

Convex Polyhedra of Doubly Stochastic Matrices. I. Applications of the Permanent Function

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The permanent function is used to determine geometrical properties of the set Ω_n of all $n \times n$ nonnegative doubly stochastic matrices. If \mathcal{F} is a face of Ω_n , then \mathcal{F} corresponds to an $n \times n$ (0, 1)-matrix A , where the permanent of A is the number of vertices of \mathcal{F} . If A is fully indecomposable, then the dimension of \mathcal{F} equals $\sigma(A) - 2n + 1$, where $\sigma(A)$ is the number of 1's in A . The only two-dimensional faces of Ω_n are triangles and rectangles. For $n \geq 6$, Ω_n has four types of three-dimensional faces. The facets of the faces of Ω_n are characterized. Faces of Ω_n which are simplices are determined. If \mathcal{F} is a face of Ω_n which is two-neighborly but not a simplex, then \mathcal{F} has dimension 4 and six vertices. All k -dimensional faces with $k + 2$ vertices are determined. The maximum number of vertices of a k -dimensional face is 2^k . All k -dimensional faces with at least $2^{k-1} + 1$ vertices are determined.

1. INTRODUCTION

In this paper we investigate some geometrical properties of the set Ω_n of all $n \times n$ nonnegative doubly stochastic matrices. It is a well-known fact that Ω_n is a closed bounded convex polyhedron in Euclidean n^2 -space whose dimension is $(n - 1)^2$ and whose vertices are the $n \times n$ permutation matrices (see, e.g., [11, pp. 95–101]). One reason for the interest in Ω_n is that it is the polyhedron that arises in the optimal assignment problem [5, pp. 111–112]. Doubly stochastic matrices have been studied quite extensively,

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especially in their relation with the van der Waerden conjecture for the permanent [12]. Apart from some applications in combinatorics and probability, the permanent, unlike the determinant, appears to be a specialized function with limited application in mathematics. However, in our approach to the investigation of properties of Ω_n , the permanent turns out to be a very useful tool.

The nonempty faces of Ω_n correspond to $n \times n$ matrices of 0's and 1's whose rows and columns can be permuted to give a direct sum of fully indecomposable matrices. Let A be an $n \times n$ fully indecomposable matrix of 0's and 1's. The number of vertices of the face \mathcal{F} corresponding to A equals the permanent of A . We shall see that the dimension of \mathcal{F} is given by $\sigma(A) - 2n + 1$, where $\sigma(A)$ is the total number of 1's in A . Since the number of vertices of \mathcal{F} exceeds its dimension, we conclude that per $A \geq \sigma(A) - 2n + 2$, which is an inequality for the permanent due to Minc [13]. We determine when equality holds in this inequality and in this way determine all faces of Ω_n which are simplices.

The faces of Ω_n are themselves convex polyhedra, and we determine their facets. The faces of Ω_n have rather special properties which, in general, are not shared by other convex polyhedra. The only two-dimensional faces of Ω_n are triangles and rectangles. Also, there are only four types of three-dimensional faces, and we determine them. The convex polyhedron Ω_3 is four-dimensional and two-neighborly, and has six vertices. We shall show that any face of Ω_n which is two-neighborly but not a simplex must have dimension 4 and six vertices. Other special faces of Ω_n are investigated. We characterize the k -dimensional faces of Ω_n with $k + 2$ vertices. We show that a k -dimensional face of Ω_n has at most 2^k vertices and determine when equality holds.

In subsequent parts of this paper, other properties of Ω_n are investigated. In particular, a number of results concerning the vertex-edge graph of faces of Ω_n are derived. We also look more thoroughly at general combinatorial and affine properties of faces of Ω_n .

2. BASIC PROPERTIES

An $n \times n$ real matrix A is *doubly stochastic* provided all row and column sums equal 1. Let Ω_n denote the compact convex polyhedron of all $n \times n$ nonnegative doubly stochastic matrices. As already pointed out, the vertices of Ω_n are the $n \times n$ permutation matrices [4], and the dimension of Ω_n is $(n - 1)^2$ [11, pp. 99-101]. We adopt the geometric terminology of Grünbaum [7]; in particular, a compact convex polyhedron with a finite number of vertices is called a *polytope*. If P is a fixed $n \times n$ permutation matrix and X is an $n \times n$ real matrix, then $X \rightarrow PX$ defines an orthogonal transformation of Euclidean n^2 -space which maps Ω_n onto Ω_n . Since

given $n \times n$ permutation matrices Q_1 and Q_2 there exists a permutation matrix P such that $PQ_1 = Q_2$, there exists such an orthogonal transformation which maps Q_1 to Q_2 . We shall often use this homogeneity property by fixing a particular permutation matrix.

The polytope Ω_n consists of all $n \times n$ real matrices $X = [x_{ij}]$ which satisfy the following constraints:

$$x_{ij} \geq 0 \quad (i, j = 1, \dots, n), \tag{2.1}$$

$$\sum_{k=1}^n x_{ik} = 1 = \sum_{k=1}^n x_{ki} \quad (i = 1, \dots, n). \tag{2.2}$$

Thus Ω_n is a polytope in the $(n - 1)^2$ -dimensional linear manifold defined by Eqs. (2.2). We obtain the faces of Ω_n by replacing some of the inequalities of (2.1) by equalities. Note that because of the constraints given in (2.2), the constraints of (2.1) are not independent. Let $K \subseteq \{(i, j) : i, j = 1, \dots, n\}$ and determine a face \mathcal{F} of Ω_n by replacing (2.1) by

$$\begin{aligned} x_{ij} &\geq 0 && ((i, j) \in K), \\ x_{ij} &= 0 && ((i, j) \notin K). \end{aligned}$$

Let $B = [b_{ij}]$ be the $n \times n$ $(0, 1)$ -matrix, where $b_{ij} = 1$ if and only if $(i, j) \in K$. Thus the face \mathcal{F} consists of all $n \times n$ nonnegative matrices $X = [x_{ij}]$ satisfying (2.2) and $x_{ij} \leq b_{ij}$ ($i, j = 1, \dots, n$). We denote this face by $\mathcal{F}(B)$. The $n \times n$ permutation matrices P such that $P \leq B$ are precisely the vertices of the faces $\mathcal{F}(B)$. Hence if B' is an $n \times n$ $(0, 1)$ -matrix such that for each permutation matrix P , $P \leq B$ if and only if $P \leq B'$, then $\mathcal{F}(B) = \mathcal{F}(B')$. As a consequence, if there exist $r, s \in \{1, \dots, n\}$ with $b_{rs} = 1$ for which there is no permutation matrix $P = [p_{ij}]$ with $p_{rs} = 1$ and $P \leq B$, then $\mathcal{F}(B) = \mathcal{F}(B')$, where B' is obtained from B by replacing b_{rs} by 0. Hence in determining the nonempty faces of Ω_n we need only consider those $n \times n$ nonzero $(0, 1)$ -matrices $B = [b_{ij}]$, with the property that $b_{rs} = 1$ implies there exists a permutation matrix $P = [p_{ij}]$ with $p_{rs} = 1$ and $P \leq B$. Such matrices B are said to have *total support* [17].

Let A be an $n \times n$ nonnegative matrix. If $n > 1$, A is *fully indecomposable*, provided there do not exist permutation matrices P and Q such that

$$PAQ = \begin{bmatrix} A_1 & 0 \\ A_3 & A_2 \end{bmatrix}, \tag{2.3}$$

where A_1 and A_2 are square matrices. If $n = 1$, A is fully indecomposable if and only if A is positive. If A is doubly stochastic matrix of the form (2.3), then it follows easily that $A_3 = 0$. Hence it follows that if A is a doubly stochastic matrix, then there exist permutation matrices P and Q such that PAQ is a direct sum of fully indecomposable matrices. It follows from [16]

that a $(0, 1)$ -matrix has total support if and only if there exist permutation matrices P and Q such that PAQ is a direct sum of fully indecomposable matrices. Thus the nonempty faces of Ω_n are in one-to-one correspondence with the $n \times n$ $(0, 1)$ -matrices which can be permuted to direct sums of fully indecomposable matrices. If A is an $n \times n$ $(0, 1)$ -matrix with total support, then we say that the matrix A and the face $\mathcal{F}(A)$ correspond to each other. The matrix corresponding to Ω_n is the $n \times n$ matrix $K_{n,n}$ all of whose entries are 1. Note that if P and Q are $n \times n$ permutation matrices and A^T is the transpose of A , then $\mathcal{F}(A)$, $\mathcal{F}(PAQ)$, and $\mathcal{F}(A^T)$ are all congruent.

Let $A = [a_{ij}]$ be an $n \times n$ real matrix. Then the permanent of A is defined by

$$\text{per } A = \sum_{\sigma \in S_n} \prod_{i=1}^n a_{i\sigma(i)},$$

where S_n is the symmetric group on $\{1, \dots, n\}$. The basic properties of the permanent can be found in the survey article [12]. If A is an $n \times n$ $(0, 1)$ -matrix, then the permanent of A is the number of permutation matrices P with $P \leq A$. Thus if A is an $n \times n$ $(0, 1)$ -matrix with total support, then the permanent of A is the number of vertices of the face of Ω_n corresponding to A .

THEOREM 2.1. *Let P_1, \dots, P_t be distinct $n \times n$ permutation matrices. Let $A = [a_{ij}]$ be the $n \times n$ $(0, 1)$ -matrix such that $a_{ij} = 1$ if and only if the (i, j) -entry of at least one of the P_k 's is 1. Then A has total support and $\mathcal{F}(A)$ is the smallest face of Ω_n which contains the vertices P_1, \dots, P_t . Moreover, P_1, \dots, P_t are the vertices of a face of Ω_n if and only if $\text{per } A = t$.*

Proof. From the definition of A , A has total support. Clearly, $\mathcal{F}(A)$ is a face containing P_1, \dots, P_t . Now suppose that B is a $(0, 1)$ -matrix with total support such that $\mathcal{F}(B)$ contains P_1, \dots, P_t . Then $P_i \leq B$ ($i = 1, \dots, t$), so that $A \leq B$. Hence $\mathcal{F}(A) \subseteq \mathcal{F}(B)$. Therefore $\mathcal{F}(A)$ is the smallest face containing P_1, \dots, P_t . This, in turn, implies that P_1, \dots, P_t are the vertices of a face if and only if they are the vertices of $\mathcal{F}(A)$. Since $\text{per } A$ equals the number of vertices of $\mathcal{F}(A)$, P_1, \dots, P_t are the vertices of $\mathcal{F}(A)$ if and only if $\text{per } A = t$.

Let P be an $n \times n$ permutation matrix. Then the cycles of P are defined to be the cycles of the permutation in S_n which corresponds to P . If Q is also an $n \times n$ permutation matrix, then the cycle number of P and Q , $\nu(P, Q)$, equals the number of cycles of length greater than 1 of $P^{-1}Q$. Observe that $\nu(P, Q) = \nu(Q, P)$ and that if I is the $n \times n$ identity matrix, then $\nu(I, P)$ equals the number of cycles of length greater than 1 of P . If R is an $n \times n$ permutation matrix, then it is easy to verify that $\nu(RP, RQ) = \nu(P, Q) = \nu(PR, QR)$. The following result is essentially the same as [1, Theorem 2].

THEOREM 2.2. *Let P, Q be distinct $n \times n$ permutation matrices. Then $\nu(P, Q) \geq 1$ with equality if and only if P and Q are the vertices of a (one-dimensional) face of Ω_n .*

Proof. Clearly, $\nu(P, Q) \geq 1$ if $P \neq Q$. By the homogeneity of Ω_n and the invariance properties of the cycle number we may take $P = I$. Let A be the matrix obtained from I and Q as in Theorem 2.1. There exists a permutation matrix R such that $RAR^{-1} = L_1 \oplus \cdots \oplus L_k \oplus I$, where $k = \nu(I, Q)$ and for each $i = 1, \dots, k$, L_i has order at least 2 and is of the form

$$L = \begin{bmatrix} 1 & 0 & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 1 & 1 \end{bmatrix}. \tag{2.4}$$

Since R is a permutation matrix, $\text{per } A = \text{per } RAR^{-1} = 2^k$. Hence, from Theorem 2.1, I and Q are the vertices of a face if and only if $k = 1$.

