Algorithms to compute the topology of orientable real algebraic surfaces

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Received 30 October 2002; accepted 7 March 2003

Abstract

We present constructive algorithms to determine the topological type of a non-singular orientable real algebraic projective surface $S$ in the real projective space, starting from a polynomial equation with rational coefficients for $S$. We address this question when there exists a line in $\mathbb{RP}^3$ not intersecting the surface, which is a decidable problem; in the case of quartic surfaces, when this condition is always fulfilled, we give a procedure to find a line disjoint from the surface. Our algorithm computes the homology of the various connected components of the surface in a finite number of steps, using as a basic tool Morse theory. The entire procedure has been implemented in Axiom. © 2003 Elsevier Ltd. All rights reserved.

1. Introduction

In this paper we present constructive algorithms to determine the topological type of a non-singular orientable real algebraic projective surface $S$ in the real projective space starting from an equation for it. By the classification theorem for surfaces, we know that any compact connected orientable surface is homeomorphic to a sphere or to the connected sum of $g$ tori, i.e. to a sphere or to a torus with $g$ holes. Therefore we can determine the topology of $S$ by computing the number of its connected components and, for each of them, the number $g$ of holes. Since the first homology group $H_1$ of a torus with $g$ holes is a free Abelian group of rank $2g$, a way to determine the topological type of $S$ is that of computing the group $H_1$ for each of its connected components.

We address the mentioned question when there exists a line in $\mathbb{RP}^3$ not intersecting the surface $S$. In Section 7 we will see that the existence of such a line is a decidable problem;

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we will also show that for quartic surfaces a disjoint line always exists and give a procedure to find it.

Our algorithm, starting from a polynomial equation with rational coefficients of a surface, returns as output a list of as many integers as the connected components of $S$, where the integers represent the ranks of the groups $H_1$ relative to the various connected components. Our strategy will consist in reconstructing in a finite number of steps the homology of $S$ using as a basic tool Morse theory (see for instance Milnor (1963) or Hirsch (1976)), as first proposed in Gianni and Traverso (1983). The basic step, to be used iteratively, is described in Section 4, while the main algorithm is presented in Section 5.

We have developed algorithms for performing all the tasks and we have produced a working implementation in Axiom of the entire procedure. To do this we assume that the given system of coordinates is a “good frame”, in the sense defined in Section 2, which is always true up to a linear change of coordinates. In Section 6 we will describe algorithms to test whether the hypotheses both on the surface and on the system of coordinates are fulfilled or not. In this paper we do not deal with the complexity aspects of our algorithms.

For a proof of the classical results about surfaces, homotopy and homology already recalled and of those we will use throughout the paper, we refer to Massey (1977) or to Massey (1991).

2. First definitions and notations

Let $\phi(x, y, z, t)$ be a square-free homogeneous polynomial of degree $d$ with rational coefficients defining a non-singular orientable real algebraic projective surface; since the surface is orientable, $d$ has to be even. By an abuse of language, by the same term “surface” we will indicate also the set $S$ of the points in $\mathbb{P}^3$ satisfying the equation $\phi(x, y, z, t) = 0$.

Let us emphasize that, when we say that the surface $S$ is non-singular, we mean that no point in $\mathbb{P}^3$ annihilates $\phi$ and all its first partial derivatives. Note that, since $S$ can be seen as the real part of the complex zero-set $S_{\mathbb{C}}$ of $\phi$ in $\mathbb{C}P^3$, according to our assumptions $S_{\mathbb{C}}$ may contain non-real singular points.

Recall that, if $f : S \to \mathbb{R}$ is a differentiable function defined on a smooth surface, a point $P \in S$ is said to be critical for $f$ if, working in local coordinates $u_1, u_2$, both first partial derivatives of $f$ vanish at $P$; then $f(P)$ is said to be a critical value. $P$ is said to be non-degenerate if the Hessian form of $f$ at $P$ is a non-degenerate quadratic form, i.e. the $2 \times 2$ matrix $H_P = (\partial^2 f / \partial u_i \partial u_j)(P)$ is invertible; the number of negative eigenvalues of $H_P$ is called the index of the critical point $P$. If all critical points of $f$ are non-degenerate, $f$ is called a Morse function.

In this paper we will describe an algorithm to study the topology of $S = \{\phi(x, y, z, t) = 0\}$ assuming that there exists a line $L \subset \mathbb{P}^3$ such that $L \cap S = \emptyset$ and that in our system of homogeneous coordinates $[x, y, z, t]$ we have

1. $L = \{z = 0, t = 0\}$
2. If $Z = \{x = 0, y = 0\}$, the function $\pi : S \to Z \cong \mathbb{R}P^1$, restriction to $S$ of the projection $p : \mathbb{P}^3 \setminus L \to Z$ of $\mathbb{P}^3$ onto $Z$ with centre $L$, is a Morse function and $[0, 0, 1, 0]$ is not a critical value for $\pi$
3. Whenever $P, Q \in S$ are critical points for $\pi$, we have $\pi(P) \neq \pi(Q)$. 
As a first step our algorithm checks that all these requirements are satisfied in the given system of coordinates; in Section 6 we will describe the strategy used to check that \( S \) is non-singular, to compute the critical points and to test that all of them are non-degenerate. Observe that, up to a generic linear change of coordinates, one can assume that the assumption (3) is fulfilled. Thus the only essential hypothesis is the existence of a line disjoint from the surface.

Let us denote by \( H_\infty \) the plane \( \{ t = 0 \} \) and by \( U \) the open set \( \mathbb{R}^3 \setminus H_\infty \), which can be identified with \( \mathbb{R}^3 \) by means of the map \( [x, y, z, t] \to (\frac{x}{t}, \frac{y}{t}, \frac{z}{t}) \). Then \( F(x, y, z) = \phi(x, y, z, 1) = 0 \) is an equation of the affine part \( S \cap U \) of \( S \). Let us identify the line \( Z \) with \( \mathbb{R} \cup \{ \infty \} \) associating to any point \([0, 0, z, t]\) of \( Z \) the “affine coordinate” \( z/t \) if \( t \neq 0 \) and \( \infty \) if \( t = 0 \). Since in the affine chart \( U \) we have that \( \pi(x, y, z) = z \), it is possible to check, using local coordinates, that the critical points of \( \pi \) in \( U \) are the points where the first partial derivatives of \( F \) with respect to \( x \) and \( y \) vanish, that is the points where the tangent plane to \( S \) has an equation \( z = h \) for a suitable constant \( h \); by our hypotheses such a plane is tangent to \( S \) only in one point and here it has a contact of order 2 with the surface. Recall that \( \pi \) can have only a finite number of critical values (for a proof, see Milnor, 1968, Corollary 2.8); since by hypothesis \( \infty \) is not critical, there exists an interval \( (-N, N) \subset \mathbb{R} \) containing all the critical values, with \( N \in \mathbb{Q} \). If \( \pi \) has no critical value at all, conventionally we fix \( N = 1 \).

![Diagram](image)

For all \( a \in (-N, N] \) we will denote \( S_a = \pi^{-1}([-N, a]) = S \cap p^{-1}([-N, a]) \) and, for any \( a \in \mathbb{R} \cup \{ \infty \} \), \( C_a \) will denote the level curve \( \pi^{-1}(a) = S \cap p^{-1}(a) \). With this notation, the level surface \( S_a \) is a surface with boundary \( C_{-N} \cup C_a \). Since in \( U \) the plane \( p^{-1}(a) \) is given by the equation \( z = a \), then \( F(x, y, a) = 0 \) is an equation for the curve \( C_a \). If \( a \) is a non-critical value for \( \pi \), the curve \( C_a \) is non-singular.

For our algorithm we will need to investigate a finite number of non-singular level curves and in particular to compute the mutual disposition of their connected components. Recall (see for instance Viro, 1986 or Wilson, 1978) that any connected component \( X \) of a non-singular real algebraic curve \( C \) in the real projective plane is homeomorphic to a circle and can be embedded in \( \mathbb{R}P^2 \) in two different ways: it can disconnect or not the projective plane. In the first case \( X \) is called an oval and it disconnects \( \mathbb{R}P^2 \) into two connected components: one is homeomorphic to a disc and is called the interior part of the oval, the other is homeomorphic to a Möbius band and is called the exterior part of \( X \). An oval is said to be empty if no other oval is contained in its interior part. A list \([X_1, \ldots, X_k]\) of ovals of a curve \( C \) is called a nest of depth \( k \) if \( X_1 \) is empty, \( X_i \) is contained in the interior...
part of $X_{i+1}$ for all $i = 1, \ldots, k - 1$ and $X_k$ is not contained in the interior part of any oval of the curve. If $X$ does not disconnect $\mathbb{R}P^2$, it is called a one-sided component or a pseudo-line. As is well known, a non-singular real algebraic curve can contain at most one pseudo-line and this occurs if and only if the degree of the curve is odd. In our situation any level curve has an even degree; thus all its connected components are ovals and the shape of the curve is completely determined when the set of its nests is known.

3. Our strategy

As explained in the introduction, we want to determine algorithmically the topology of $S$ by computing the number of its connected components and, for each component, the first homology group with rational coefficients; for that, we will use a Morse-type investigation of the projection $\pi$ introduced in the previous section and given in the affine chart $U$ by $\pi(x, y, z) = z$.

