian $\mathfrak{b}$-ideals in $\mathfrak{g}$ and the $\mathfrak{b}^\vee$-stable commutative subalgebras of $\mathbb{V}(\theta^\vee)$.

3.2. Theorem. $I \subset \Delta_l^+$ *is an Abelian ideal* $\Leftrightarrow$ $\mathfrak{a}(I^\vee)$ *is a* $\mathfrak{b}^\vee$*-stable commutative subalgebra in* $\mathbb{V}(\theta^\vee)$.

Proof. '$\Rightarrow$' Given an Abelian ideal $I \subset \Delta_l^+$, we are to prove that $\mathfrak{a}(I^\vee)$ is $\mathfrak{b}^\vee$-stable and commutative.

1. Suppose $\mu \in I, \beta \in \Delta^+$, and $\mu^\vee + \beta^\vee \in \Delta^\vee$.

- If β is long, then $\mu^\vee + \beta^\vee = (\mu + \beta)^\vee \in I^\vee$.
- If β is short, then $\mu + \beta \in I$ is a short root, which is impossible.

Thus, the space $\mathfrak{a}(I^\vee)$ is $\mathfrak{b}^\vee$-stable.

2. Assume $\mathfrak{a}(I^\vee)$ is not commutative, i.e., there are some $\mu, \beta \in I$ such that $\mu^\vee + \beta^\vee \in \Delta^\vee$. Since I is Abelian, we have $\mu + \beta \notin \Delta$. Then $\mu + \beta = 2\gamma$, where $\gamma \in \Delta_s$. Therefore $(\mu, \beta) = 0$ and $(\mu, \gamma) = (\beta, \gamma) > 0$. Hence $\gamma - \mu, \gamma - \beta \in \Delta_s$ and $(\gamma - \mu) + (\gamma - \beta) = 0$. Because one of these roots is positive, we conclude that $\gamma \in I \cap \Delta_s$. This contradiction proves that $\mu^\vee + \beta^\vee \notin \Delta^\vee$, i.e., $\mathfrak{a}(I^\vee)$ is a commutative subalgebra.

'$\Leftarrow$' The argument is similar. One has to only use the following property of this $\mathbb{Z}_2$-grading:

Suppose $\nu, \gamma \in (\Delta^\vee)_s$ and v_ν, v_γ are corresponding weight vectors in $\mathbb{V}(\theta^\vee)$. If $\nu + \gamma \in \Delta^\vee$, then $0 \neq [v_\nu, v_\gamma] \in \mathfrak{g}^\vee$. $\square$

Since $(\mathfrak{so}_{2p+1})^\vee = \mathfrak{sp}_{2p}$ and $\mathbf{F}_4^\vee = \mathbf{F}_4$, we see that the number of $\mathfrak{b}$-stable commutative subalgebras is equal to

- 2^{p-1} for the $\mathfrak{sp}_{2p}$-module $\mathbb{V}(\varphi_2)$;
- 2 for the $\mathfrak{so}_{2p+1}$-module $\mathbb{V}(\varphi_1)$;
- 2^2 for the $\mathbf{F}_4$-module $\mathbb{V}(\varphi_1)$.

One should not think, however, that if $\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{V}$ is a $\mathbb{Z}_2$-grading, then the number of $\mathfrak{b}(\mathfrak{h})$-stable commutative subalgebras of $\mathbb{V}$ is always a power of 2.

Example. Straightforward computations give us the following.

1. If $\widehat{\mathfrak{h}} = \mathfrak{gl}_n$ and $\mathfrak{h} = \mathfrak{gl}_r \times \mathfrak{gl}_{n-r}$, then there are $\binom{n}{r} + (n-r)\binom{n-1}{r-1}$ commutative $\mathfrak{b}(\mathfrak{h})$-stable subalgebras in $\mathbb{V}$.

2. If $\widehat{\mathfrak{h}} = \mathfrak{sl}_{2p+1}$ and $\mathfrak{h} = \mathfrak{so}_{2p+1}$, then the number of such commutative subalgebras is $2^{p+1} - 1$.

3.3. Remark. In the previous exposition the case of $\mathfrak{g} = \mathbf{G}_2$ is omitted, and the reason is that the little adjoint $\mathbf{G}_2$-module is associated with a certain $\mathbb{Z}_3$-grading. Namely, there is an automorphism σ of order 3 of $\mathfrak{so}_8$ such that the fixed-point subalgebra is $\mathbf{G}_2$ and two other eigenspaces of σ are little adjoint $\mathbf{G}_2$-modules. That is,

$$\mathfrak{so}_8 = \mathbf{G}_2 \oplus \mathbb{W} \oplus \mathbb{W}',$$

where $\mathbb{W} \simeq \mathbb{W}' \simeq \mathbb{V}(\theta_s)$ and $[\mathbb{W}, \mathbb{W}] \subset \mathbb{W}'$, $[\mathbb{W}', \mathbb{W}'] \subset \mathbb{W}$. Still, it makes sense to speak about $\mathfrak{b}$-stable commutative subalgebras of $\mathbb{W}$. We have $\mathbf{G}_2^\vee \simeq \mathbf{G}_2$. Then completely the same argument shows that the $\mathfrak{b}$-stable commutative subalgebras in $\mathbb{W}$ are in one-to-one correspondence with the long Abelian $\mathfrak{b}$-ideals in $\mathbf{G}_2$.

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