We determine the relationship between the number of 1's in a $(0, 1)$ -matrix A with total support and the dimension of $\mathcal{F}(A)$ by first considering nearly decomposable matrices. An $n \times n$ $(0, 1)$ -matrix A is called *nearly decomposable* provided it is fully indecomposable and no matrix obtained by replacing a 1 in A with a 0 is fully indecomposable. Given a fully indecomposable nonnegative integral matrix B there exists a nearly decomposable $(0, 1)$ -matrix A with $A \leq B$. It has been shown [9, 18] that given an $n \times n$ nearly decomposable matrix A with $n > 1$ there exist permutation matrices P and Q such that PAQ has the form

$$\left[\begin{array}{ccc|ccc} 1 & & & & & 1 \\ 1 & 1 & & & & \\ & 1 & \cdot & & & \\ & & \cdot & \cdot & 1 & \\ & & & 1 & 1 & \\ \hline & & & 1 & & \\ & & & & & A_1 \end{array} \right], \tag{2.5}$$

where A_1 is an $m \times m$ nearly decomposable matrix with $1 \leq m < n$ and where unspecified entries are 0. Let A be a nearly decomposable matrix of order $n \geq 3$. It has been shown [10, 14] that $\sigma(A) \leq 3(n - 1)$. Hence A has at least three rows (columns) with exactly two entries equal to 1.

If \mathcal{F} is a face of Ω_n , then $\dim \mathcal{F}$ denotes the dimension of \mathcal{F} . If A is any matrix, then $\sigma(A)$ denotes the sum of the entries of A .

LEMMA 2.3. *Let A be an $n \times n$ nearly decomposable matrix. Then $\dim \mathcal{F}(A) \geq \sigma(A) - 2n + 1$.*

Proof. We induct on n . Clearly, the lemma is true for $n = 1$. Let $n > 1$. We may assume A has the form (2.5), where A_1 is an $m \times m$ nearly decomposable matrix with $m < n$. Let B be the matrix obtained from A by replacing the 1 in the $(1, n)$ -position by a 0. Then clearly $\dim \mathcal{F}(B) = \dim \mathcal{F}(A_1)$. Since A is fully indecomposable, there exists a permutation matrix P with $P \leq A$ and $P \not\leq B$. Hence $\dim \mathcal{F}(A) \geq \dim \mathcal{F}(A_1) + 1$. Observing that $\sigma(A) - 2n + 1 = (\sigma(A_1) - 2m + 1) + 1$, we conclude from the inductive assumption that $\dim \mathcal{F}(A) \geq \sigma(A) - 2n + 1$.

LEMMA 2.4. *Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $n \times n$ fully indecomposable $(0, 1)$ -matrices with $A \leq B$, $A \neq B$. Then $\dim \mathcal{F}(A) < \dim \mathcal{F}(B)$.*

Proof. Choose r, s such that $a_{rs} = 0, b_{rs} = 1$. Since B is fully indecomposable, there exists a permutation matrix $P = [p_{ij}]$ such that $P \leq B$ and $p_{rs} = 1$. Since $P \not\leq A$, P is not in $\mathcal{F}(A)$. Thus $\mathcal{F}(A)$ is a proper subspace of $\mathcal{F}(B)$, so $\dim \mathcal{F}(A) < \dim \mathcal{F}(B)$.

THEOREM 2.5. *Let A be an $n \times n$ fully indecomposable $(0, 1)$ -matrix. Then*

$$\dim \mathcal{F}(A) = \sigma(A) - 2n + 1.$$

Proof. Let B be a nearly decomposable matrix with $B \leq A$. Set $t = n^2 - \sigma(B)$. Then there exist fully indecomposable matrices A_0, A_1, \dots, A_t such that $A_0 = B, A_t = K_{n,n}, A_j = A$ for some j and

$$A_i \leq A_{i+1}, \quad \sigma(A_{i+1}) = \sigma(A_i) + 1 \quad (i = 0, \dots, t - 1).$$

By Lemma 2.3, $\dim \mathcal{F}(B) \geq \sigma(B) - 2n + 1$. Hence it follows from Lemma 2.4 that

$$\dim \mathcal{F}(A) \geq \sigma(A) - 2n + 1.$$

From Eqs. (2.2) we conclude that $\dim \mathcal{F}(K_{n,n}) \leq (n - 1)^2$. So from Lemma 2.4 again,

$$\dim \mathcal{F}(A) \leq \sigma(A) - 2n + 1.$$

This proves the theorem.

Since $\dim \mathcal{F}(A_1 \oplus \dots \oplus A_t) = \dim \mathcal{F}(A_1) + \dots + \dim \mathcal{F}(A_t)$ for non-zero square $(0, 1)$ -matrices, we have the following.

COROLLARY 2.6. *Let A be an $n \times n$ $(0, 1)$ -matrix with total support. Let P and Q be permutation matrices such that $PAQ = A_1 \oplus \cdots \oplus A_t$, where A_i is a fully indecomposable matrix ($i = 1, \dots, t$). Then*

$$\dim \mathcal{F}(A) = \sigma(A) - 2n + t.$$

Let A be an $n \times n$ $(0, 1)$ -matrix with total support, and let v equal the number of vertices of $\mathcal{F}(A)$ and d equal the dimension of $\mathcal{F}(A)$. We define $\Delta(A)$ by $\Delta(A) = v - d$. Clearly $\Delta(A) \geq 1$ with equality if and only if $\mathcal{F}(A)$ is a simplex. Since $v = \text{per } A$ and $d = \sigma(A) - 2n + t$, where t is the number of fully indecomposable components of A , $\Delta(A) = \text{per } A - (\sigma(A) - 2n + t)$. In particular, we obtain the following inequality, due to Minc [13].

COROLLARY 2.7. *If A is an $n \times n$ fully indecomposable $(0, 1)$ -matrix, then $\text{per } A \geq \sigma(A) - 2n + 2$.*

In the next section, our study of faces of Ω_n which are simplices will enable us to determine when equality holds in Minc's inequality.

Let A be an $n \times n$ $(0, 1)$ -matrix with total support which is not a permutation matrix. Then there exist permutation matrices P and Q such that $PAQ = A_1 \oplus \cdots \oplus A_s \oplus I$, where A_i is a fully indecomposable $(0, 1)$ -matrix of order at least 2 ($i = 1, \dots, s$) and I is a (possibly vacuous) identity matrix. The matrices A_1, \dots, A_s are called the *nontrivial fully indecomposable components* of A . They are unique up to permutations of their rows and columns. Clearly, $\dim \mathcal{F}(A) \geq s$.

THEOREM 2.8. *Let A be an $n \times n$ $(0, 1)$ -matrix with total support having s nontrivial fully indecomposable components. Then $\Delta(A) \geq s$. Equality holds if and only if either $s = 2$ and each of the nontrivial fully indecomposable components corresponds to a face of dimension 1, or $s \leq 1$ and $\mathcal{F}(A)$ is a simplex.*

Proof. We may assume $A = A_1 \oplus \cdots \oplus A_t$, where A_i is an $n_i \times n_i$ fully indecomposable matrix ($i = 1, \dots, t$) with $n_i > 1$ if and only if $1 \leq i \leq s$. The conclusions of the theorem clearly hold if $s = 0$. Suppose $s > 0$. We have

$$\begin{aligned} \Delta(A) &= \text{per } A - (\sigma(A) - 2n + t) \\ &= \prod_{i=1}^t \text{per } A_i - \sum_{i=1}^t (\sigma(A_i) - 2n_i + 1). \end{aligned}$$

Since $n_i = 1$ for $i = s + 1, \dots, t$, and $\text{per } A_i \geq 2$ for $i = 1, \dots, s$, it follows that

$$\begin{aligned} \Delta(A) &= \prod_{i=1}^s \text{per } A_i - \sum_{i=1}^s (\sigma(A_i) - 2n_i + 1) \\ &\geq \sum_{i=1}^s \text{per } A_i - \sum_{i=1}^s (\sigma(A_i) - 2n_i + 1) \\ &= \sum_{i=1}^s \Delta(A_i) \geq s. \end{aligned}$$

The theorem now follows.

As an immediate consequence we have the following.

COROLLARY 2.9. *Let A be an $n \times n$ $(0, 1)$ -matrix with total support and let A be different from a permutation matrix. Then $\mathcal{F}(A)$ is a simplex if and only if A has exactly one nontrivial fully indecomposable component A_1 and $\Delta(A_1) = 1$.*

The special nature of the polytope Ω_n is pointed out by the fact that the only planar faces that can arise are triangles (2-simplices) and rectangles. Let A be an $n \times n$ $(0, 1)$ -matrix with total support, and suppose $\dim \mathcal{F}(A) = 2$. Suppose first that A has exactly one nontrivial fully indecomposable component. Then we may assume $A = A_1 \oplus I$, where A_1 is fully indecomposable of order $n_1 > 1$. From Corollary 2.6 we conclude that $\sigma(A_1) = 2n_1 + 1$. Since A_1 is fully indecomposable, A_1 has exactly one row sum equal to 3 and all others equal to 2. But this implies that $\text{per } A_1 = 3$ (see, e.g., [15]). Thus $\mathcal{F}(A_1)$ is a simplex.

Now suppose that A has more than one nontrivial fully indecomposable component. Then A has exactly two nontrivial fully indecomposable components, and we may assume that $A = L_1 \oplus L_2 \oplus I$, where L_1 and L_2 have the form of the matrix L of (2.4). Let P_1 and P_2 be the permutation matrices such that $I + P_1 = L_1$ and $I + P_2 = L_2$. Then $\mathcal{F}(A)$ has exactly four vertices, namely,

$$\begin{aligned} V_1 &= I \oplus I \oplus I, & V_2 &= P_1 \oplus I \oplus I, \\ V_3 &= P_1 \oplus P_2 \oplus I, & V_4 &= I \oplus P_2 \oplus I. \end{aligned}$$

It is easy to verify that V_1, V_2, V_3, V_4 are the vertices of a polygon with consecutive sides orthogonal. Thus $\mathcal{F}(A)$ is a rectangle.

Given a polytope of dimension k , we define a *facet* to be a face of dimension $k - 1$. For $n > 2$, Ω_n has exactly n^2 facets and they correspond to the $n \times n$ $(0, 1)$ -matrices obtained from $K_{n,n}$ by replacing a 1 with a 0. However, Ω_2 has only two facets, since the replacement of any 1 in $K_{2,2}$ gives a matrix

which does not have total support. More generally, we now determine the facets of the faces of Ω_n . Let A be an $n \times n$ $(0, 1)$ -matrix with total support. Then the faces of $\mathcal{F}(A)$ are the faces $\mathcal{F}(B)$ of Ω_n corresponding to $(0, 1)$ -matrices B for which $B \leq A$. Since A has total support, if B is obtained from A by replacing a 1 by a 0, then $\dim \mathcal{F}(B) < \dim \mathcal{F}(A)$. Hence all of the facets of $\mathcal{F}(A)$ can be found among the faces $\mathcal{F}(B)$, where B is obtained from A by replacing a 1 by a 0.

THEOREM 2.10. *Let $A = [a_{ij}]$ be a fully indecomposable $(0, 1)$ -matrix of order $n > 1$, and let $a_{rs} = 1$. Let A' be the matrix obtained from A by replacing a_{rs} by 0. Then $\mathcal{F}(A')$ is a facet of $\mathcal{F}(A)$ if and only if either A' is a fully indecomposable matrix or there exist permutation matrices P and Q such that*

$$PAQ = \begin{bmatrix} A_1 & 0 & \cdots & 0 & E_k \\ E_1 & A_2 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & A_{k-1} & 0 \\ 0 & 0 & \cdots & E_{k-1} & A_k \end{bmatrix}, \quad PA'Q = \begin{bmatrix} A_1 & 0 & \cdots & 0 & 0 \\ E_1 & A_2 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & A_{k-1} & 0 \\ 0 & 0 & \cdots & E_{k-1} & A_k \end{bmatrix}, \tag{2.6}$$

where $k \geq 2$, A_1, \dots, A_k are fully indecomposable, and $\sigma(E_i) = 1$ for $i = 1, \dots, k$.