From Morse theory let us recall that if $[a, b]$ contains no critical value for $\pi$, then $S_b$ is homotopically equivalent to $S_a$; instead, if $[a, b]$ contains exactly one critical value $c$ for $\pi$, with $a < c < b$, and $k$ is the index of the relative non-degenerate critical point, then $S_b$ is homotopically equivalent to the space obtained attaching a $k$-cell to $S_a$. This information is crucial for us, because homotopically equivalent spaces have the same homology groups.

Even if we know the critical points and the relative indexes, that is not sufficient to determine the shape of the surface. For instance, both surfaces in the figure have six critical points with the same sequence of indexes $(0, 0, 1, 1, 2, 2)$ and, still, they do not even have the same number of connected components. It is therefore necessary to enrich the list of data that will allow us to reconstruct algorithmically the homology of the surface $S$.

Definition 3.1. For any non-critical value $a$ for $\pi$, a data system $DS(S_a)$ for the surface $S_a$ consists of

1. a list $Data(C_a) = [(P_1, n_1), \ldots, (P_k, n_k)]$, where $n_1, \ldots, n_k$ are the nests of the curve $C_a$ and $P_i$ is a point internal to all the ovals of the nest $n_i$;
2. a list $Hom(S_a)$ of integers, where the length of the list is the number of connected components of $S_a$ and the $k$th element of the list represents the rank of the group $H_1$ relative to the $k$th connected component;
3. a function $\mu_a : H_0(C_a) \rightarrow H_0(S_a)$ that associates to each oval $\omega$ of the curve $C_a$ the connected component of $S_a$ that contains $\omega$ in its boundary.
If \( \omega_1, \ldots, \omega_r \) are the ovals of \( C_0 \), we will identify the oval \( \omega_i \) by means of the index \( i \); in particular a nest \([\omega_1, \ldots, \omega_i]\) will be represented by \([i_1, \ldots, i_k]\). Similarly if \( W_1, \ldots, W_p \) are the connected components of \( S_0 \), then the function \( \mu_i \) can be seen as a map \( [1, \ldots, r] \rightarrow [1, \ldots, p] \) which in general is neither injective nor surjective.

In Section 4 we will see that it is possible to compute a data system for \( S_0 \) from a data system for \( S_m \) for any interval \([a, b]\) containing at most one critical value. After subdividing \([-N, N]\) by means of finitely many points \( a_0 = -N < a_1 < \cdots < a_m < a_{m+1} = N \), in such a way that any subinterval \([a_i, a_{i+1}]\) contains exactly one critical value in its interior part, our algorithm will iteratively reconstruct the data systems for all the \( S_m \)'s and thus eventually the homology groups of \( S \). The main algorithm will be described in detail in Section 5.

From the definition of “data system” and the strategy outlined above, it should be clear that our algorithm requires two main steps of investigation: first, we need to study the shape of the level curves \( C_{a_i} \); second, we have to lift the needed information from one level \( a_i \) to the higher level \( a_{i+1} \).

As for the first task, for each level curve \( C_{a_i} \) we have to determine the set of its ovals and their mutual disposition, i.e. the list of its nests. The algorithm we used to do this follows the ideas presented in Gianni and Traverso (1983) and Cellini et al. (1991), although other solutions have been given by other authors (see e.g. Arnon and McCallum, 1988; Cucker et al., 1991; Roy, 1990; Arnborg and Feng, 1988; Winkler, 2000). Whatever algorithm is used to determine the shape of a plane curve, it needs to be augmented by two special functions: \textit{findOvals} and \textit{findPoint}.

Before describing these functions, we observe that, since \( S \cap L = \emptyset \), all our level curves \( C_{a_i} \) are contained in the affine plane \( \mathbb{P}^2 \) and we can consider on them the projection \( \pi_a : C_{a_i} \rightarrow \mathbb{R} \) given by \( \pi_a(x, y) = x \). Also in this case, up to a linear change of coordinates, we can assume that \( \pi_a \) is a Morse function and that, for any critical points \( P, Q \in C_{a_i} \), we have that \( \pi_a(P) \neq \pi_a(Q) \).

The function \textit{findOvals}, given a point \( P \in \mathbb{R}^2 \) and a level curve \( C \), returns the list of ovals of \( C \) containing \( P \) ordered by inclusion starting from the innermost oval. As observed above, in our situation all the level curves do not meet the line at infinity; thus we can assume that both \( P \) and \( C \) are contained in \( \mathbb{R}^2 \). Let \( P = (x_P, y_P) \) and denote by \( x_1, \ldots, x_m \) the critical values of \( \pi : C \rightarrow \mathbb{R}, \pi(x, y) = x \). Our curve algorithm recognizes the ovals of \( C \) passing through the points in the fibres \( \pi^{-1}(x_i) \) and computes the index of the unique critical point in any critical fibre. Using these data, it is an easy matter to reconstruct the fibre \( \pi^{-1}(x_P) \), recognizing the different ovals passing through its points. Thus we get two lists \( L_1 = [l_1, \ldots, l_s] \), \( L_2 = [l_{s+1}, \ldots, l_t] \) representing the labels (possibly with multiplicity, if \( x_P = x_i \) for some \( i \)) of the ovals passing through the points in \( \pi^{-1}(x_P) \) respectively below \( P \) and above \( P \) (following the positive direction along the \( y \) axis). Observe that any oval meets any line an even number of times and that \( P \) is internal to an oval labelled \( k \) if and only if the label \( k \) appears an odd number of times in the list \( L_1 \) (and consequently an odd number of times also in \( L_2 \)). Using this easy but crucial observation, it is easy to compute from the lists \( L_1 \) and \( L_2 \) the ordered list of the ovals containing \( P \).

The second function, called \textit{findPoint}, given an oval \( \omega \) of \( C \), returns a point lying inside \( \omega \), more precisely a point \( Q \) such that \( \omega \) is the first oval of the sequence.
findOvals(\(Q\)). A way to do this is to compute the point \(P = (x_P, y_P)\) having the smallest \(x_P\)-coordinate among the critical points on \(\omega\) of index 1. Then consider the non-empty set \(A = \{(x, y_P) \in \mathcal{C} \mid x > x_P\}\) and let \(R = (x_R, y_P)\) be the point in \(A\) having the smallest first coordinate. The point \(Q = ((x_P + x_R)/2, y_P)\) is a point with the requested property.

As we said, the second step we have to perform is to lift the information from one level to the next. Consider for instance the quite simple situation in which a level curve \(C_a\) consists of only two ovals \(\omega_1\) and \(\omega_2\) external each to the other and therefore the list \(\text{Data}(C_a)\) contains only two elements \((P_1, [\omega_1]), (P_2, [\omega_2])\). If \([a, b]\) contains no critical value, necessarily the curve \(C_b\) contains only two ovals mutually external, say \(\omega_1'\) and \(\omega_2'\), and \(S_b\) is given by the union of \(S_a\) and two cylinders “connecting” the ovals \(\omega_1, \omega_2\) with the ovals \(\omega_1', \omega_2'\). It is however necessary to know whether the cylinder containing \(\omega_1\) in its boundary has as its other boundary component \(\omega_1'\) or \(\omega_2'\). It is possible to detect this, for instance, by means of a continuous path never intersecting the surface \(S\) and connecting the point \(P_1\) with a point \(Q_1\) in the plane \(p^{-1}(b)\). Necessarily \(Q_1\) is internal either to \(\omega_1'\) or to \(\omega_2'\); this can be decided by using the function \(\text{findOvals}\) and thus we can reconstruct algorithmically the list \(\text{Data}(C_b)\) and the function \(\mu_b\).

If we denote by \(W\) the connected component of \((U \setminus S) \cap [a \leq z \leq b]\) that contains the point \(P_1\), the possibility of reaching a point \(Q_1 \in W \cap p^{-1}(b)\) by means of a connecting path lying in \(W\) is guaranteed by the existence of roadmaps. According to the standard definition, a roadmap for our connected component \(W\) is a 1-dimensional connected semialgebraic set \(R \subseteq W\) which intersects any connected component of \(W \cap p^{-1}(c)\) for all \(c \in [a, b]\). In our case the situation is quite simple because, since \([a, b]\) contains no critical value, for any \(c \in [a, b]\) the set \(W \cap p^{-1}(c)\) is connected, hence \(R\) has only to intersect \(W\) at any level \(c\). In our procedure we will also use connecting paths in the opposite direction, i.e. to go from an upper level towards a lower level. Precisely we will use the following

**Remark 3.2.** Let \(P\) be a point in the plane \(p^{-1}(a)\). Given \(b > a\), let \(a : [0, 1] \rightarrow [a \leq z \leq b]\) be a continuous semialgebraic path such that \(a(0) = P, a(1) \in p^{-1}(b)\) and \(a([0, 1]) \cap S = \emptyset\). We will denote by \(\text{roadMap}(P, b)\) the final point \(a(1)\) of the path \(a\). Similarly, if \(b < a\), we will denote by \(\text{invRoadMap}(P, b)\) the final point \(\beta(1)\) of a continuous semialgebraic path \(\beta : [0, 1] \rightarrow [b \leq z \leq a]\) fulfilling the same conditions as \(a\) here above.

The notion of the roadmap was first introduced in Canny (1988) in relation with the problem of deciding connectivity of semialgebraic sets. Since then, many authors have addressed the question and given alternative algorithms for computing roadmaps (see for instance Heintz et al., 1994; Gourary and Risler, 1993; Basu et al., 2000). Since all of these algorithms are equally applicable, we do not think it is necessary to describe in detail the one we used.

4. Iterative step

In this section, given an interval \([a, b]\) containing at most one critical value for \(\pi\), with both \(a\) and \(b\) non-critical for \(\pi\), and assuming to know a data system for \(S_a\), we will see how it is possible to reconstruct a data system for \(S_b\), that is the lists \(\text{Data}(C_b)\) and \(\text{Hom}(S_b)\).
and the function \( \mu_b \). As a first step we study the shape of the level curve \( C_b \) and thus we determine the list of its nests.

Let us first examine the case when \([a, b]\) contains no critical value; in this case \( S_a \) is a deformation retract of \( S_b \), so the number of connected components and the relative homology groups do not change, i.e. \( \text{Hom}(S_b) = \text{Hom}(S_a) \). To complete the computation of a data system \( DS(S_b) \) it is sufficient to compute a finite number of roadmaps. Namely, for each \((P_i, n_i) \in Data(C_a)\) let \( Q_i = \text{roadMap}(P_i, b) \); the point \( Q_i \) is internal to all the ovals of a nest \( m_i \) of \( C_b \) that we can find by means of the function \( \text{findOvals} \). We have thus determined \((Q_i, m_i)\) as an element of the list \( Data(C_b) \). The nests \( n_i \) and \( m_i \) contain the same number of ovals and evidently the \( k \)th oval of \( m_i \) belongs to the boundary of the connected component of \( S_b \) whose retraction in \( S_a \) has in its boundary the \( k \)th oval of \( n_i \). It is so possible to reconstruct the value of the function \( \mu_b \) on the ovals in the nest \( m_i \). After repeating the same procedure for all \((P_i, n_i) \in Data(C_a)\), we have completely reconstructed \( Data(C_b) \) and \( \mu_b \); we have so computed the data system \( DS(S_b) \) from \( DS(S_a) \) by means of a “data lifting process” based on roadmaps.

Let us now see how we can modify the previous procedure of reconstruction of \( DS(S_b) \) when \([a, b]\) contains one single critical value \( c \), where \( a < c < b \) and \( c = \pi(P) \) with \( P \) non-degenerate critical point. If \( P = (x_0, y_0, z_0) \) it will be useful to consider the two points \( P^+ = (x_0, y_0, z_0 + \epsilon) \) and \( P^- = (x_0, y_0, z_0 - \epsilon) \) with \( \epsilon > 0 \) sufficiently small so that the segment joining \( P^+ \) and \( P^- \) does not contain any point of the surface except \( P \).

The idea, at least when the index of the critical point is 0 or 2, is that of modifying \( DS(S_a) \) constructing a fictitious data system \( \tilde{DS}(S_a) \) so that \( DS(S_b) \) can be reconstructed from \( \tilde{DS}(S_a) \) as in the case when \([a, b]\) contains no critical value. We can have three different cases depending on the index of \( P \).

Case 1. \( P \) has index 0:

![Diagram](where \approx \) means “homotopically equivalent”). In this case \( S_b \) has an additional connected component with respect to \( S_a \) and \( S_b \) is homeomorphic to the disjoint union of \( S_a \) and a disc \( D \). Thus, in order to obtain \( \text{Hom}(S_b) \) it is enough to add to the list \( \text{Hom}(S_a) \) the integer 0 corresponding to the first homology group of the disc. To compute \( DS(S_b) \) it will be necessary to detect the position of \( D \) with respect to the cylinders of \( S_b \backslash (S_a \cup D) \) and of its boundary \( \omega \) with respect to the ovals of \( C_b \backslash \omega \). For that, we can construct the point \( P_a = \text{invRoadMap}(P^-, a) \) and determine, by using the function \( \text{findOvals} \), the position of \( P_a \) with respect to the ovals of \( C_a \).

(i) If \( P_a \) is external to all the ovals of \( C_a \), then the disc \( D \) is external to all the tubes of \( S_b \backslash (S_a \cup D) \) and \( C_b \) contains \( \omega \) as an additional nest of depth 1 with respect to \( C_b \). Then we add to the list \( Data(C_a) \) a fictitious pair \((P^+, n^+)\) where \( n^+ \) represents an oval (nest of depth 1) containing \( P^+ \).

(ii) If \( P_a \) is internal to all the ovals of a nest \( n_i \) of \( C_a \), then the disc \( D \) is internal to a “nest” of tubes of \( S_b \backslash (S_a \cup D) \) and \( C_b \) has the same number of nests as \( C_a \), but one
of them has a depth increased by 1. We modify \( Data(C_a) \) replacing the pair \((P_i, n_i)\) by the pair \((P^+, n_i^+)\), where \( n_i^+ \) is obtained from the nest \( n_i \) by adding, in the first position, a further label representing a fictitious innermost oval.

(iii) If \( P_a \) is not external to \( C_a \) but it is not internal to all the ovals of any nest of \( C_a \), then the disc \( D \) is internal to some tube in \( S_0 \setminus (S_a \cup D) \) and its boundary \( \omega \) gives origin to a new nest in \( C_b \), in which \( \omega \) is the innermost oval. In this case we add to the list \( Data(C_a) \) a fictitious pair \((P^+, n^+)\) where \( n^+ \) represents the nest obtained adding, in the first position, an additional label (i.e. a new oval) to the sequence of ovals determined by \( \text{findOvals}(P_a) \).

After having so constructed the data system \( \widehat{DS}(S_0) \) modifying in a suitable way \( DS(S_0) \), we complete the reconstruction of \( DS(S_0) \) by means of roadmaps as in the case with no critical value.

Case 2. \( P \) has index 2:

\[
\begin{array}{c}
\bigcirc \\
\vdots
\end{array}
\]

In this case we obtain \( S_0 \) from \( S_a \) by attaching a 2-cell \( D \) along an oval \( \omega \) of \( C_a \) which was the innermost oval of a nest. It is possible to recognize such an oval: if \( P_a = \text{invRoadMap}(P^-, a) \) and \( \omega \) is the innermost of the ovals containing \( P_a \), determined by the function \( \text{findOvals}(P_a) \), then \( \omega \) is the oval that disappears. Since \( \omega \) is the first oval of a nest (possibly of depth 1), it appears in \( Data(C_a) \) only in one pair \((P_i, n_i)\) we have to modify as follows. If the depth of the nest \( n_i \) is 1, we remove the pair \((P_i, n_i)\) from \( Data(C_a) \). If the depth of the nest \( n_i \) is \( > 1 \), we remove from \( n_i \) the oval \( \omega \) and we choose \( P^+ \) as the innermost point of the new fictitious nest, except that, if removing \( \omega \) from \( n_i \), we find a list of ovals which are part of another nest, then we again remove the pair \((P_i, n_i)\) from \( Data(C_a) \). At this point, as before, starting from the fictitious data system \( \widehat{DS}(S_0) \) so constructed, we reconstruct \( Data(C_b) \) and \( \mu_b \) by means of roadmaps.

As for \( \text{Hom}(S_0) \), since \( S_0 \) has the same number of connected components as \( S_a \), the list \( \text{Hom}(S_0) \) contains as many elements as \( \text{Hom}(S_a) \). If \( W_\omega \) denotes the connected component of \( S_0 \) in whose boundary \( \omega \) lies (and that we can recognize by means of \( \mu_a \)), the only number in the list of integers \( \text{Hom}(S_0) \) that can change is the one relative to the component \( W_\omega \) and the way such a number can change or not depends on the position of the oval \( \omega \) along which the 2-cell \( D \) is attached. Let us denote by \( v : H_0(C_a) \to H_1(S_a) \) the homomorphism obtained by seeing any oval of \( C_a \) as a loop in \( S_a \) and by \( \widehat{W}_\omega \) the connected component of \( S_0 \) obtained attaching \( D \) to \( W_\omega \). Then it is easy to see that

(i) if \( v(\omega) = 0 \), then \( H_1(\widehat{W}_\omega) \cong H_1(W_\omega) \)

(ii) if \( v(\omega) \neq 0 \), then \( H_1(\widehat{W}_\omega) \cong H_1(W_\omega)/(v(\omega)) \).

This can be rigorously proved using the Mayer–Vietoris sequence. Intuitively this result is due to the fact that attaching a 2-cell along \( \omega \) does not alter \( H_1(W_\omega) \) if \( \omega \) already was homologically zero in \( H_1(S_0) \); otherwise, if \( \omega \) becomes homologically zero only
because one can retract it to the constant loop along the 2-cell, then a generator of $H_1(W_\omega)$ disappears and thus the rank of that group decreases by 1.

Observe that $\nu(\omega) = 0$ if and only if $\omega$ is the only boundary component of $W_\omega$, which can be recognized using the function $\mu_d$; thus it is possible to decide whether we are in case (i) or (ii).

**Case 3.** $P$ has index 1:

In this case $S_b$ is obtained from $S_a$ by attaching a 1-cell to two ovals $\omega_1$ and $\omega_2$ possibly coincident. If $\omega_1 \neq \omega_2$, then the two ovals $\omega_1$ and $\omega_2$ are glued by the 1-cell and they give origin in $C_b$ to a single oval, say $\omega$. If on the contrary $\omega_1 = \omega_2$, then in the passage through the critical value $c$, the oval $\omega_1 = \omega_2$ gives origin to two distinct ovals in $C_b$. It is possible to realize which is the situation simply computing the number of ovals of the level curves $C_a$ and $C_b$: if $C_b$ has fewer ovals than $C_a$, then $\omega_1 \neq \omega_2$; if $C_b$ has more ovals than $C_a$, then $\omega_1 = \omega_2$.