Proof. Observe that $\mathcal{F}(A')$ is a facet of $\mathcal{F}(A)$ if and only if $\dim \mathcal{F}(A') = \dim \mathcal{F}(A) - 1$. If A' is fully indecomposable, then by Theorem 2.5,

$$\dim \mathcal{F}(A) = \sigma(A) - 2n + 1 = \sigma(A') + 1 - 2n + 1 = \dim \mathcal{F}(A') + 1.$$

If A' is not fully indecomposable, then there exist permutation matrices P and Q such that

$$PAQ = \begin{bmatrix} A_1 & 0 & \cdots & 0 & E_k \\ X_{21} & A_2 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ X_{k-1,1} & X_{k-1,2} & \cdots & A_{k-1} & 0 \\ X_{k1} & X_{k2} & \cdots & X_{k,k-1} & A_k \end{bmatrix}, \tag{2.7}$$

$$PA'Q = \begin{bmatrix} A_1 & 0 & \cdots & 0 & 0 \\ X_{21} & A_2 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ X_{k-1,1} & X_{k-1,2} & \cdots & A_{k-1} & 0 \\ X_{k1} & X_{k2} & \cdots & X_{k,k-1} & A_k \end{bmatrix},$$

where $k \geq 2$ and A_1, \dots, A_k are fully indecomposable. From Theorem 2.5,

$$\dim \mathcal{F}(A) = \left(\sum_{i=1}^k \sigma(A_i) + \sum_{j < i} \sigma(X_{ij}) + 1 \right) - 2n + 1.$$

Since $\mathcal{F}(PA'Q) = \mathcal{F}(A_1 \oplus \cdots \oplus A_k)$, from Corollary 2.6 we have

$$\dim \mathcal{F}(A') = \sum_{i=1}^k \sigma(A_i) - 2n + k.$$

Thus $\mathcal{F}(A')$ is a facet of $\mathcal{F}(A)$ if and only if

$$\sum_{j < i} \sigma(X_{ij}) = k - 1. \tag{2.8}$$

Since A is fully indecomposable, there must be a 1 below each of A_1, \dots, A_{k-1} and a 1 to the left of each of A_2, \dots, A_k in the form (2.7). Thus (2.8) holds if and only if we have the form (2.6) as given in the statement of the theorem.

COROLLARY 2.11. *Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $n \times n$ $(0, 1)$ -matrices, where A is fully indecomposable and B has total support. Then $\mathcal{F}(B)$ is a facet of $\mathcal{F}(A)$ if and only if one of the following holds.*

(i) *B is fully indecomposable and there exist r and s such that $a_{rs} = 1$ and B is obtained from A by replacing a_{rs} by 0.*

(ii) *B is not fully indecomposable and there exist permutation matrices P and Q such that*

$$PAQ = \begin{bmatrix} A_1 & 0 & \cdots & 0 & E_k \\ E_1 & A_2 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & A_{k-1} & 0 \\ 0 & 0 & \cdots & E_{k-1} & A_k \end{bmatrix}, \quad PBQ = A_1 \oplus \cdots \oplus A_k, \tag{2.9}$$

where for $i = 1, \dots, k$, A_i is fully indecomposable and $\sigma(E_i) = 1$.

Proof. Suppose $\mathcal{F}(B)$ is a facet of $\mathcal{F}(A)$. Then $B \leq A$ and $\dim \mathcal{F}(B) = \dim \mathcal{F}(A) - 1$. Hence there exist r and s such that $a_{rs} = 1, b_{rs} = 0$. Let A' be the matrix obtained from A by replacing a_{rs} by 0. Then

$$\mathcal{F}(B) \subseteq \mathcal{F}(A') \subseteq \mathcal{F}(A), \quad \mathcal{F}(A') \neq \mathcal{F}(A).$$

Therefore, since $\mathcal{F}(B)$ is a facet of $\mathcal{F}(A)$, we conclude that $\mathcal{F}(B) = \mathcal{F}(A')$. It now follows from Theorem 2.10 that (i) or (ii) holds.

The converse follows easily using Corollary 2.6.

If A_1, \dots, A_t are fully indecomposable $(0, 1)$ -matrices then

$$\dim \mathcal{F}(A_1 \oplus \cdots \oplus A_t) = \dim \mathcal{F}(A_1) + \cdots + \dim \mathcal{F}(A_t).$$

Thus if A is an $n \times n$ $(0, 1)$ -matrix with total support, Corollary 2.11 can be used to determine the facets of $\mathcal{F}(A)$.

Let $A = [a_{ij}]$ be a fully indecomposable $(0, 1)$ -matrix of order $n > 1$. Let $K = \{(i, j): a_{ij} = 1\}$. The face $\mathcal{F}(A)$ consists of all those real $n \times n$ matrices $X = [x_{ij}]$ satisfying

$$x_{ij} \geq 0 \quad ((i, j) \in K), \tag{2.10}$$

$$x_{ij} = 0 \quad ((i, j) \notin K), \tag{2.11}$$

$$\sum_{r=1}^n x_{ir} = 1 = \sum_{r=1}^n x_{ri} \quad (i = 1, \dots, n). \tag{2.12}$$

We now show how Corollary 2.11 can be used to determine a minimal subset M of K such that $\mathcal{F}(A)$ consists of all those real $n \times n$ matrices $X = [x_{ij}]$ satisfying (2.11), (2.12), and

$$x_{ij} \geq 0 \quad ((i, j) \in M). \tag{2.13}$$

Because $\mathcal{F}(A)$ is a polytope in the linear manifold of Euclidean n^2 -space defined by (2.11) and (2.12), there may be several minimal sets M , but they all have the same cardinality.

Let $\mathcal{F}_1, \dots, \mathcal{F}_m$ be the facets of $\mathcal{F}(A)$. From Corollary 2.11 we see that for each facet \mathcal{F}_i of $\mathcal{F}(A)$ there exists a nonempty subset $S_i = \{(i_1, j_1), \dots, (i_k, j_k)\}$ of K such that if $(u, v) \in S_i$ then $a_{uv} = 1$ and $\mathcal{F}_i = \mathcal{F}(B)$, where B is the $(0, 1)$ -matrix with total support obtained from A by replacing $a_{i_r j_r}$ by 0 for $r = 1, \dots, k$. Moreover, if $X = [x_{ij}]$ is an $n \times n$ real matrix satisfying (2.10), (2.11), and (2.12), then $x_{i_1 j_1} = \dots = x_{i_k j_k}$. If $k = 1$, then B is a fully indecomposable matrix. If $k > 1$, then A and B have the form of (2.9) where $a_{i_1 j_1}, \dots, a_{i_k j_k}$ are the 1's appearing in E_1, \dots, E_k in some order. The sets S_1, \dots, S_m are pairwise disjoint. The minimal subsets M of K such that (2.11), (2.12), and (2.13) are equivalent to (2.10), (2.11), and (2.12) are obtained by choosing one element from each of S_1, \dots, S_m .

3. MATRICES CORRESPONDING TO SIMPLICES

In this section we determine all $n \times n$ $(0, 1)$ -matrices A with total support such that $\mathcal{F}(A)$ is a simplex. We do this by determining all faces of Ω_n which are two-neighborly polytopes. It turns out that almost all of these are simplices. Moreover, we shall see that a face \mathcal{F} of Ω_n is a two-neighborly polytope if and only if \mathcal{F} is two simplicial.

Let $n > 1$ and let $\alpha = \{i_1, \dots, i_k\}$ and $\beta = \{j_1, \dots, j_k\}$ be nonempty subsets of $\{1, \dots, n\}$ of cardinality $k < n$. Let A be an $n \times n$ matrix. Then $A[\alpha; \beta]$ denotes the $k \times k$ submatrix of A lying in rows i_1, \dots, i_k and columns j_1, \dots, j_k , and $A(\alpha; \beta)$ denotes the $n - k \times n - k$ submatrix lying in rows outside rows i_1, \dots, i_k and columns outside columns j_1, \dots, j_k . We call $A[\alpha; \beta]$ and

$A(\alpha; \beta)$ complementary submatrices of A . If A is an $n \times n$ nonnegative matrix then A is said to have property N provided for every pair A_1, A_2 of complementary submatrices of A either $\text{per } A_1 \leq 1$ or $\text{per } A_2 \leq 1$. It is easy to see that if A has property N and P and Q are permutation matrices, then PAQ has property N . Also, A^T has property N whenever A does. A polytope \mathcal{P} is said to be *two-neighborly* [7, pp. 122–129] provided every pair of distinct vertices determines an edge (one-dimensional face) of \mathcal{P} . Thus \mathcal{P} is a two-neighborly polytope if and only if the vertex–edge graph of \mathcal{P} is a complete graph.

If P and Q are distinct 3×3 permutation matrices, then it is clear that $\nu(P, Q) = 1$. Hence Ω_3 is a two-neighborly polytope of dimension 4 with six vertices. We shall see that Ω_3 is essentially the only two-neighborly polytope which is a face of an Ω_n but is not a simplex. Observe that the matrix $K_{3,3}$ corresponding to Ω_3 has property N . More generally, we have the following.

THEOREM 3.1. *Let A be a $(0, 1)$ -matrix of order $n > 1$. Then A has property N if and only if $\mathcal{F}(A)$ is two-neighborly.*

Proof. Suppose A does not have property N . Then there exist complementary submatrices A_1 and A_2 of A such that $\text{per } A_1 \geq 2$ and $\text{per } A_2 \geq 2$. We may assume that

$$A = \begin{bmatrix} A_1 & B_1 \\ B_2 & A_2 \end{bmatrix}.$$

Since $\text{per } A_i \geq 2$, there exist distinct permutation matrices $P_i \leq A_i$ and $Q_i \leq A_i$ ($i = 1, 2$). If $P = P_1 \oplus P_2$ and $Q = Q_1 \oplus Q_2$, then clearly $P, Q \leq A$ and $\nu(P, Q) \geq 2$. Hence $\mathcal{F}(A)$ is not two neighborly.

Now suppose that $\mathcal{F}(A)$ is not two neighborly. Thus there exist permutation matrices $P \leq A$ and $Q \leq A$ such that $\nu(P, Q) \geq 2$. We may assume that $P = I$ and $Q = Q_1 \oplus Q_2 \oplus Q_3$, where Q_1 and Q_2 are permutation matrices which correspond to cycles of length greater than 1. Since $\text{per}(I + Q_1) \geq 2$ and $\text{per}(I + Q_2) \geq 2$, we see that there exist complementary submatrices A_1 and A_2 of A such that $\text{per } A_1 \geq 2$ and $\text{per } A_2 \geq 2$. Hence A does not have property N .

Let $A = [a_{ij}]$ be an $m \times n$ real matrix with row vectors $\alpha_1, \dots, \alpha_m$. We say A is *contractible* on column (respectively, row) k if column (respectively, row) k contains exactly two nonzero entries. Suppose A is contractible on column k with $a_{ik} \neq 0 \neq a_{jk}$ and $i \neq j$. Then the $(m - 1) \times (n - 1)$ matrix $A_{ij;k}$ obtained from A by replacing row i with $a_{jk}\alpha_i + a_{ik}\alpha_j$ and deleting row j and column k is called the *contraction* of A on column k relative to rows i and j . If A is contractible on row k with $a_{ki} \neq 0 \neq a_{kj}$ and $i \neq j$, then the matrix $A_{k;i,j} = (A_{ij;k}^T)^T$ is called the contraction of A on row k relative to columns i and j . We say that A can be *contracted* to a matrix B if either $B = A$

or there exist matrices A_0, A_1, \dots, A_t ($t \geq 1$) such that $A_0 = A$, $A_t = B$, and A_r is a contraction of A_{r-1} for $r = 1, \dots, t$. We observe that if a $(0, 1)$ -matrix B is a contraction of a fully indecomposable nonnegative integral matrix A , then A is a $(0, 1)$ -matrix. On the other hand, a contraction of a $(0, 1)$ -matrix is not necessarily a $(0, 1)$ -matrix. For this reason it is convenient to formulate some of our theorems for nonnegative integral matrices even though they do not have an obvious geometrical interpretation.