Let us start to consider the case $\omega_1 \neq \omega_2$. As a first step, it is possible to detect the oval $\omega$ in $C_b$ originated by the glueing of $\omega_1$ and $\omega_2$. Namely, if $Q = \text{roadMap}(P^+, b)$, then $\omega$ is the first oval in the list $Ov(Q) = \text{findOvals}(Q)$. It is possible that $Ov(Q)$ is not a nest in $C_b$; precisely $Ov(Q)$ is a nest in $C_b$ if and only if both $\omega_1$ and $\omega_2$ contained no ovals in their interior parts. In this case the number of nests of $C_b$ decreases by 1 with respect to the number of nests of $C_a$, a situation which however occurs also when only one of the two ovals, say for instance $\omega_1$, is empty (but then $Ov(Q)$ is not a nest in $C_b$). When neither $\omega_1$ nor $\omega_2$ are empty, then $C_b$ has as many nests as $C_a$ but, again, $Ov(Q)$ is not a nest in $C_b$. We can therefore reconstruct algorithmically $\text{Data}(C_b)$ proceeding as follows:

1. Study the curve $C_b$ and compute $Q$, $Ov(Q)$ and $\omega$.
2. Compare $Ov(Q)$ with the nests of $C_b$: if $Ov(Q)$ is one of the nests of $C_b$, insert the pair $(Q, Ov(Q))$ in the list $\text{Data}(C_b)$; otherwise leave $\text{Data}(C_b)$ temporarily empty.
3. For each $(P_i, m_i)$ in $\text{Data}(C_a)$, compute the point $Q_i = \text{roadMap}(P_i, b)$ and $m_i = \text{findOvals}(Q_i)$. If $m_i \notin Ov(Q)$, add $(Q_i, m_i)$ to $\text{Data}(C_b)$, otherwise discard the pair $(Q_i, m_i)$.

It is now also clear how we can reconstruct $\mu_b$: for the ovals of $C_b$ belonging to a nest $n$ different from $Ov(Q)$, since such a nest has been reconstructed by means of a roadmap, it is possible to reconstruct the value of $\mu_b$ on the ovals of $n$ as in the case when $[a, b]$ contains no critical value. Thus at most we have still to deal with the ovals in $Ov(Q)$. The innermost
of them is \( a_1 \), which is therefore in the boundary both of the connected component \( H_{\mu_a}(a_1) \) and of the connected component \( H_{\mu_a}(a_2) \): notice that, when \( H_{\mu_a}(a_1) \neq H_{\mu_a}(a_2) \), we are in the situation in which \( a_1 \) and \( a_2 \) lie in different connected components of \( S \) that are glued by the attachment of the 1-cell. Such an attachment and the consequent relation that, when passing through the critical value \( c \), identifies the components \( H_{\mu_a}(a_1) \) and \( H_{\mu_a}(a_2) \), must be taken into account in the process of labelling the components of \( S \) and in the definition of \( H_{\mu_a} \).

Let us now see how we can reconstruct \( Data(C_b) \) and \( H_{\mu_b} \) when \( a_1 = a_2 \). In this case for each \((P_i, n_i)\) in \( Data(C_a) \), compute \( Q_i = roadMap(P_i, b) \) and \( m_i = findOvals(Q_i) \), and insert \((Q_i, m_i)\) in the list \( Data(C_b) \). It is possible that there exists an oval in \( C_b \) that does not appear in the list of nests of \( Data(C_b) \) so constructed; that happens if and only if the oval \( a_1 = a_2 \) splits into two distinct ovals of \( C_b \), say \( a_1 \) and \( a_2 \), and all the points \( Q_i \) obtained starting from points \( P_i \) in \( a_1 = a_2 \) lie inside the same oval, say for instance \( a_2 \). In this case we compute, using the function \( findPoint \), a point \( Q_i \), internal to the empty oval \( a_1 \) and we add the pair \((Q_i, findOvals(Q_i))\) to the list \( Data(C_b) \). Moreover, as usual, taking into account the roadmaps already computed, we can reconstruct the value of \( H_{\mu_b} \) on all the ovals of \( C_b \); the only possible oval not obtained in this way is \( a_1 \), for which we already know that \( H_{\mu_b}(a_1) = H_{\mu_a}(a_1) \).

Recall that any compact surface \( M \) with a non-empty boundary is homotopically equivalent to an "\( n \)-leafed rose", i.e. a union of \( n \) circles with a single point in common; the first homology group of such a space has rank \( n \), so this determines the rank of the group \( H_1 \) of the surface \( M \). This observation can be useful for reconstructing \( Hom(S_b) \): if \( H_{\mu_b}(a_1) = H_{\mu_b}(a_2) \) (which in particular occurs when \( a_1 = a_2 \), the attachment of the 1-cell can be seen as attaching a further leaf to the rose homotopically equivalent to \( S_{\mu_a}(a_1) \). Hence in this case the rank of the group \( H_1 \) of this component increases by 1. If \( H_{\mu_b}(a_1) \neq H_{\mu_b}(a_2) \), the 1-cell connects two different components into a new one homotopically equivalent to a rose having as many leaves as the sum of the ranks of \( H_1(S_{\mu_a}(a_1)) \) and of \( H_1(S_{\mu_a}(a_2)) \).

5. The surface algorithm

Assume that all the requirements on the surface have been positively tested and that the critical points have been computed; how to do that will be examined in Section 6.

Let us start to describe the way our algorithm works when the surface \( S \) does not intersect the plane at infinity \( H_\infty \), i.e. when \( S \) is contained in the affine chart \( U \) and \( S = S_N \).

Subdivide \([-N, N] = [-N, a_1] \cup [a_1, a_2] \cup \cdots \cup [a_m, N] \) choosing finitely many rational numbers \( a_i \) in such a way that each \( a_i \) is non-critical for \( \pi \) and each interval \([a_i, a_{i+1}]\) contains only one critical value in its interior part. The level curve \( C_{-N} \) is empty; so we initialize the lists \( Data(C_{-N}) \) and \( Hom(S_{-N}) \) as empty lists. After calling a finite number of times the algorithmic step described in the previous section, eventually the list \( Hom(S_{N}) \) gives the first homology groups of the connected components of \( S \) we were looking for.

When \( C_\infty = S \cap H_\infty \) is not empty, \( S_N \) is a differentiable surface with boundary \( C_{-N} \cup C_N \) and \( S' = S \setminus S_N \) is diffeomorphic to \( C_N \times [0, 1] \), i.e. to the disjoint union of a finite number of cylinders. If we denote by \( \hat{S} \) the topological surface obtained
attaching a 2-cell to each oval of the boundary of \( S_N \), then \( \hat{S} \) is compact in \( U \) and \( S \) can be seen topologically as the surface obtained attaching to \( \hat{S} \) the handles of \( S' \) along the ovals of the boundary of \( S_N \). We will see that, in order to compute the homology of \( S \), essentially we have only to compute the homology of \( \hat{S} \) and a finite number of roadmaps.

Notice that the surface \( \hat{S} \) is not algebraic, or at least we do not have an equation for it; however, if we denote \( \hat{S}_a = \hat{S} \cap p^{-1}((-\infty, a]) \), we know that \( \hat{S} \cap p^{-1}(-N) = S \cap p^{-1}(-N) = \hat{S}_N \) and therefore, studying the non-empty curve \( C_{-N} \), we can initialize the list \( Data(C_{-N}) \). By construction \( \hat{S}_{-N} \) is a disjoint union of \( k \) discs, where \( k \) is the number of ovals in \( C_{-N} \); so we initialize \( \text{Hom}(\hat{S}_{-N}) \) as the list consisting of \( k \) copies of the integer 0. Now, proceeding as in the compact case and since \( \hat{S} \cap p^{-1}([-N, N]) = S \cap p^{-1}([-N, N]) \), we can reconstruct \( DS(\hat{S}_N) \). Note that, if \([-N, N]\) contains no critical values, the computation of \( DS(\hat{S}_N) \) is particularly simple and, as seen before, it requires only a finite number of roadmaps. Since \( \hat{S} \) is obtained from \( \hat{S}_N \) attaching a 2-cell along the ovals of \( C_N \), the list \( \text{Hom}(\hat{S}) \) can be obtained from \( \text{Hom}(\hat{S}_N) \) as described in Case 2 of Section 4.

As already said, \( S \) can be obtained from \( \hat{S} \) attaching a finite number of cylinders. The way in which the homology groups vary depends on the way each cylinder is attached. If it is attached to one single connected component of \( \hat{S} \) as a handle, then the group \( H_1 \) of that component increases its rank by 2; if its attachment connects two distinct components of \( \hat{S} \) (and in other words we obtain the connected sum of those components), then the group \( H_1 \) of the unified component has a rank equal to the sum of the ranks of the connected components that have been linked.

Thus the only matter we have to deal with is to recognize the components of \( \hat{S} \) that any cylinder of \( S' \) connects, which can be detected by means of roadmaps. Any such cylinder intersect the plane \( H_\infty \), but the previous problem can be dealt with, working in non-homogeneous coordinates and so using the function \( \text{roadMap} \), in the affine chart of \( \mathbb{R}^3 \) given by \( \mathbb{R}^3 \setminus \{(x, y, z, 1) \mid z = 0\} \).

**Example 1.** Consider \( F(x, y, z) = z^4 + (2y^2 + 2x^2 - 10)z^2 + y^4 + (2x^2 + 6)y^2 + x^4 - 10x^2 + 9 \) and let us show how our algorithm recognizes it defines a torus.

First of all we find four critical values: \(-3, -1, 1, 3\). The interval \([-5, 5]\) containing all the critical values is subdivided as \([-5, -2] \cup [-2, 0] \cup [0, 2] \cup [2, 5]\). The curve \( C_{-5} \) is empty (hence the surface does not meet the plane at infinity), so \( Data(C_{-5}) \) and \( \text{Hom}(S_{-5}) \) are empty lists.

**Step 1** (Reconstruction of \( DS(S_{-2}) \)). The curve \( C_{-2} \) has only one oval and the critical point corresponding to the value \(-3\) is \( P = (0, 0, -3) \) which has index 0. Following Case 1 of Section 4, we modify \( Data(C_{-5}) \) into the fictitious list \( (P^+, [1]) \) with \( P^+ = (0, 0, -3 + \epsilon) \) and compute \( \text{roadMap}(P^+, -2) = (0, 0, -2) \). Since it is internal to the nest \([1]\) of \( C_{-2} \), we set \( Data(C_{-2}) = [(0, 0, -2), [1]], \mu_{-2}(1) = 1 \) and \( \text{Hom}(S_{-2}) = [0] \).

**Step 2.** The interval \([-2, 0]\) contains the only critical point \( P = (0, 0, -1) \) which has index 1. The curve \( C_0 \) has two ovals labelled 1 and 2, so we realize that the only oval in \( C_{-2} \) splits into two ovals at level 0. If we compute \( \text{roadMap}((0, 0, -2), 0) \), we get a point \( Q_1 = (x_0, 0, 0) \) lying in the oval 1 of \( C_0 \); using the function \( \text{findPoint} \) we find a point
The interval $[0, 2]$ contains the critical point $P = (0, 0, 1)$ which has index 1. The curve $C_2$ has one oval labelled 1 so, again according to Case 3 in Section 4, we get a point $R$ in the oval 1 of $C_2$ and we reconstruct $Data(C_2) = [(R, [1])], \mu_2(1) = 1$ and $Hom(S_2) = [2]$.

Example 2. Consider $F(x, y, z) = x^2 + y^2 - z^2 - 4$. In this case we find that the curve $C_{\infty}$ is an oval; according to what was seen earlier in this section, we first compute the homology of the compact affine surface $\tilde{S}$. Computations yield that there are no critical values, so we set $N = 1$. The level curve $C_{-1}$ has one oval containing the point $Q = (0, 0, -1)$; we initialize $Data(C_{-1}) = ([Q, [1]]), \mu_{-1}(1) = 1$ and $Hom(\tilde{S}_{-1}) = [0]$.

Then we compute $R = roadMap(Q, 1) = (0, 0, 1)$ and reconstruct $Data(C_1) = [(R, [1]), \mu_1(1) = 1$ and $Hom(\tilde{S}_1) = [0]$. Attaching a fictitious 2-cell along the oval at level 1, we get that $Hom(\tilde{S}) = [0]$, so $\tilde{S}$ is a sphere. We have only to compute a roadmap that, passing through the plane at infinity, connects the point $R$ to a point in the plane $z = -1$. Since evidently we are attaching a handle to the sphere, we get that $Hom(S) = [2]$, so $S$ is again a topological torus embedded in a different way with respect to that of Example 1.

Example 3. Let $F(x, y, z) = z^8 + 4z^7 + (4y^2 + 4x^2 - 126)z^6 + (12y^2 + 12x^2 - 392)z^5 + (6y^4 + 12x^2 - 158)y^2 + 6x^4 - 222x^2 + 3745)z^4 + (12y^4 + 24x^2 - 336)y^2 + 12x^4 - 464x^2 + 8148)z^3 + (4y^6 + (12x^2 + 62)y^4 + (12x^4 - 4x^2 + 3562)y^2 + 4x^6 - 66x^4 - 2646x^2 - 28260)z^2 + (4y^6 + (12x^2 + 56)y^4 + (12x^4 - 16x^2 + 3732)y^2 + 4x^6 - 72x^4 - 2412x^2 - 32400)z + y^8 + (4x^2 + 94)y^6 + (6x^4 + 218x^2 + 1993)y^4 + (4x^6 + 154x^4 + 978x^2 - 7560)y^2 + 9x^8 + 30x^6 + 9x^4 - 3240x^2 - 1$. Here the algorithm computes eight critical points contained in the interval $[-12, 12]$, which is subdivided as $[-12, -8] \cup [-8, -5] \cup [-5, -2] \cup [-2, -1/2] \cup [-1/2, 1] \cup [1, 4] \cup [4, 7] \cup [7, 12]$ so that each subinterval contains exactly one critical value. The curve $C_{\infty}$ is empty, so $Data(C_{-12})$ and $Hom(S_{-12})$ are empty lists.

Step 1. The interval $[-12, -8]$ contains one critical point of index 0; by the usual procedure we get $Data(C_{-8}) = [(0, 0, -8), [1]], \mu_{-8}(1) = 1$ and $Hom(S_{-8}) = [0]$.

Step 2. The interval $[-8, -5]$ contains one critical point of index 1. The curve $C_{-5}$ has two ovals: the point $Q_2 = roadMap((0, 0, -8), -5)$ lies in the oval 1 of $C_{-5}$ by using $findPoint$ we find a point $R_2$ in the oval 2. So $Data(C_{-5}) = [(Q_2, [1]), (R_2, [2])], \mu_{-5}(1) = \mu_{-5}(2) = 1$ and $Hom(S_{-5}) = [1]$.

Step 3. In $[-5, -2]$ we pass through a point of index 1, the two ovals of the previous level glue together and a hole appears in the surface. Computations yield $Data(C_{-2}) = [(Q_3, [1]), (R_2, [1])], \mu_{-2}(1) = 1$ and $Hom(S_{-2}) = [2]$.
Step 4. Here a second critical point of index 0 gives origin to a second connected component and \( C_{-1/2} \) is a nest of depth 2. The algorithm output is \( \text{Data}(C_{-1/2}) = \{(Q_4, [2, 1])\}, \mu_{-1/2}(1) = 1, \mu_{-1/2}(2) = 2 \) and \( \text{Hom}(S_{-2}) = [2, 0] \).

Step 5. In \([-1/2, 1]\) we pass through a critical point of index 2, which attaches a 2-cell to the connected component of \( S_{-1/2} \) labelled 2. We get \( \text{Data}(C_1) = \{(Q_5, [1])\}, \mu_1(1) = 1 \) and \( \text{Hom}(S_1) = [2, 0] \).

Step 6. In \([-1, 4]\) we find again a point of index 1, which causes the splitting of the oval \( C_{-1} \) into two ovals: the reconstructed data system is now \( \text{Data}(C_4) = \{(Q_6, [1]), (R_6, [2])\}, \mu_4(1) = \mu_4(2) = 1 \) and \( \text{Hom}(S_4) = [3, 0] \).

Step 7. \([4, 7]\) contains one more point of index 1, where two ovals glue together and give origin to a second hole in the level surface \( S_7 \): \( \text{Data}(C_7) = \{(Q_7, [1])\}, \mu_7(1) = 1 \) and \( \text{Hom}(S_7) = [4, 0] \).

Step 8. The last critical point has index 2 and corresponds to the attachment of a 2-cell along the boundary of \( S_7 \). The last call of the procedure yields \( \text{Hom}(S_{12}) = [4, 0] \); this says that \( S = S_{12} \) is a compact surface with two connected components: one of them is a torus with two holes, the other one is a sphere.

6. Singular and critical points

In this section we describe how to check that the surface is non-singular and that it has, with respect to the projection \( \pi \) fixed in Section 2, only a finite number of real critical points, all of which are required to be non-degenerate; we also show how to compute them. Let us emphasize that our procedure to test the singularities and to compute the critical points does not require that the surface is disjoint from the line \( L = \{z = 0, t = 0\} \).

Let \( \phi(x, y, z, t) \) be a square-free homogeneous polynomial representing our surface \( S \). As a first step, we test that \( \infty \) is not a critical value for \( \pi \) checking that the curve \( C_\infty \) is non-singular.

Denote by \( J \) the homogeneous ideal generated by all four partial derivatives of \( \phi \) and by \( K \) the homogeneous ideal generated by \( \phi \) and its partial derivatives with respect to \( x \) and \( y \). We also denote by \( V(J) \) and \( V(K) \) (resp. \( V_\mathbb{R}(J) \) and \( V_\mathbb{R}(K) \)) the sets of the complex zeros (resp. real zeros) of \( J \) and \( K \). Then \( V_\mathbb{R}(J) \) is the real singular locus of \( S \) and, after checking that \( V_\mathbb{R}(J) \) is empty, \( V_\mathbb{R}(K) \) will be the real critical locus. Since we only require that \( S \) has no real singularities and that \( \pi \) has no real degenerate critical points, we need to distinguish between the real and complex solutions of the polynomial systems defining \( V(J) \) and \( V(K) \), where \( J \) and \( K \) are in general 1-dimensional ideals.