LEMMA 3.2. *Let A be a nonnegative integral matrix of order $n > 1$ and let B be a contraction of A . Then the following hold.*

(i) $\text{per } A = \text{per } B$.

(ii) *Let B be obtained from A by a contraction relative to rows (respectively, columns) i and j , where rows (respectively, columns) i and j each contain at least two positive entries. Then A is fully indecomposable if and only if B is fully indecomposable.*

(iii) *If A has property N , then B has property N .*

(iv) *Let B be obtained from A by a contraction on a column or row whose two nonzero entries are 1's. If B has property N , then A has property N .*

Proof. It suffices to consider the case where B is the contraction of A on column 1 relative to rows 1 and 2. Thus A and B have the form

$$A = \begin{bmatrix} a & \alpha \\ b & \beta \\ 0 & C \end{bmatrix}, \quad B = \begin{bmatrix} a\beta + b\alpha \\ C \end{bmatrix}, \quad (3.1)$$

where $a \neq 0 \neq b$.

(i) Using the Laplace expansion of the permanent with respect to column 1, we obtain

$$\text{per } A = a \text{ per } \begin{bmatrix} \beta \\ C \end{bmatrix} + b \text{ per } \begin{bmatrix} \alpha \\ C \end{bmatrix}.$$

Hence by the linearity of the permanent, $\text{per } A = \text{per } B$.

(ii) Suppose B is not fully indecomposable. Then there exists an $r \times s$ zero submatrix $0_{r,s}$ of B where $r + s = n - 1$. If $0_{r,s}$ is a submatrix of C , then clearly A has an $r \times (s + 1)$ zero submatrix where $r + (s + 1) = n$. Hence in this case A is not fully indecomposable. Suppose $0_{r,s}$ is not a submatrix of C . Since a and b are positive while α and β are nonnegative, A has an $(r + 1) \times s$ zero submatrix where $(r + 1) + s = n$. Therefore A is not fully indecomposable.

Now suppose A is not fully indecomposable. Thus A contains an $r \times s$ zero submatrix $0_{r,s}$ with $r + s = n$. If $0_{r,s}$ is contained in the last $n - 2$ rows of A , then C , and thus B , contains an $r \times (s - 1)$ zero submatrix

with $r + (s - 1) = n - 1$. Let $0_{r,s}$ not be contained in the last $n - 2$ rows of A . Then, since a and b are positive, $0_{r,s}$ is contained in the last $n - 1$ columns of A . Since rows 1 and 2 of A each contain at least two positive entries, $0_{r,s}$ is a submatrix of neither α nor β . Hence B contains an $r - 1 \times s$ zero submatrix with $r - 1 + s = n - 1$. Therefore B is not fully indecomposable.

(iii) Suppose A has property N . Let B_1 and B_2 be complementary submatrices of B . We may assume that B_2 is a submatrix of C . Then there exists a submatrix

$$A_1 = \begin{bmatrix} a & \alpha' \\ b & \beta' \\ 0 & C' \end{bmatrix}$$

of A such that B_1 is the contraction of A_1 on column 1 relative to rows 1 and 2. From part (i) we see that $\text{per } B_1 = \text{per } A_1$. Moreover, A_1 and B_2 are complementary submatrices of A . Since A has property N , either $\text{per } B_1 \leq 1$ or $\text{per } B_2 \leq 1$. Therefore B has property N .

(iv) Suppose B has property N . We can assume A and B have the form given in (3.1) with $a = b = 1$. Let A_1 and A_2 be complementary submatrices of A . If either A_1 or A_2 is a submatrix of C , then it follows as in the proof of (iii) that there exist complementary submatrices B_1 and B_2 of B such that $\text{per } A_1 = \text{per } B_1$ and $\text{per } A_2 = \text{per } B_2$. Since B has property N , we conclude that $\text{per } A_1 \leq 1$ or $\text{per } A_2 \leq 1$. Now consider the case where neither A_1 nor A_2 is a submatrix of C . First, suppose that A_1 or A_2 is 1×1 . It suffices to assume that $A_1 = [a_{1j}]$ where $1 < j \leq n$. Then $\text{per } A_2 = \text{per } B_2$ where B_2 is the submatrix of B complementary to the 1×1 submatrix $B_1 = [a_{1j} + a_{2j}]$. Since B has property N and A is nonnegative, $a_{1j} \leq 1$ or $\text{per } A_2 \leq 1$. Now suppose that A_1 and A_2 have order at least 2. We may assume that A_1 intersects row 1 and column 1 and A_2 intersects row 2. Then $\text{per } A_1 = \text{per } B_1$, where B_1 is a submatrix of B of order one less than that of A_1 , and $A_2 \leq B_2$, where B_2 is the complementary submatrix of B_1 in B . Since $\text{per } A_2 \leq \text{per } B_2$ and B has property N , it follows that $\text{per } A_1 \leq 1$ or $\text{per } A_2 \leq 1$.

For an $n \times n$ matrix A , $r_i(A)$ denotes the sum of the entries in row i of A ($i = 1, \dots, n$) and $c_i(A)$ denotes the sum of the entries in column i of A ($i = 1, \dots, n$).

LEMMA 3.3. *Let $A = [a_{ij}]$ be a nonnegative integral matrix of order $n > 1$, and let $B = [b_{ij}]$ be obtained from A by a contraction on column 1 relative to rows 1 and 2, where $a_{11} = a_{21} = 1$. Then $\sigma(B) = \sigma(A) - 2$, and*

$$\begin{aligned} r_1(B) &= r_1(A) + r_2(A) - 2, \\ r_i(B) &= r_{i+1}(A), \quad i = 2, \dots, n - 1, \\ c_i(B) &= c_{i+1}(A), \quad i = 1, \dots, n - 1. \end{aligned}$$

An analogous statement holds for a contraction on any row or column whose two positive entries are 1's.

Proof. The matrices A and B have the form given in (3.1) with $a = b = 1$. It is clear from these forms that the lemma holds.

Observe that if A is an $n \times n$ $(0, 1)$ -matrix with total support which has property N , then A has at most one nontrivial fully indecomposable component. Thus, when considering property N we shall restrict our attention to fully indecomposable matrices.

THEOREM 3.4. *Let A be a fully indecomposable nonnegative integral matrix of order $n \geq 2$. Then A has property N if and only if one of the following holds.*

(i) A can be contracted to $K_{3,3}$.

(ii) There exists an integer p with $0 \leq p \leq n - 1$ and permutation matrices P and Q such that PAQ has the form

$$\begin{bmatrix} A_3 & A_1 \\ A_2 & 0 \end{bmatrix}, \quad (3.2)$$

where A_3 is an $n - p \times p + 1$ nonnegative integral matrix, and A_1 and A_2^T are $(0, 1)$ -matrices with exactly two 1's in each column.

Proof. We first show that if A satisfies (i) or (ii), then A has property N . Suppose A can be contracted to $K_{3,3}$. As already noted, $K_{3,3}$ has property N . Since $K_{3,3}$ is a $(0, 1)$ -matrix and A is fully indecomposable, A is a $(0, 1)$ -matrix, and it follows from (iv) of Lemma 3.2 that A has property N . Now suppose A satisfies (ii). We show by induction on n that A has property N . This is clear for $n = 2$, since then (3.2) takes one of the two forms

$$\begin{bmatrix} a & b \\ 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} a & 1 \\ b & 1 \end{bmatrix}.$$

Let $n > 2$. We may assume that A is of the form (3.2). First, suppose that $0 \leq p \leq n - 2$. Then A_1 is not vacuous and the matrix A' obtained from A by a contraction on column n has the form

$$\begin{bmatrix} A_3' & A_1' \\ A_2 & 0 \end{bmatrix}.$$

By Lemma 3.2, A' is a fully indecomposable nonnegative integral matrix of order $n - 1$. The matrix A_3' is an $(n - 1) - p \times p + 1$ matrix, where $0 \leq p \leq n - 2$. If $p = n - 2$, then A_1' is vacuous and it follows that A' satisfies (ii) with n replaced by $n - 1$. Now let $0 \leq p \leq n - 3$. Then the last $n - p - 2$ column sums of A' are 2. Since A' is a fully indecomposable

nonnegative integral matrix, A_1' is a $(0, 1)$ -matrix with exactly two 1's in each column. Hence A' satisfies (ii) with n replaced by $n - 1$. Therefore by the inductive assumption A' has property N . Thus by (iv) of Lemma 3.2, A has property N . If $p = n - 1$, then a similar argument using a contraction on row n of A shows that A has property N . Therefore if A satisfies (i) or (ii), A has property N .

Now suppose A has property N . We show by induction on n that A satisfies (i) or (ii). If $n = 2$, this is clear. Now let $n = 3$. Suppose $A \neq K_{3,3}$. Then it is easy to see that $A = [a_{ij}]$ has a zero entry. It suffices to assume $a_{33} = 0$. Further, suppose A does not satisfy (ii) with $p = 1$. Then row 3 or column 3 contains an entry greater than 1. By symmetry we may assume $a_{13} = b > 1$. Then, since A has property N , per $A(1; 3) = 1$. Therefore we may assume A has the form

$$\begin{bmatrix} d & g & b \\ 1 & 0 & h \\ c & 1 & 0 \end{bmatrix}.$$

Since A is fully indecomposable, $cg h > 0$. By property N at most one of c, g, h is greater than 1. Since A does not satisfy (ii) with $p = 1$, either $g > 1$ or $h > 1$. If $g > 1$, then $c = h = 1$ and A satisfies (ii) with $p = 2$. Using property N , we see that if $h > 1$, then $d = 0$ and $c = g = 1$. Hence in this case A satisfies (ii) with $p = 0$. Thus if $n = 3$ and A has property N , then A satisfies (i) or (ii).

Let $A = [a_{ij}]$ be of order $n > 3$ and have property N . We first show that the inductive assumption implies that A has a row or column with two entries equal to 1 and all other entries equal to 0. Suppose A does not have this property. Let R (respectively, S) be the set of all i (respectively, j) such that there exists a fully indecomposable matrix H where $H \leq A$ and H has only two positive entries in row i (respectively, column j). Since a nearly decomposable matrix has at least one row and column with exactly two positive entries, the sets R and S are nonempty. For each $i \in R$ let m_i be the number of positive entries in row i of A . For each $j \in S$ let n_j be the number of positive entries in column j of A . Let

$$r = \min\{m_i : i \in R\}, \quad s = \min\{n_j : j \in S\}.$$

Without loss in generality we may assume $s \leq r$. Let k be an element of S with $n_k = s$. Thus there exists a fully indecomposable matrix $H = [h_{ij}]$ such that $H \leq A$ and H has exactly two positive entries in column k , say, h_{uk} and h_{vk} . Let A' be the matrix obtained from A by replacing a_{ik} by 0 for all $i \neq u, v$. Since $H \leq A'$ and H is fully indecomposable, we see that A' is fully indecomposable. Since $A' \leq A$ and A has property N , it follows that A' has property N . With no loss in generality we assume that $k = u = 1$

and $v = 2$. Let B be obtained from A' by contracting on column 1 relative to rows 1 and 2. Then

$$A' = \begin{bmatrix} a & \alpha \\ b & \beta \\ 0 & C \end{bmatrix}, \quad B = \begin{bmatrix} a\beta + b\alpha \\ C \end{bmatrix}.$$

By (iii) of Lemma 3.2, B has property N . By the inductive assumption, B satisfies property (i) or (ii). Suppose that B can be contracted to $K_{3,3}$. If $n = 4$, then $B = K_{3,3}$ and it is easy to see that A' is equal to one of the matrices

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

Direct verification shows that if either of the 0's in column 1 of these two matrices is replaced by a positive integer, then the resulting matrix does not have property N . Since A has property N , $A = A'$. Now suppose $n > 4$. Since B is a fully indecomposable matrix of order $n - 1$ which can be contracted to $K_{3,3}$, it follows from Lemma 3.3 that $\sigma(B) = 2(n - 1) + 3$. Hence, since $n - 1 > 3$, B has a column with exactly two entries equal to 1. Therefore, whenever B can be contracted to $K_{3,3}$, A has a column with two entries equal to 1 and all other entries equal to 0.