The problem of deciding whether a real algebraic variety is empty is one of the basic questions in computational real algebraic geometry. Several algorithms proposed by different authors are available in the literature, from the classical cylindrical algebraic decomposition (Collins, 1975), which solves a much more general problem, to the very recent paper (Aubry et al., 2002), which also contains an extensive bibliography on the subject. Our situation, though, is very special; therefore we will present a simpler special purpose approach in order to avoid these more complicated comprehensive algorithms.
We will take advantage not only of the low dimension, but also of the special geometric properties of our varieties.

As $\infty$ is not a critical value for $\pi$, the curve $C_\infty$ is non-singular. So there exists no real point $[x, y, z, 0] \in H_\infty$ that annihilates $\phi$ and its partial derivatives with respect to $x$, $y$ and $z$ and hence there exists no real singular point for $S$ on $H_\infty$. Thus it is enough to investigate the structure of $V_\mathbb{R}(J)$ and of $V_\mathbb{R}(K)$ in the affine chart $U$, i.e. we consider the non-homogeneous ideals in three variables $J_A$ and $K_A$ obtained respectively from $J$ and $K$ by evaluating their generators for $t = 1$. If, as usual, we denote $F(x, y, z) = \phi(x, y, z, 1)$, then $J_A = (F, \partial F/\partial x, \partial F/\partial y, \partial F/\partial z)$ and $K_A = (F, \partial F/\partial x, \partial F/\partial y)$.

Let $f$ denote the product of all the univariate factors of $F$ in the variable $z$. If $f$ has a real root $z_0$, then the plane $[z = z_0]$ is contained in $S \cap U$ and therefore $S$ has real singularities; in this case the surface does not fulfil the requirements and will be discarded. If, on the contrary, $f$ has no real root, we can divide $F$ by $f$ without modifying the real zero-set. Thus we can assume that $F$ is not divisible by any univariate polynomial in $z$.

Now, the first question we have to deal with is to decide whether $V_\mathbb{R}(J_A) = V_\mathbb{R}(F, \partial F/\partial x, \partial F/\partial y, \partial F/\partial z) = \emptyset$ or not (Singularity test).

The ideal $J_A$ cannot have dimension 2, because $F$ is square-free; moreover, without changing the zero-set, we can assume that $J_A$ is radical. The ideal $J_A$ can define the empty set, a finite set of points or a curve (respectively when the dimension of the ideal is $-1$, 0 or 1). If the ideal has dimension 0, it is easy to decide whether the corresponding points are real or not (see e.g. Aubry et al., 1999; Auzinger and Stetter, 1988; Gonzalez-Vega et al., 1999; Cohen et al., 1999).

The situation is more delicate when the dimension is 1, i.e. when the ideal defines a space curve; thus finding an answer to our Singularity test reduces to solving the following problem:

**Empty real test**: Given a (complex) affine space curve defined by a 1-dimensional radical ideal $J_A \subset \mathbb{Q}[x, y, z]$ and such that the projective closure of $V_\mathbb{R}(J_A)$ in $\mathbb{P}^3$ does not meet the plane at infinity, decide whether $V_\mathbb{R}(J_A)$ is empty or not.

We will decide about emptiness of $V_\mathbb{R}(J_A)$ by constructing an ideal $D$ of dimension $\leq 0$ such that $V_\mathbb{R}(J_A)$ is empty if and only if $V_\mathbb{R}(D)$ is empty, which is easy to test. The ideal $D$ is constructed choosing a “good projection” of the space curve $C_A = V(J_A)$ on a plane in such a way that we can recover the real points on the space curve from the study of the plane one. We will say that a projection $\sigma : \mathbb{C}^3 \to \mathbb{C}^2$ is a good projection for $C_A$ if generically it gives a one-to-one correspondence between the points of $C_A$ and the points of $\sigma(C_A)$ and only for at most a finite number of points $P$ in $\sigma(C_A)$ the fibre $\sigma^{-1}(P) \cap C_A$ consists of finitely many (>1) points.

It is possible to see that, up to a generic linear change of coordinates, the projection on the plane $[x = 0]$ can be assumed to be a good projection for $C_A$ and that such a condition can be easily tested using a lexicographic Gröbner basis (with $x > y > z$) for the ideal $J_A$. One proof of these facts can be found in Fortuna et al. (2003), where the structure of the reduced lex Gröbner basis for a radical ideal of pure dimension 1 is discussed. The proof, that is based on the Shape lemma (Gianni and Mora, 1989), can be easily generalized with similar arguments to hold for 1-dimensional radical ideals containing also zero-dimensional components. Using these results we can test whether the projection
Hence dim \( q \) and such that the gcd ep rojection we can try to test if the basis contains at least one polynomial coefficients of the linear equations for \( x \) precisely,

Remark 6.1. If the Gröbner basis does not fulfil these conditions because all the leading coefficients of the linear equations for \( x \) are not relatively prime with \( g \), before changing the projection we can try to test if the basis contains at least one polynomial \( h \) linear in \( x \) and such that the gcd \( q \) between \( g \) and the leading coefficient of \( h \) satisfies the condition \( V_\mathbb{R}(q) = \emptyset \). In such a case, we have that \( V_\mathbb{R}(J_A, g) = V_\mathbb{R}(J_A, g/q) \); therefore, instead of \( (J_A, g) \) we can use the ideal \( J_A' = (J_A, g/q) \) which defines the same real locus and for which \( J_A' \cap \mathbb{Q}[y, z] = (g/q) \). It is then clear that the projection \( (x, y, z) \to (y, z) \) is good.

We want now to relate the existence of real points on the space curve \( C_A \) to the existence of real points on \( \tilde{C}_A = V(I) \). With our notations \( V(I) = V(I_0) \cup V(g) \); without changing the zero-set, we can assume that \( g \) is square-free, and hence that \( V(g) \) has at most finitely many singular points.

Remark 6.2. We have that \( \sigma(V_\mathbb{R}(J_A)) \subseteq V_\mathbb{R}(I) \); thus, if \( V_\mathbb{R}(I) \) is empty, then \( V_\mathbb{R}(J_A) \) is empty. The opposite inclusion is not true, because \( V_\mathbb{R}(I) \) can contain real points which are not the image of any real point in \( V_\mathbb{R}(J_A) \), but which are the projection of complex points in \( V(J_A) \). Since the projection is good, this can happen only for finitely many points. More precisely, \( Z = V_\mathbb{R}(I) \setminus \sigma(V_\mathbb{R}(J_A)) \) consists of finitely many isolated points. In particular, recalling that \( V_\mathbb{R}(J_A) \) has no point at infinity, \( V_\mathbb{R}(g) \) cannot contain any 1-dimensional connected component intersecting the line at infinity.

As a consequence, if \( g \) is divisible by a univariate polynomial in the variable \( y \), say \( f(y) \), \( f \) cannot admit any real root \( y_0 \), because otherwise the line \( \{y = y_0\} \) would be contained in \( V_\mathbb{R}(g) \). So, up to dividing \( g \) by its univariate factors in \( y \), which does not modify the real zero-set, we can assume that \( g \) is not divisible by any univariate polynomial in \( y \).

Lemma 6.3. Let \( K_1 = (g, \partial g/\partial z) \). Then the ideal \( K_1 \) has dimension \( \leq 0 \).

Proof. The zero-set defined by \( K_1 \) contains the points of \( V(g) \) which are either singular or critical with respect to the projection on the \( y \) axis. We already know that \( V(g) \) has at most finitely many singular points, because \( g \) is square-free. Moreover, since \( g \) has no univariate factors in \( y \), \( V(g) \) cannot contain a 1-dimensional irreducible component of critical points. Hence \( \dim K_1 \leq 0 \). □

In order to give a computational answer to the Empty real test, we need only to distinguish if the presence of real points in \( V_\mathbb{R}(I) \) originates through projection from real or complex points.
Proposition 6.4. Let $K_1 = (g, \partial g/\partial z)$ and $D = (J_A, I_0 \cdot K_1)$. Then

(i) the ideal $D$ has dimension $\leq 0$,
(ii) $V_{\mathbb{R}}(J_A)$ is empty if and only if $V_{\mathbb{R}}(D)$ is empty.

Proof. (i) The ideal $D$ is at most zero-dimensional because $\dim(I_0 \cdot K_1) \leq 0$ and the projection is good.
(ii) If $V_{\mathbb{R}}(J_A) = \emptyset$, evidently $V_{\mathbb{R}}(D) = \emptyset$. Conversely assume that $V_{\mathbb{R}}(D)$ is empty. The thesis is trivial if $V_{\mathbb{R}}(I) = \emptyset$. Otherwise consider $V_{\mathbb{R}}(g)$ and recall that it cannot contain any 1-dimensional connected component intersecting the line at infinity. Moreover $V_{\mathbb{R}}(g)$ cannot contain any 1-dimensional compact component: otherwise such a component should contain a non-isolated singular or critical point $w \in V_{\mathbb{R}}(K_1)$. Since $Z = V_{\mathbb{R}}(I) \setminus \sigma(V_{\mathbb{R}}(J_A))$ consists only of isolated points (see Remark 6.2), then $w \notin Z$, i.e. the fibre $\sigma^{-1}(w)$ should contain a point in $V_{\mathbb{R}}(J_A)$, while $V_{\mathbb{R}}(D)$ is empty. Thus $V_{\mathbb{R}}(g)$ can contain at most finitely many points, that are necessarily singular and hence contained in $V_{\mathbb{R}}(K_1)$. Then $V_{\mathbb{R}}(I) = V_{\mathbb{R}}(I_0) \cup V_{\mathbb{R}}(K_1)$ and therefore by the hypothesis $V_{\mathbb{R}}(J_A) = \emptyset$. □

After testing that the surface has no real singular points, we start to investigate the real critical locus $V_{\mathbb{R}}(K_A)$.