Now suppose B satisfies property (ii). Thus there exist permutation matrices P and Q such that PBQ has the form (3.2), where A_3 is an $(n - 1) - p \times p + 1$ matrix with $0 \leq p \leq n - 2$. If $p \neq n - 2$, then B has a column with two entries equal to 1 and all other entries equal to 0. Hence A has a column with this property. Now suppose $p = n - 2$. Then

$$PBQ = \begin{bmatrix} A_3 \\ A_2 \end{bmatrix},$$

where A_2 is an $n - 2 \times n - 1$ (0, 1)-matrix with exactly two 1's in each row. If the first row of B corresponds to a row of A_2 , then the first row of A has the property that two entries equal 1 and all other entries equal 0. Now suppose that the first row of B corresponds to the row A_3 of PBQ . Thus we may assume that $A_3 = a\beta + b\alpha$ and $A_2 = C$. We now show that $a_{i1} = 0$ for some $i = 3, \dots, n$ by proving that $s \leq 3$. Let H be a nearly decomposable matrix with $H \leq A$. Since H has at least three rows with exactly two positive entries, there exists a $t \in \{3, \dots, n\}$ such that row t of H has exactly two positive entries. Hence $t \in R$ and $s \leq r \leq m_t \leq 3$. Since $n > 3$, it now follows that for some $i = 3, \dots, n$, $a_{i1} = 0$ and row i of A has two entries equal to 1 and all other entries equal to 0.

By what we have just shown, A has a row or column with two entries equal to 1 and all other entries equal to 0. Let B be the matrix obtained from A by a contraction on a row or column with this property. With no loss in generality, we may assume that

$$A = \begin{bmatrix} 1 & \alpha \\ 1 & \beta \\ 0 & C \end{bmatrix}, \quad B = \begin{bmatrix} \alpha + \beta \\ C \end{bmatrix}.$$

Since B has property N , by the inductive assumption B satisfies property (i) or (ii). If B satisfies (i), then it is clear that A satisfies (i). Suppose B satisfies (ii). Then there exists an integer q with $0 \leq q \leq n - 2$ and permutation matrices P' and Q' such that

$$P'BQ' = B' = \begin{bmatrix} B_3 & B_1 \\ B_2 & 0 \end{bmatrix},$$

where B_3 is an $(n - 1) - q \times q + 1$ nonnegative integral matrix and B_1 and B_2^T are $(0, 1)$ -matrices with exactly two 1's in each column. If the first row of B corresponds to one of the first $(n - 1) - q$ rows of B' , then A satisfies property (ii), where $p = q$; that is, A_3 is an $n - q \times q + 1$ matrix. If the first row of B corresponds to one of the last q rows of B' , then A satisfies property (ii), where $p = q + 1$; that is, A_3 is an $n - (q + 1) \times (q + 1) + 1$ matrix. Thus the theorem follows by induction on n .

Using Theorem 3.4, we can now characterize faces of Ω_n which are simplices. Because of Corollary 2.9 we may restrict ourselves to faces corresponding to fully indecomposable matrices.

THEOREM 3.5. *Let A be a fully indecomposable $(0, 1)$ -matrix of order $n > 2$. Then $\mathcal{F}(A)$ is a simplex if and only if A satisfies property (ii) of Theorem 3.4 with $1 \leq p \leq n - 2$.*

Proof. First suppose $\mathcal{F}(A)$ is a simplex. Then $\mathcal{F}(A)$ is clearly two neighborly, and hence by Theorem 3.1 A has property N . Thus A satisfies properties (i) or (ii) of Theorem 3.4. Assume that A satisfies (i). It follows from Lemmas 3.2 and 3.3 that $\Delta(A) = \Delta(K_{3,3})$. Therefore $\Delta(A) = 2$, which contradicts $\mathcal{F}(A)$ being a simplex. Thus A satisfies property (ii) with $0 \leq p \leq n - 1$. Suppose A satisfies (ii) with $p = 0$. Then A_1 is an $n \times n - 1$ $(0, 1)$ -matrix with exactly two 1's in each column. Hence, since A is a fully indecomposable $(0, 1)$ -matrix, there exists an i with $1 \leq i \leq n$ such that row i of A_1 contains exactly one 1 and it follows that the entry in row i of A_3 is a 1. Hence A satisfies (ii) with $p = 1$. A similar argument shows that if A satisfies (ii) with $p = n - 1$, then it also satisfies (ii) with $p = n - 2$.

Therefore if $\mathcal{F}(A)$ is a simplex, A satisfies property (ii) of Theorem 3.4 with $1 \leq p \leq n - 2$.

The converse follows from the fact that if A satisfies (ii) of Theorem 3.4, then $\text{per } A = \sigma(A_3)$. More generally, it can be shown by induction on n that if A is an $n \times n$ fully indecomposable nonnegative integral matrix which satisfies (ii) of Theorem 3.4, then $\text{per } A = \sigma(A_3)$. Since the induction is essentially the same as the induction used in the proof of Theorem 3.4 to conclude that such a matrix has property N , we omit it. From $\text{per } A = \sigma(A_3)$ we see that $\Delta(A) = 1$, and thus by Theorem 2.8 $\mathcal{F}(A)$ is a simplex.

We now characterize faces of Ω_n which are simplices but which are not properly contained in any other face which is a simplex. The $r \times s$ matrix, all of whose entries are 1, is denoted by $K_{r,s}$.

THEOREM 3.6. *Let A be a $(0, 1)$ -matrix of order $n > 2$ with total support such that $\mathcal{F}(A)$ is a simplex. Then there does not exist a $(0, 1)$ -matrix B of order n such that $\mathcal{F}(B)$ is a simplex, and $\mathcal{F}(A)$ is a proper subface of $\mathcal{F}(B)$ if and only if*

(*) *A is fully indecomposable and there exist an integer p with $1 \leq p \leq n - 2$ and permutation matrices P and Q such that PAQ has the form*

$$\begin{bmatrix} K_{n-p, p+1} & A_1 \\ A_2 & 0 \end{bmatrix}, \quad (3.3)$$

where A_1 and A_2^T have exactly two 1's in each column.

Proof. Suppose (*) holds. Since $1 \leq p \leq n - 2$, each row and column sum of $K_{n-p, p+1}$ is at least 2. Let B be an $n \times n$ matrix obtained from A by replacing at least one 0 by a 1. Then B is fully indecomposable, and the total number of rows and columns of B with exactly two 1's is at most $n - 2$. Hence by Theorem 3.5 $\mathcal{F}(B)$ is not a simplex.

Conversely, suppose (*) is not satisfied. If A is not fully indecomposable, it is not difficult to show that there exists a $(0, 1)$ -matrix B of order n such that $\mathcal{F}(B)$ is a simplex and $\mathcal{F}(A)$ is a proper subface of $\mathcal{F}(B)$. Suppose A is fully indecomposable. By Theorem 3.5, A satisfies (ii) of Theorem 3.4 with $1 \leq p \leq n - 2$. Since (*) does not hold, $A_3 \neq K_{n-p, p+1}$. Let B be an $n \times n$ matrix obtained by replacing a 0 of A corresponding to a 0 of A_3 by a 1. Then B is fully indecomposable, and it follows from Theorem 3.5 that $\mathcal{F}(B)$ is a simplex. Since $\mathcal{F}(A)$ is a proper subface of $\mathcal{F}(B)$, this proves the theorem.

For a real number x , $[x]$ denotes the greatest integer which does not exceed x . From Theorem 3.6 we obtain the following upper bound for the number of vertices of a face of Ω_n which is a simplex.

COROLLARY 3.7. *Let A be a $(0, 1)$ -matrix of order $n > 2$ such that $\mathcal{F}(A)$ is a simplex. Then*

$$\text{per } A \leq [(n + 1)/2]^2$$

with equality if and only if A satisfies () of Theorem 3.6 with $|n - 2p - 1| \leq 1$.*

Corollary 3.7 gives an upper bound for the permanent of a fully indecomposable $(0, 1)$ -matrix of order $n > 2$ for which equality holds in Minc's inequality (Corollary 2.7). We now show that Minc's inequality can be extended from $(0, 1)$ -matrices to nonnegative integral matrices and characterize those matrices for which equality holds.

THEOREM 3.8. *Let A be a fully indecomposable nonnegative integral matrix of order $n \geq 2$. Then*

$$\text{per } A \geq \sigma(A) - 2n + 2, \tag{3.4}$$

with equality if and only if A satisfies (ii) of Theorem 3.4.

Proof. We prove by induction on $\sigma(A)$ that $\text{per } A \geq \sigma(A) - 2n + 2$ and if equality holds then A has property N . First suppose that A is a $(0, 1)$ -matrix. By Corollary 2.7, (3.4) holds. If equality holds, $\mathcal{F}(A)$ is a simplex and by Theorems 3.4 and 3.5, A has property N . Now suppose $A = [a_{ij}]$, where $a_{rs} > 1$. Let B be the matrix obtained from A by replacing a_{rs} by $a_{rs} - 1$. Then B is a fully indecomposable nonnegative integral matrix with $\sigma(B) = \sigma(A) - 1$. Since A is fully indecomposable, $\text{per } A(r; s) \geq 1$. Hence using the inductive assumption, we see that

$$\text{per } A = \text{per } B + \text{per } A(r; s) \geq \sigma(A) - 2n + 2.$$

Suppose equality holds in this inequality. Then $\text{per } A(r; s) = 1$ and by the inductive assumption B has property N . Let A_1 and A_2 be complementary submatrices of A . If a_{rs} appears in neither a nonzero term of $\text{per } A_1$ nor a nonzero term of $\text{per } A_2$, then since B has property N , $\text{per } A_1 \leq 1$ or $\text{per } A_2 \leq 1$. Now suppose that a_{rs} appears in a nonzero term of $\text{per } A_1$ or $\text{per } A_2$, say $\text{per } A_1$. Let A_1' be the submatrix of A_1 which is complementary to the 1×1 submatrix $[a_{rs}]$ of A_1 . Then A_1' and A_2 are complementary submatrices of $A(r; s)$ with $\text{per } A_1' \geq 1$. Thus since $\text{per } A(r; s) = 1$, we see that $\text{per } A_2 \leq 1$. Hence, by induction on n , (3.4) holds with equality only if A has property N . Suppose equality holds in (3.4). Then A has property N , and it follows that A satisfies (i) or (ii) of Theorem 3.4. However, as we saw in the proof of Theorem 3.5, (i) does not hold. Hence (3.4) holds with equality only if A satisfies (ii) of Theorem 3.4. Now suppose A satisfies (ii) of Theorem 3.4. Then as we pointed out in the proof of Theorem 3.5, $\text{per } A = \sigma(A_3)$, and it follows that equality holds in (3.4).

We now characterize those matrices satisfying (ii) of Theorem 3.4 which are fully indecomposable. In view of Theorem 3.5 this characterization shows how to construct all faces of Ω_n which are simplices. Later we shall characterize fully indecomposable matrices which can be contracted to $K_{3,3}$. It follows from Theorems 3.1, 3.4, and 3.5 that this will show how to determine all two-neighborly faces of Ω_n which are not simplices.

Let G be a simple graph [3] with vertices v_1, \dots, v_k and edges e_1, \dots, e_m . Then the *vertex-edge incidence matrix* of G is the $k \times m$ $(0, 1)$ -matrix $C = [c_{ij}]$, where $c_{ij} = 1$ if and only if v_i is incident with e_j . Observe that C has exactly two 1's in each column. If G is a tree [3], then $m = k - 1$.

THEOREM 3.9. *Let n and p be integers with $n \geq 2$ and $0 \leq p \leq n - 1$, and let A be an $n \times n$ matrix with*

$$A = \begin{bmatrix} A_3 & A_1 \\ A_2 & 0 \end{bmatrix}, \quad (3.5)$$

where A_3 is an $n - p \times p + 1$ nonnegative integral matrix and A_1 and A_2^T are $(0, 1)$ -matrices with exactly two 1's in each column. Then A is fully indecomposable if and only if A_1 and A_2^T are vertex-edge incidence matrices of trees and A has at least two positive entries in each row and column.

Proof. First suppose that A is fully indecomposable. Then clearly A has at least two positive entries in each row and column. If $p = n - 1$, then A_1 does not appear in A . Assume that $p \leq n - 2$. Since A_1 is a $(0, 1)$ -matrix with exactly two 1's in each column, A_1 is the vertex-edge incidence matrix of a graph G . Suppose G is not a tree. Since G has $n - p$ vertices and $n - p - 1$ edges, G is not connected. Thus the vertices of G can be partitioned into two nonempty sets so that no edge of G joins a vertex of one set with a vertex of the other. This implies that there exist permutation matrices P and Q such that

$$PA_1Q = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix},$$

where B_1 is an $r \times s$ matrix with $0 < r < n - p$. Hence A contains an $n - r \times s$ zero matrix and a $p + r \times (n - p - 1) - s$ zero matrix. Observe that $n - r + s \geq n$ if $s \geq r$, while $p + r + (n - p - 1) - s \geq n$ if $s \leq r - 1$. This contradicts the fact that A is fully indecomposable. Hence G is a tree. A similar argument shows that if A_2 appears in A then A_2^T is the vertex-edge incidence matrix of a tree.

Now suppose that A_1 and A_2^T are vertex-edge incidence matrices of trees and A has at least two positive entries in each row and column. We show by induction on n that A is fully indecomposable. This is clearly so if $n = 2$

or 3. Let $n > 3$. Then either A_1 has at least two columns or A_2 has at least two rows. It suffices to assume that A_1 has at least two columns. Let G be a tree with vertex-edge incidence matrix A_1 . There is no loss in generality in assuming the last column of A_1 corresponds to a pendant edge of G and that the vertices incident with this edge correspond to the first two rows of A_1 . Let B be the matrix obtained from A by a contraction on column n relative to rows 1 and 2. Then

$$B = \begin{bmatrix} A_3' & A_1' \\ A_2 & 0 \end{bmatrix}.$$

Since A_1' is obtained from A_1 by a contraction on a column of A_1 corresponding to a pendant edge of the tree G , it is clear that A_1' is the vertex-edge incidence matrix of a tree. Since A_2^T is the vertex-edge incidence matrix of a tree, every column of A_2 contains at least one 1. Suppose that column k of A_2 contained only one 1. Then column k of A_3 , and hence column k of A_3' , contains at least one positive entry. Therefore the first $p + 1$ columns of B each contain at least two positive entries. Moreover, since A_1' is the vertex-edge incidence matrix of a tree, the other columns of B each contain at least two positive entries. Since A has at least two positive entries in each row, it is clear that each row of B other than possibly row 1 contains at least two positive entries. Since row 1 or row 2 of A_1 corresponds to a pendant vertex of G , either row 1 or row 2 of A_3 contains a positive entry. Therefore row 1 of A_3' contains a positive entry, and it follows that row 1 of B contains at least two positive entries. Hence by the inductive assumption B is fully indecomposable. Since A has at least two positive entries in each row, it follows from (ii) of Lemma 3.2 that A is fully indecomposable. This completes the proof of the theorem.

From Theorem 3.9, and Cayley's formula for the number of labeled trees, we conclude that the number of fully indecomposable matrices of the form (3.3), where A_1 and A_2^T have exactly two 1's in each column, is equal to $(p + 1)^{p-1}(n - p)^{n-p-2}$.

Let $B = [b_{ij}]$ be a nearly decomposable $(0, 1)$ -matrix of order $n > 2$ such that $\text{per } B = \sigma(B) - 2n + 2$. It follows from Theorem 3.5 that there exist permutation matrices P and Q such that $PBQ = A = [a_{ij}]$ is a nearly decomposable matrix satisfying the hypotheses of Theorem 3.9. Let $a_{rs} = 1$. If a_{rs} corresponds to an entry of A_1 , then column s of A contains exactly two 1's. If a_{rs} corresponds to an entry of A_2 , then row r of A contains exactly two 1's. Suppose that a_{rs} corresponds to an entry of A_3 . If row r and column s of A each contain at least three 1's, then it follows from Theorem 3.9 that the matrix A' obtained from A by replacing a_{rs} by 0 is fully indecomposable. Therefore since A is nearly decomposable, either row or column s of A contains exactly two 1's. Hence $b_{ij} = 1$ implies either row i or column j of B contains exactly two 1's. This property was obtained also

by Hedrick [10], and it forms part of his characterization of nearly decomposable $(0, 1)$ -matrices B of order n for which $\text{per } B = \sigma(B) - 2n + 2$.

A polytope is called *two-simplicial* provided every two-dimensional face is a simplex. Let \mathcal{P} be a polytope which is not two-simplicial. Hence \mathcal{P} has a two-dimensional face \mathcal{F} which is not a simplex. Thus two vertices of \mathcal{F} do not determine an edge of \mathcal{P} , so that \mathcal{P} is not two-neighborly. Hence if a polytope is two-neighborly, it is two-simplicial. For faces of Ω_n the converse holds.

THEOREM 3.10. *A face \mathcal{F} of Ω_n is two-simplicial if and only if it is two-neighborly.*

Proof. Suppose \mathcal{F} is not two-neighborly. Then there exist vertices P and Q of \mathcal{F} such that $\nu(P, Q) \geq 2$. Hence there exist vertices P' and Q' of \mathcal{F} such that $\nu(P', Q') = 2$. Thus \mathcal{F} has a two-dimensional face which is a rectangle, and \mathcal{F} is not two-simplicial. In view of the discussion preceding the theorem, this completes the proof.

THEOREM 3.11. *Let A be an $n \times n$ $(0, 1)$ -matrix with total support such that A cannot be contracted to $K_{3,3}$. Then the following are equivalent.*

- (i) $\mathcal{F}(A)$ is two-simplicial.
- (ii) $\mathcal{F}(A)$ is two-neighborly.
- (iii) $\mathcal{F}(A)$ is a simplex.

This theorem is a direct consequence of Theorem 3.1, 3.4, 3.5, and 3.10. We conclude this section with a characterization of those $(0, 1)$ -matrices which can be contracted to $K_{3,3}$.

THEOREM 3.12. *Let A be a fully indecomposable nonnegative integral matrix of order $n \geq 3$. The following are equivalent.*

- (i) A can be contracted to $K_{3,3}$.
- (ii) $\text{per } A = 6$, $\sigma(A) = 2n + 3$, and there exist permutation matrices P and Q such that $H = PAQ$, where $\text{per } H[1, \dots, n-3; 1, \dots, n-3] > 0$ (for $n > 3$), and the last three row and column sums of H each equal 3.

Proof. We first prove by induction on n that (i) implies (ii). If $n = 3$, this is trivial. Now let $n \geq 4$ and suppose A_0, A_1, \dots, A_{n-3} are matrices such that $A_0 = A$, $A_{n-3} = K_{3,3}$, and A_r is a contraction of A_{r-1} for $r = 1, \dots, n-3$. Since $K_{3,3}$ is a $(0, 1)$ -matrix, A must be $(0, 1)$ -matrix. Hence it follows from

Lemmas 3.2 and 3.3 that per $A = 6$ and $\sigma(A) = 2n + 3$. Without loss in generality we may assume that

$$A = \begin{bmatrix} 1 & \alpha \\ 1 & \beta \\ 0 & C \end{bmatrix}, \quad A_1 = \begin{bmatrix} \alpha + \beta \\ C \end{bmatrix}.$$

Since A_1 can be contracted to $K_{3,3}$, it follows by the inductive assumption that there exist permutation matrices P_1 and Q_1 such that

$$P_1 A_1 Q_1 = G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix},$$

where G_{22} is 3×3 , the last three row and column sums of G each equal 3, and per $G_{11} > 0$. If $\sigma(\alpha + \beta) = 3$, then we may assume that $\sigma(\alpha) = 1$ and $\sigma(\beta) = 2$. It is then not difficult to show that A satisfies (ii). Now suppose $\sigma(\alpha + \beta) = 2$, so that $\sigma(\alpha) = \sigma(\beta) = 1$. It is clear that there exist permutation matrices P and Q such that

$$PAQ = H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix},$$

where H_{22} is a 3×3 matrix, the last three row and column sums of H each equal 3, and G_{11} is a contraction of H_{11} . Since per $G_{11} > 0$, it follows from (i) of Lemma 3.2 that per $H_{11} > 0$. Therefore (i) implies (ii).

We now prove by induction on n that (ii) implies (i). Let $n = 3$. Suppose $A = [a_{ij}]$ satisfies (ii) but $A \neq K_{3,3}$. Then A must have an entry equal to 0. There is no loss in generality in assuming that $a_{11} = 0$. Moreover, since A is fully indecomposable with first row sum and first column sum equal to 3, we may assume that $a_{12} = a_{21} = 1$ and $a_{13} = a_{31} = 2$. Now since all other row and column sums of A equal 3, we see that A must be one of the two matrices

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix}.$$

Since neither of these matrices has permanent equal to 6, this is a contradiction. Thus (ii) implies (i) for $n = 3$. Now let $n \geq 4$, and suppose that A satisfies (ii). It follows that the matrix $H = [h_{ij}]$ of (ii) is a fully indecomposable nonnegative integral matrix of order $n \geq 4$ with $\sigma(H) = 2n + 3$ and the last three column sums of H each equal to 3. Therefore there exist integers r and s with $1 \leq r < s \leq n$ such that $h_{r1} = h_{s1} = 1$ and $h_{i1} = 0$ for $i \neq r, s$. Let G be the matrix obtained from H by a contraction on column 1 relative to rows s and r . Using Lemmas 3.2 and 3.3, we see that G

is a fully indecomposable nonnegative integral matrix with $\text{per } G = 6$, $\sigma(G) = 2(n - 1) + 3$, and the last three column sums each equal to 3. Let $H_1 = H[1, \dots, n - 3; 1, \dots, n - 3]$ and $G_1 = G[1, \dots, n - 4; 1, \dots, n - 4]$ (for $n > 4$). Since $\text{per } H_1 > 0$, we see that $r \leq n - 3$. Therefore by Lemma 3.3 it follows that the last three row sums of G each equal 3. If $n = 4$, then G satisfies (ii). Let $n > 4$. First suppose that $s > n - 3$. Then h_{r1} is the only positive entry contained in the first column of H_1 , and it follows that $\text{per } H_1(r; 1) = h_{r1} \text{per } H_1(r; 1) = \text{per } H_1 > 0$. Therefore, since $G_1 = H_1(r; 1)$, G satisfies (ii). Now suppose that $s \leq n - 3$. Then G_1 is a contraction of H_1 . By (i) of Lemma 3.2, $\text{per } G_1 = \text{per } H_1 > 0$. Therefore G satisfies (ii). By the inductive assumption G can be contracted to $K_{3,3}$. Hence H , and thus A , can be contracted to $K_{3,3}$. Therefore (ii) implies (i).

Observe that Theorem 3.12 implies that a fully indecomposable nonnegative integral matrix of order $n \geq 3$ which satisfies (ii) must be a $(0, 1)$ -matrix.

4. k -DIMENSIONAL FACES WITH $k + 2$ VERTICES

In Section 3 it was shown that if \mathcal{F} is a two-neighborly face of Ω_n which is not a simplex, then \mathcal{F} has dimension 4 and six vertices. In this section all k -dimensional faces of Ω_n with $k + 2$ vertices are determined.

We shall use the following lemma in obtaining our characterization of k -dimensional faces of Ω_n with $k + 2$ vertices. Its statement and proof are similar to those of Theorem 3.9.

LEMMA 4.1. *Let n and r be integers with $n \geq 2$ and $1 \leq r \leq n - 1$, and let A be a nonnegative integral matrix of the form*

$$\begin{bmatrix} A_3 & A_1 & A_4 \\ A_2 & 0 & 0 \\ A_5 & 0 & 1 \end{bmatrix}, \tag{4.1}$$

where A_3 is an $r \times n - r$ matrix, and A_1 and A_2^T are $(0, 1)$ -matrices with exactly two 1's in each column. Then A is fully indecomposable if and only if A_1 and A_2^T are vertex-edge incidence matrices of trees, A has at least two positive entries in each row and column, and at least one entry of A_3 is positive.

Proof. First suppose that A is fully indecomposable. Then it follows that A has at least two positive entries in each row and column and that A_3 is not a zero matrix. As in the proof of Theorem 3.9, it follows that A_1 and A_2^T are vertex-edge incidence matrices of trees.