Also the ideal $K_A$ cannot have dimension 2: otherwise, if $h = 0$ is an equation of an irreducible component of $V(K_A)$ of dimension 2, then $h$ divides both $F$ and its derivatives with respect to $x$ and $y$. Hence $h$ necessarily is a univariate polynomial in $z$, while we know that $F$ does not admit any such factor.

Again the only subtle point is when the ideal $K_A$ has dimension 1. The situation in this case is slightly different, because we will construct an ideal $G$ of dimension $\leq 0$ such that $V_{\mathbb{R}}(G)$ contains exactly the real critical non-degenerate points.

First of all we want to exclude the existence of real degenerate critical points, that is critical points that annihilate the determinant of $H$, where $H$ is the matrix

$$
\begin{pmatrix}
\frac{\partial^2 F}{\partial x^2} & \frac{\partial^2 F}{\partial x \partial y} \\
\frac{\partial^2 F}{\partial x \partial y} & \frac{\partial^2 F}{\partial y^2}
\end{pmatrix}
$$

So we have to check that $V_{\mathbb{R}}(K_A, \det(H))$ is empty, which can be done by the same strategy we used earlier for $J_A$. If this is the case, we have only to compute the real critical points.

Proposition 6.5. Any point $P$ lying on a 1-dimensional (complex) component of $V(K_A)$ is necessarily either singular for $S$ or degenerate.

Proof. Recall that $K_A = (F, \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y})$ and consider the $3 \times 3$ Jacobian matrix Jac of $K_A$. At any critical point $P$, the determinant of Jac is the product of $\partial F/\partial z$ and the determinant of $H$ evaluated at $P$. Since $P$ lies on a 1-dimensional component of $V(K_A)$, this determinant vanishes in $P$ and therefore either $\partial F/\partial z(P) = 0$, and hence $P$ is singular for $S$, or the determinant of $H$ vanishes at $P$, that is $P$ is a degenerate critical point. □

Since we already know that $S$ has neither real singular points nor real degenerate critical points, then there exist only finitely many real critical points. Moreover, if we remove from $V(K_A)$ the points lying in $V(\partial F/\partial z \det(H))$, we are sure that we are not missing any real critical point and also that we are removing from $V(K_A)$ all the 1-dimensional components.
Thus the ideal $G$ defining the set $V(K_A) \setminus V(\partial F / \partial z \det(H))$ is at most zero-dimensional and $V_{\mathbb R}(G)$ contains precisely the real non-degenerate critical points. Since it is well known how to compute $G$, for instance using Gröbner bases (see Gianni et al., 1988), we can compute the points in the critical locus.

**Example 4.** Let $S$ be the projective closure of the affine surface defined by $F = x^6 - 12x^3 + 3x^4y^2 - 4x^2z^2 + 22x^4 - 24x^3y^2 + 60x^3z^2 + 156x^3 + 3x^2y^4 - 8x^2y^2z^2 + 44x^2y^2 - x^2z^4 - 163x^2z^2 - 419x^2 - 12x^2y + 60x^2z^2 + 156y^2^2 - 48x^2z^2 - 300x^2z^2 - 432x + y^6 - 4y^4z^2 + 22y^4 - y^2z^2 - 163y^2z^2 - 419y^2 + 4z^6 + 165z^4 + 911z^2 + 1260.

The curve $S \cap H_{\mathbb R}$ is non-singular, so we can check whether $S$ fulfills all the requirements working in the affine chart $U$.

Consider first the Jacobian ideal $J_A = (F, \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z})$ defining the singular locus. The ideal $J_A$ has dimension 1; so, to show that $V_{\mathbb R}(J_A)$ is empty, we need to project the space curve on a plane. In order to see whether the projection $(x, y, z) \to (y, z)$ is good, we compute the lexicographic Gröbner basis for $J_A$ with $x > y > z$ and we get

$$(36x^3 + \ldots , (3z^2 + 5)(7056y^3 + 66313)x + \ldots ,
\times (3z^2 + 5)(7z^2 - 61)x + \ldots , (3z^2 + 5)\widetilde{g}).$$

So $J_A \cap \mathbb Q[y, z] = (g) = ((3z^2 + 5)\widetilde{g})$. In this case the lex Gröbner basis contains a monic polynomial for $x$ and also two polynomials linear in $x$, but for both of them the gcd between the leading coefficient and $g$ is $3z^2 + 5$. Since $V_{\mathbb R}(3z^2 + 5) = \emptyset$, by Remark 6.1 instead of $V_{\mathbb R}(J_A)$ we can study $V_{\mathbb R}(J_A, \widetilde{g})$ for which the projection is good.

Thus we consider the projected plane curve $\tilde{C}_A$ given by $\tilde{g} = 0$ and the ideal $K_1 = (\tilde{g}, \frac{\partial \tilde{g}}{\partial y})$ defining the set of the singular or critical points of $\tilde{C}_A$ (w.r.t. the projection on the $y$ axis). Since it turns out that $V_{\mathbb R}(K_1)$ is empty, then $V_{\mathbb R}(J_A)$ is also empty, i.e. $S$ is non-singular.

At this point we can begin the analysis of the critical points of the surface (w.r.t. the projection $\pi(x, y, z) = z$). The first thing we need to check is that there is no real degenerate critical point for $S$. For that, we consider the ideal $K_A = (F, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z})$ defining the critical locus and we add to it the Hessian determinant $\det(H) = \frac{\partial^2 F}{\partial z^2} - \left(\frac{\partial^2 F}{\partial x^2}\right)^2$; we want to see that $V_{\mathbb R}(K_A, \det(H))$ is empty.

The ideal $(K_A, \det(H))$ is one-dimensional, but if we compute the lex Gröbner basis for it with $x > y > z$, we find in it a univariate polynomial in $z$ that does not have any real root. This tells us immediately that $S$ has no real degenerate critical point and that the real critical locus is given by $V(K_A) \setminus V(\partial F / \partial z \det(H))$.

Then we compute the saturation of $K_A$ with respect to $(\partial F / \partial z \det(H))$, that is

$$\left(K_A : \left(\frac{\partial F}{\partial z} \det(H)\right) \cap \mathbb Q[x, y, z]\right).$$

This yields the zero-dimensional ideal $(x - 24/65z^4 - 30/13z^2 - 216/65, y, z^6 + 21/4z^4 + 11/4z^2 - 9)$, whose zero-set gives the two real critical points

$$P_1 = (6, 0, -1) \quad P_2 = (6, 0, 1).$$
7. Existence of a disjoint line

Our algorithm works when the surface is disjoint from some projective line. Whether a line disjoint from \( S \) exists or not in \( \mathbb{RP}^3 \) is a decidable problem. If we write parametrically a line as \( \{ \lambda P + \mu Q \mid [\lambda, \mu] \in \mathbb{RP}^1 \} \) with \( P = [a, b, c, 0] \in H_\infty \) and \( Q = [\alpha, \beta, \gamma, \delta] \in \mathbb{RP}^3 \), the previous problem is equivalent to deciding whether there exist values for the parameters \( a, b, c, \alpha, \beta, \gamma, \delta \in \mathbb{R} \) such that \([a, b, c, 0] \notin S \) and the equation
\[
\psi(s) = \phi(as + \alpha, bs + \beta, cs + \gamma, \delta) = 0
\]
has no real solution. In the literature one can find methods to determine the number of real roots of polynomials with parametric coefficients, using for instance the good properties of the Sturm-Habicht sequence (see for instance Gonzalez-Vega et al., 1989, 1998; Habicht, 1948; Roy, 1996). For low degree polynomials, one can also find a list of explicit expressions in terms of the coefficients that is sufficient to determine the number and multiplicities of the real and complex roots, i.e. the complete root classification (see Arnon, 1988; Yang et al., 1996a,b; Yang, 1999).

It is clear that, even though the previous methods guarantee that the problem of the existence of the line is decidable, they are not very practical. For sure they can be avoided.

We shall give a special purpose procedure to find a line not intersecting the surface \( S \) in a curve having no real singularities.

If the section curve \( C_\infty = S \cap H_\infty \) has two complex conjugate singular points, then the line connecting the two points has a real equation and is disjoint from \( C_\infty \) in \( H_\infty \) and from \( S \) in \( \mathbb{RP}^3 \).

If \( C_\infty \) has neither real nor complex singular points (in particular it is irreducible), then we can use the fact, proved by Zeuthen (1874), see also Degtyarev et al. (2000), that the real quartic curve has a real double tangent intersecting the curve in two real points (maybe coinciding) or in a pair of complex conjugate points. In the latter case, the double tangent is already disjoint from the real locus of the quartic; let us show that, in the case of real points of tangency, it is possible to obtain a disjoint line by slightly perturbing the double tangent.