Now suppose that A_1 and A_2^T are vertex-edge incidence matrices of trees, A has at least two positive entries in each row and column, and at least one entry of A_3 is positive. We show, by induction on n , that A is fully indecom-

posable. It is not difficult to verify this for $n = 2$ and 3 . Now let $n > 3$. First suppose that A_1 has at least two columns. Let G be the tree whose vertex-edge incidence matrix is A_1 . Let B be the matrix obtained from A by a contraction on a column of A corresponding to a pendant edge of G . As in the proof of Theorem 3.9, it follows that we can apply the inductive assumption to B to conclude that B is fully indecomposable. Using (ii) of Lemma 3.2, we conclude that A is fully indecomposable. If A_2 has at least two rows, a similar argument can be used to show A is fully indecomposable. The only case left to consider is when $n = 4$ and $r = 2$. In this case it can be verified that a contraction on column 3 yields a fully indecomposable matrix. Hence the theorem follows.

Let A be an $n \times n$ nonnegative integral matrix. We define $\tilde{\Delta}(A)$ to be $\text{per } A - (\sigma(A) - 2n + 1)$. It follows from Theorem 3.8 that if A is fully indecomposable, $\tilde{\Delta}(A) \geq 1$. It follows from Section 2 that if A is a fully indecomposable $(0, 1)$ -matrix, then $\tilde{\Delta}(A) = \Delta(A)$, where $\Delta(A)$ was defined to be $v - d$ for v and d , respectively, the number of vertices and the dimension of $\mathcal{F}(A)$.

LEMMA 4.2. *Let A be a fully indecomposable nonnegative integral matrix of order $n \geq 2$. Let b be a positive integer, and let r, s be integers with $1 \leq r, s \leq n$. Let B be the matrix obtained by adding b to the (r, s) -entry of A . Then $\tilde{\Delta}(B) \geq \tilde{\Delta}(A)$ with equality if and only if $\text{per } A(r; s) = 1$.*

Proof. Using elementary properties of the permanent, we see that

$$\tilde{\Delta}(B) - \tilde{\Delta}(A) = b(\text{per } A(r; s) - 1).$$

Since A is fully indecomposable, $\text{per } A(r; s) \geq 1$ and the lemma follows.

LEMMA 4.3. *Let n and r be integers with $n \geq 1$ and $1 \leq r \leq n$, and let A be an $n \times n$ nonnegative integral matrix with*

$$A = \begin{bmatrix} A_3 & A_1 \\ A_2 & 0 \end{bmatrix},$$

where A_3 is an $r \times (n - r) + 1$ matrix, and A_1 and A_2^T are vertex-edge incidence matrices of trees. Then $\text{per } A = \sigma(A_3)$ and $\tilde{\Delta}(A) = 1$.

The lemma follows by induction on n using a contraction on a row or column of A corresponding to a pendant edge of a tree associated with A_1 or A_2^T .

THEOREM 4.4. *Let A be a fully indecomposable nonnegative integral matrix of order $n \geq 2$. Then $\tilde{\Delta}(A) = 2$ if and only if one of the following holds.*

(i) A can be contracted to $K_{3,3}$.

(ii) There exist positive integers r, s , and t with $r + s + t = n + 1$ and permutation matrices P and Q such that PAQ has the form

$$\begin{bmatrix} M_3 & M_1 & M_4 \\ M_2 & 0 & 0 \\ M_5 & 0 & M_6 \end{bmatrix}, \quad (4.2)$$

where M_3 is an $r \times s$ matrix, M_6 is a $t \times t$ $(0, 1)$ -matrix with $\sigma(M_6) = 2t - 1$, M_1 , and M_2^T are $(0, 1)$ -matrices with exactly two 1's in each column, and the first column sum of M_4 and M_5^T is 2 and all other column sums of M_4 and M_5^T are 0.

Proof. We first show that if A satisfies (i) or (ii), then $\tilde{\Delta}(A) = 2$. If A can be contracted to $K_{3,3}$, then $\tilde{\Delta}(A) = \Delta(A) = 2$. Now suppose A satisfies (ii). Let B be the matrix given in (4.2). Suppose $t > 1$. Since B is fully indecomposable and $\sigma(M_6) = 2t - 1$, a contraction on the last column of B results in a fully indecomposable matrix satisfying (ii) with t replaced by $t - 1$. Hence we may assume that $t = 1$. Let B' be the matrix obtained from B by replacing the 1 in the (n, n) -position of B by 0. Then

$$\begin{aligned} \text{per } B &= \text{per } B(n, n) + \text{per } B' \\ &= \text{per } B(n, n) + \text{per}[M_1 \ M_4] \text{per} \begin{bmatrix} M_2 \\ M_5 \end{bmatrix}. \end{aligned}$$

Using Lemmas 4.1 and 4.3, we see that

$$\text{per } B = \sigma(M_3) + \sigma(M_4) \sigma(M_5) = \sigma(M_3) + 4.$$

Since $\sigma(B) = \sigma(M_3) + 2n - 1$, it follows that $\tilde{\Delta}(B) = 2$. Therefore if (i) or (ii) holds, $\tilde{\Delta}(A) = 2$.

Now suppose $\tilde{\Delta}(A) = 2$. We show by induction on n that A satisfies (i) or (ii). Suppose $n = 2$. Then

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

where a, b, c , and d are positive integers. Since $\tilde{\Delta}(A) = 2$, it follows that

$$(a - 1)(d - 1) + (b - 1)(c - 1) + 1 = 2.$$

Therefore there exist permutation matrices P and Q such that

$$PAQ = \begin{bmatrix} x & 2 \\ 2 & 1 \end{bmatrix},$$

where x is a positive integer. Hence A satisfies (ii) with $r = s = t = 1$.

Let $n \geq 3$. We first show that if $A \neq K_{3,3}$, then A has a row or column which contains exactly two positive entries. Suppose $A \neq K_{3,3}$ and every row and column of A contains at least three positive entries. Let G be the $(0, 1)$ -matrix obtained from A by replacing each positive entry by 1. Then G is fully indecomposable and each row and column of B contains at least three 1's. Suppose that $n = 3$. Then $G = K_{3,3}$. Since $A \neq K_{3,3}$, some entry of A is greater than 1. Using Lemma 4.2, we conclude that $\tilde{\Delta}(A) > \tilde{\Delta}(K_{3,3}) = 2$. Therefore $n > 3$. Let H be a fully indecomposable matrix obtained from G by replacing some 1 by a 0. Since $n > 3$, it follows from Theorem 3.5 that $\mathcal{F}(H)$ is not a simplex. Hence $\tilde{\Delta}(H) = \Delta(H) \geq 2$. Since each row of the fully indecomposable matrix G contains at least three 1's, it follows from Hall's inequality [8] that $\text{per } G(i, j) \geq 2$ for $i, j = 1, \dots, n$. Hence by Lemma 4.2, $\tilde{\Delta}(A) \geq \tilde{\Delta}(G) > \tilde{\Delta}(H) \geq 2$. Therefore if $A \neq K_{3,3}$, some row or column of A contains exactly two positive entries.

We now show that the inductive assumption implies that if A cannot be contracted to $K_{3,3}$, then A has a row or column with two entries equal to 1 and all other entries equal to 0. Assume A cannot be contracted to $K_{3,3}$. Without loss in generality we may assume that the first two entries in column 1 of A are positive and all other entries in column 1 are 0. Let B be the matrix obtained from A by a contraction on column 1 relative to rows 1 and 2. Then

$$A = \begin{bmatrix} a & \alpha \\ b & \beta \\ 0 & C \end{bmatrix}, \quad B = \begin{bmatrix} a\beta + b\alpha \\ C \end{bmatrix}.$$

It is easy to verify that

$$\tilde{\Delta}(A) - \tilde{\Delta}(B) = (a - 1)(\sigma(\beta) - 1) + (b - 1)(\sigma(\alpha) - 1). \tag{4.3}$$

Hence $2 = \tilde{\Delta}(A) \geq \tilde{\Delta}(B) \geq 1$. Suppose first that $\tilde{\Delta}(B) = 1$. Then by Theorem 3.8, B satisfies property (ii) of Theorem 3.4. In this case it follows that A has a row or column with two entries equal to 1 and all other entries equal to 0. Now suppose that $\tilde{\Delta}(A) = \tilde{\Delta}(B) = 2$. We consider two cases according to whether $n = 3$ or $n > 3$. Let $n = 3$. Since A is a fully indecomposable nonnegative integral matrix, it follows from (4.3) that $a = 1$ or $\sigma(\beta) = 1$, and $b = 1$ or $\sigma(\alpha) = 1$. Hence if $a = 1$ or $b = 1$, A has a row or column with two entries equal to 1 and all other entries equal to 0. Suppose that $a > 1$ and $b > 1$. Then $\sigma(\alpha) = \sigma(\beta) = 1$ and we may assume A equals

$$\begin{bmatrix} a & 1 & 0 \\ b & 0 & 1 \\ 0 & c_1 & c_2 \end{bmatrix}.$$

Since $\tilde{\Delta}(A) = 2$, it follows that $(a - 1)(c_1 - 1) + (b - 1)(c_2 - 1) = 1$. Hence $c_1 = 1$ or $c_2 = 1$, and A has a column with two 1's and one 0. Now suppose $n > 3$. Since A cannot be contracted to $K_{3,3}$, by the inductive assumption B satisfies (ii). If $r > 1$, then M_1 of (4.2) is nonvacuous, and each column of M_1 corresponds to a column of B with two entries equal to 1 and all other entries equal to 0. If $s > 1$, then each row of M_2 corresponds to a row of B with two entries equal to 1 and all other entries equal to 0. Since $n > 3$, if $r = s = 1$, then $t \geq 2$ and the last column of M_6 corresponds to a column of B with two entries equal to 1 and all other entries equal to 0. It follows that A has a row or column with two entries equal to 1 and all other entries equal to 0.

We now show that the inductive assumption implies that if A does not satisfy (i), then A satisfies (ii). Suppose that A cannot be contracted to $K_{3,3}$. With no loss in generality, we may assume that the first two entries in column 1 of A are 1 and all other entries in column 1 are 0. Let B be the matrix obtained from A by a contraction on column 1 relative to rows 1 and 2. From (4.3), it follows that $\tilde{\Delta}(B) = 2$. Since B cannot be contracted to $K_{3,3}$, by the inductive assumption there exist positive integers u, v , and w with $u + v + w = n$ and permutation matrices P' and Q' such that

$$P'BQ' = B' = \begin{bmatrix} B_3 & B_1 & B_4 \\ B_2 & 0 & 0 \\ B_5 & 0 & B_6 \end{bmatrix},$$

where B_3 is a $u \times v$ matrix, B_6 is a $w \times w$ (0, 1)-matrix with $\sigma(B_6) = 2w - 1$, B_1 and B_2^T are (0, 1)-matrices with exactly two 1's in each column, and the first column sum of B_4 and B_5^T is 2 and all other column sums of B_4 and B_5^T are 0. If the first row of B corresponds to one of the first u rows of B' , then A satisfies property (ii) with $r = u + 1, s = v, t = w$. If the first row of B corresponds to one of rows $u + 1, \dots, u + v - 1$ of B' , then A satisfies (ii) with $r = u, s = v + 1, t = w$. If the first row of B corresponds to row $u + v$ of B' , then A satisfies (ii) with $r = u, s = v + 1, t = w$ or with $r = u, s = v, t = w + 1$. If $w > 1$ and the first row of B corresponds to one of the last $w - 1$ rows of B' , then A satisfies (ii) with $r = u, s = v, t = w + 1$. Thus the theorem follows by induction on n .

The d -dimensional faces of Ω_n with $d + 1$ vertices (the simplices) were characterized in Section 3. We now characterize those d -dimensional faces of Ω_n which have $d + 2$ vertices.