Denote by \( C_A \) a non-singular real affine plane curve of degree 4 and by \( C \) its projective closure; let \( r \) be a real double tangent to \( C \) touching the curve in two real points \( P, Q \in C \).

Consider first the case when \( P \neq Q \). Up to changing coordinates, we can assume that \( r \) has equation \( y = 0 \) and that \( P, Q \in C_A \). By Bézout’s theorem, \( r \) and \( C \) cannot meet at infinity, i.e. \([1, 0, 0] \notin C \). Since the multiplicity of intersection of \( r \) and \( C \) in \( P \) and \( Q \) is 2, locally at the two points the curve lies in one of the two half-planes bounded by \( r \).

Assume first that locally at both points the curve is contained in the same half-plane, say for instance \( y < 0 \). Consider, for all \( a \in \mathbb{R} \), the lines \( s_a = \{ y = az \} \). Since \([1, 0, 0] \notin C \), \( s_a \) and \( C \) cannot meet at infinity.

Let \( \Gamma = \{ a \in \mathbb{R} \mid a > 0, s_a \cap C \neq \emptyset \} \). If \( \Gamma = \emptyset \), then we have a lot of disjoint lines. Otherwise, let \( b = \inf \Gamma \); it is sufficient to prove that \( b > 0 \), because then \( s_b \) is disjoint from \( C \) for all \( a \in (0, b) \). Assume by contradiction that \( b = 0 \). Then we might find a sequence of points \( P_n = (x_n, y_n) \in C_A \), with \( 0 < y_n < 1 \) and \( y_n \) converging to 0. Since \([1, 0, 0] \notin C \), there exists \( M > 0 \) such that \( P_n \in [-M, M] \times [0, 1] \) for all \( n \in \mathbb{N} \). Thus, up to extracting
a subsequence, we can assume that \( \{P_n\} \) converges to a point \( R \in C_A \cap \{y = 0\} \), hence \( R = P \) or \( R = Q \). This is impossible because locally at \( P \) and \( Q \) the curve is contained in \( \{y < 0\} \).

Note that the same argument can be used also when the real double tangent meets the curve in a single real point \( P \), because then locally at \( P \) in a suitable system of coordinates the curve is given by \( y + x^4 = 0 \). If locally at \( P \) and \( Q \) the quartic is contained in distinct half-planes, it is sufficient to adapt the previous argument considering a pencil of lines passing through the middle point of the segment \( PQ \).

Thus, the only task we have to perform is to compute a real double tangent to a projective plane curve \( C \) having neither real nor complex singular points and given by \( G(x, y, z) = 0 \) with \( G \) a homogeneous polynomial of degree 4. To do that, we can use the dual curve \( \hat{C} \subseteq (\mathbb{CP}^2)^* \) consisting of all points corresponding to the lines tangent to \( C \) at some point. As is well known, the double tangents to \( C \) correspond to singular points of \( \hat{C} \); more precisely, the double tangents touching \( C \) in two distinct points correspond to nodes of \( \hat{C} \), while the double tangents intersecting \( C \) only in one point correspond to non-ordinary double points of type “tacnode”.

If we denote by \([u, v, w]\) homogeneous coordinates in \((\mathbb{CP}^2)^*\), a way to compute an equation for \( \hat{C} \) is to consider the ideal \( B = (G, u - \frac{\partial G}{\partial x}, v - \frac{\partial G}{\partial y}, w - \frac{\partial G}{\partial z}) \) generated by bihomogeneous polynomials in \( x, y, z \) and \( u, v, w \). Then \( \hat{C} \) is the curve defined by the ideal \( B \cap \mathbb{Q}[u, v, w] \), which is principal, generated by a polynomial \( \hat{G}(u, v, w) \); we can compute \( \hat{G} \) eliminating the variables \( x, y, z \) from \( B \) for instance by using a lex Gröbner basis with \( x > y > z > u > v > w \). For more efficient methods to compute the dual curve, see Volcheck (1997) and Bouziane and El Kahoui (2002).

The singular locus of \( \hat{C} \) is given by the ideal \( \hat{J} = (\frac{\partial \hat{G}}{\partial u}, \frac{\partial \hat{G}}{\partial v}, \frac{\partial \hat{G}}{\partial w}) \). Then the ideal \( (B, \hat{J}) \) defines the set of pairs \((P, l)\) where \( l \) is a line either tangent to \( C \) in \( P \) with intersection multiplicity \( \geq 3 \) or tangent to \( C \) in \( P \) and in other points. After discarding all the \((P, l)\) where \( l \) has complex coordinates, we consider the set \( \mathcal{P} \) of the remaining pairs; notice that, by Zeuthen’s theorem, \( \mathcal{P} \) is non-empty.

If in \( \mathcal{P} \) we find two pairs \((P_1, l), (P_2, l)\) with \( P_1 \) and \( P_2 \) complex conjugate points, then \( l \) is a real double tangent touching \( C \) in \( P_1 \) and \( P_2 \) and therefore already disjoint from \( C \) in the real projective plane.

If in \( \mathcal{P} \) we find two pairs \((P_1, l), (P_2, l)\) with \( P_1 \), \( P_2 \) distinct real points, then \( l \) is a real double tangent touching \( C \) in two real points; as seen above, it is sufficient to slightly perturb it.

It may happen that, for any two pairs \((P_i, l_i), (P_j, l_j) \in \mathcal{P} \), we have \( l_i \neq l_j \). If \( l \) appears only in one pair \((P, l)\), then there are two possibilities: either \( P \) is an inflection point for \( C \) and \( l \) intersects \( C \) in \( P \) and in another simple point, or \( l \) is a “degenerate” double tangent to \( C \) in \( P \), i.e. it meets \( C \) in \( P \) with multiplicity of intersection 4. In the former case we discard \( l \); in the latter case, which always occurs at least once by Zeuthen’s theorem, we can find a disjoint line slightly perturbing \( l \). We can decide which is the situation by checking the number of distinct intersections between \( C \) and \( l \).
Examples

(1) Let $C$ be the projective plane quartic curve defined by

$$G(x, y, z) = 4x^4 - 6x^4 + 2y z^3 + 3y^2 z^2 - 5x^2 z^2 - 4y^3 z + 10x^2 y z + y^4 - 5x^2 y^2.$$ 

$C$ has no real singular points, but for instance $P_1 = [i\sqrt{5/3}, 1 + i\sqrt{8/3}, 1]$ and $P_2 = [-i\sqrt{5/3}, 1 - i\sqrt{8/3}, 1]$ are complex conjugate singular points for $C$. Then the line connecting $P_1$ and $P_2$ has a real equation $\sqrt{8/3}x - y + z = 0$ and it is disjoint from $C$ in $\mathbb{RP}^2$.

(2) Let $G(x, y, z) = x^4 + y^4 - 3x^2 y^2 - z^4$. The curve $C = \{G = 0\}$ has neither real nor complex singular points. Among the pairs of $V(B, \hat{J})$ we find

$$(P_1, l) = \left(\left[\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 1\right], \left[\sqrt{5}, \sqrt{5}, 1\right]\right)$$

$$(P_2, l) = \left(\left[-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 1\right], \left[\sqrt{5}, \sqrt{5}, 1\right]\right).$$

Then $l = \{\sqrt{5}x + \sqrt{5}y + z = 0\}$ is a real double tangent to $C$.

(3) Let $G(x, y, z) = 2x^4 - 81x^4 - 324y^2 z^2 - 324y^3 z - 65y^4 + 32y^3 z + 24x^2 y^2 + 8x^3 y$. Working in the affine chart $[z = 1]$ of the projective plane and in the affine chart $[w = 1]$ of the dual projective plane, we find four pairs $(P, l)$ in $V(B, \hat{J})$ where $l$ has real coordinates:

$$\left([0, -3, 1], \left[\frac{1}{3}, \frac{1}{3}, 1\right]\right), \quad \left([6, -3, 1], \left[\frac{1}{3}, 1, 1\right]\right)$$

$$\left([0, -\frac{3}{5}, 1], \left[\frac{1}{3}, \frac{5}{3}, 1\right]\right), \quad \left([\frac{6}{5}, -\frac{3}{5}, 1], \left[\frac{1}{3}, 1, 1\right]\right).$$

Since each real $l$ appears only in one pair $(P, l)$, we have to check whether $l$ intersects $C$ in $P$ in an inflection point or $l$ is a degenerate double tangent for $C$ in $P$. For instance, taking $([0, -3, 1], [-1/3, 1/3, 1])$, we see that the line $l = \{-x + y + 3z = 0\}$ meets $C$ only at $[0, -3, 1]$, so it is a degenerate double tangent.

Remark 7.1. For curves $C$ of even degree $\geq 4$, it is not true anymore that there always exists a line disjoint from $C$. For instance it is possible to check that $C = \{(924166^5 + (-25600x^2 - 12604x^3)z^4 + (-5776y^4 + 11201x^2y^2 - 12625x^4)z^2 + 1600y^6 - 2500x^2y^4 - 1600x^3y^2 + 2500x^6 = 0\}$ is not disjoint from any line in the real projective plane. In any case, though, using arguments similar to the ones described before, it is possible to define a procedure for deciding whether there exists or not a line disjoint from a given curve (see Fortuna et al., 2003).

Acknowledgements

This research was partially performed with the contribution of M.U.R.S.T. and of Eurocontract HPRN-CT-2001-00271.
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