THEOREM 4.5. *Let B be a (0, 1)-matrix of order $n \geq 3$ with total support. Then $\mathcal{F}(B)$ is a d -dimensional face of Ω_n with $d + 2$ vertices if and only if one of the following is satisfied.*

- (i) $d = 2$ and $\mathcal{F}(B)$ is a rectangle.
- (ii) $d \geq 3$, B has exactly one nontrivial fully indecomposable component A , and one of the following holds:
 - (a) A can be contracted to $K_{3,3}$;
 - (b) A satisfies (ii) of Theorem 4.4 with $r \geq 2$ and $s \geq 2$.

Proof. Let k be the number of nontrivial fully indecomposable components of B . Suppose that $\mathcal{F}(B)$ is a d -dimensional face of Ω_n with $d + 2$ vertices. Clearly $k \geq 1$. By Theorem 2.8, $k \leq 2$ with $k = 2$ only if each of the nontrivial fully indecomposable components of B corresponds to a face of dimension 1. Therefore if $k = 2$, (i) holds. Suppose $k = 1$ and let A be the nontrivial fully indecomposable component of B . It follows that $\tilde{\Delta}(A) = \Delta(A) = \Delta(B) = 2$. Hence A satisfies (i) or (ii) of Theorem 4.3. Suppose A satisfies (ii) of Theorem 4.4. Since the first column sum of M_4 and M_5^T is 2 and A is a $(0, 1)$ -matrix, $r \geq 2$ and $s \geq 2$. Thus if $\mathcal{F}(B)$ is a d -dimensional face of Ω_n with $d + 2$ vertices, (i) or (ii) holds. The converse readily follows.

Let A be a fully indecomposable $(0, 1)$ -matrix of order $n \geq 3$. If A satisfies (i) of Theorem 4.4, then $\dim \mathcal{F}(A) = 4$ and $\mathcal{F}(A)$ has six vertices. If A satisfies (ii) of Theorem 4.4, then $n \geq 4$ and an easy calculation shows that $\dim \mathcal{F}(A) = \sigma(M_3) + 2$, and $\mathcal{F}(A)$ has $\sigma(M_3) + 4$ vertices. Let $\mathcal{F}(A)$ have dimension d and $d + 2$ vertices. It follows that

$$d \leq [(n^2 + 8)/4]. \tag{4.4}$$

If $n = 3$, then $A = K_{3,3}$ and equality must hold in (4.4). Let $n \geq 4$. Then equality holds in (4.4) if and only if A satisfies (ii) with $t = 1$, $|r - s| \leq 1$, and $M_3 = K_{r,s}$. For $n \geq 3$ let $f_i(n)$ be the maximum d such that there exists a d -dimensional face of Ω_n with $d + i$ vertices. From Corollary 3.7 and (4.4) we see that

$$f_1(n) = [(n^2 + 2n - 3)/4], \quad f_2(n) = [(n^2 + 8)/4],$$

and hence that $f_1(n) \leq f_2(n)$ if and only if $n \leq 6$.

Let $n \geq 4$ and let d be an integer with $2 \leq d \leq f_2(n)$. Then there exists a d -dimensional face \mathcal{F} of Ω_n with $d + 2$ vertices. For $d = 2$, \mathcal{F} is a rectangle. Let $d \geq 3$. We construct a matrix A to satisfy (ii) of Theorem 4.4 as follows. Choose r and s to be integers such that $r + s = n$ and $|r - s| \leq 1$ and let $t = 1$. Let $M_1(M_2^T)$ be a vertex-edge incidence matrix of a path. Then two of the row sums of $M_1(M_2^T)$ equal 1 and the other row sums equal 2. Choose the two 1's of $M_4(M_5^T)$ so that A must have at least two 1's in each row and column. Choose M_3 to be any $r \times s$ $(0, 1)$ -matrix with $\sigma(M_3) = d - 2$. By Lemma 4.1, A is fully indecomposable. Hence $\mathcal{F}(A)$ is a d -dimensional face of Ω_n with $d + 2$ vertices.

Note from Theorem 4.5 that there are three possible types for d -dimensional faces of Ω_n with $d + 2$ vertices. For $d = 4$, two different types may occur. For each $d \geq 2, d \neq 4$, only one type may occur. For each $n > 3$ all three types occur as faces of Ω_n .

From our results it is not difficult to see that for each positive integer m , there exist positive integers d and n such that Ω_n has a d -dimensional face with $d + m$ vertices. Let $m \geq 3$. Select an $n \times n$ fully indecomposable $(0, 1)$ -matrix A of the form (4.2) satisfying (ii) of Theorem 4.4 with $r = m$ and $t = 1$. Let B be the matrix obtained from A by replacing M_4 by $K_{m,1}$. If $\mathcal{F}(A)$ has dimension k , then $\mathcal{F}(B)$ has dimension $k + (m - 2)$. Let $d = k + (m - 2)$. It follows that $\mathcal{F}(B)$ is a d -dimensional face of Ω_n with $d + m$ vertices.

5. FURTHER CHARACTERIZATIONS OF FACES

In previous sections we have characterized four types of faces, namely, those that are simplices, two-neighborly, two-simplicial, or k -dimensional with $k + 2$ vertices. We now show that a k -dimensional face of Ω_n has at most 2^k vertices, and characterize all k -dimensional faces which have at least $2^{k-1} + 1$ vertices. Faces of Ω_n which are pyramids are also determined. We conclude by describing all three-dimensional faces of Ω_n .

We recall from Theorem 2.5 that if A is an $n \times n$ fully indecomposable $(0, 1)$ -matrix, then $\dim \mathcal{F}(A) = \sigma(A) - 2n + 1$. Hence, from Foregger's theorem [6] we immediately obtain the following.

THEOREM 5.1. *Let A be a fully indecomposable $(0, 1)$ -matrix of order $n > 1$ and let $\dim \mathcal{F}(A) = k$. Then*

$$\text{per } A \leq 2^{k-1} + 1.$$

Equality holds if and only if

(*) *there exist permutation matrices P and Q such that PAQ has the form*

$$\begin{bmatrix} C_1 & 0 & \cdots & 0 & E_r \\ E_1 & C_2 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & C_{r-1} & 0 \\ 0 & 0 & \cdots & E_{r-1} & C_r \end{bmatrix} \quad (r \geq 2), \tag{5.1}$$

where $\sigma(E_i) = 1$ for $i = 1, \dots, r$, and $r - k + 1$ of the C_i 's equal the 1×1 matrix $[1]$ while the other $k - 1$ C_i 's are matrices of order at least 2 of the form

$$L = \begin{bmatrix} 1 & 0 & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 1 & 1 \end{bmatrix}. \tag{5.2}$$

The following lemma is easily proved.

LEMMA 5.2. *Let $t \geq 2$ and n_1, \dots, n_t be positive integers. Then*

$$(2^{n_1} + 1) \cdots (2^{n_t} + 1) \leq 2^{n_1 + \cdots + n_t + (t-1)} + 1$$

with equality if and only if $t = 2$ and $n_1 = n_2 = 1$.

Let A be a $(0, 1)$ -matrix of order $n > 1$, and suppose that $\mathcal{F}(A)$ has dimension d and v vertices. If A is fully indecomposable, then according to Theorem 5.1, $v \leq 2^{d-1} + 1$. Our next theorem shows that there are at most d possible values of v with $v \geq 2^{d-1} + 1$.

THEOREM 5.3. *Let d and v be positive integers with $v \geq 2^{d-1} + 1$. There exists a d -dimensional face with v vertices of some Ω_n if and only if $v \in \{2^{d-1} + 2^m : m = 0, 1, \dots, d - 1\}$.*

For $m = 0, 1, \dots, d - 1$, let $v_m = 2^{d-1} + 2^m$ and let \mathcal{F}_m be a d -dimensional face of Ω_n with v_m vertices. Let B_m be the $n \times n$ $(0, 1)$ -matrix with total support corresponding to \mathcal{F}_m .

(i) *If $m = d - 4$, then B_m satisfies one of the following.*

(a) *It has exactly $m + 1$ nontrivial fully indecomposable components, where m of these components are of the form (5.2) and the other one satisfies (*) of Theorem 5.1 with $k = d - m$.*

(b) *It has exactly $m + 2$ nontrivial fully indecomposable components, where m of these components are of the form (5.2) and the other two satisfy (*) of Theorem 5.1 with $k = (d - m)/2 = 2$.*

(ii) *If $m \neq d - 4$, then B_m satisfies (a).*

Proof. For $m = 0, 1, \dots, d - 1$, let B_m be an $n \times n$ $(0, 1)$ -matrix with total support. If either (a) or (b) holds, it follows from Corollary 2.6 that $\dim \mathcal{F}(B_m) = d$. If (a) holds, using Theorem 5.1 we see that

$$\text{per } B_m = 2^m(2^{d-m-1} + 1) = 2^{d-1} + 2^m.$$

Similarly, if (b) holds, then $m = d - 4$ and

$$\text{per } B_m = 2^m(2 + 1)^2 = 2^m(2^3 + 1) = 2^{d-1} + 2^m.$$

Now let B be an $n \times n$ $(0, 1)$ -matrix with total support, and suppose that $\mathcal{F}(B)$ has dimension d and $v \geq 2^{d-1} + 1$ vertices. Let P and Q be permutation matrices such that $PBQ = A_1 \oplus \cdots \oplus A_s \oplus I$, where A_1, \dots, A_s are fully indecomposable matrices of order at least 2. Let $d_i = \dim \mathcal{F}(A_i)$ for $i = 1, \dots, s$. Observe that $s \leq d$. Suppose that $d_i = 1$ for $i = 1, \dots, s$. Then $s = d$ and each of A_1, \dots, A_s can be permuted to obtain a matrix of the form (5.2). In this case B satisfies (a) with $m = d - 1$. Now suppose that $d_i > 1$ for at least one i . We may assume that for some $1 \leq t \leq s$, $d_i > 1$ for $i = 1, \dots, t$ and $d_i = 1$ for $i = t + 1, \dots, s$. Let $m = s - t$. Then $d - m = d_1 + \cdots + d_t$. Applying Theorem 5.1 and Lemma 5.2, we see that

$$\begin{aligned} \text{per } B &= \prod_{i=1}^s \text{per } A_i \\ &\leq \prod_{i=1}^s (2^{d_i-1} + 1) \\ &= 2^{s-t} \prod_{i=1}^t (2^{d_i-1} + 1) \\ &\leq 2^{s-t} (2^{d_1+\cdots+d_t-1} + 1) = 2^m (2^{d-m-1} + 1). \end{aligned}$$

Suppose that $\text{per } A_i < 2^{d_i-1} + 1$ for some $1 \leq i \leq t$. It then follows that

$$\begin{aligned} \text{per } B &\leq 2^{s-t} \left(\prod_{i=1}^t (2^{d_i-1} + 1) - 1 \right) \\ &\leq 2^m 2^{d-m-1} = 2^{d-1}. \end{aligned}$$

Since $v = \text{per } B$, this contradicts $v \geq 2^{d-1} + 1$. Thus, $\text{per } A_i = 2^{d_i-1} + 1$ for $i = 1, \dots, t$. Hence A_i satisfies (*) of Theorem 5.1 with $k = d_i$ for $i = 1, \dots, t$. Now suppose that $t > 2$ or $t = 2$ with $d_1 > 2$ or $d_2 > 2$. Then by Lemma 5.2,

$$\text{per } B \leq 2^m 2^{d-m-1} = 2^{d-1}$$

and this again contradicts $v \geq 2^{d-1} + 1$. Thus $t = 1$ or $t = 2$ with $d_1 = d_2 = 2$. If $t = 2$ with $d_1 = d_2 = 2$, then $m = d - 4$ and (b) holds. If $t = 1$, then (a) holds. Hence the theorem follows.

COROLLARY 5.4. *Let A be an $n \times n$ $(0, 1)$ -matrix with total support and let $k = \dim \mathcal{F}(A)$. Then*

$$\text{per } A \leq 2^k.$$

Equality holds if and only if there exist permutation matrices P and Q such that $PAQ = L_1 \oplus \cdots \oplus L_k \oplus I$, where L_i is of the form (5.2) for $i = 1, \dots, k$.

A rectangular k -dimensional parallelotope is called a k -box [2, p. 187].

THEOREM 5.5. *Let \mathcal{F} be a face of Ω_n with dimension k and v vertices. Then $v \leq 2^k$ with equality if and only if \mathcal{F} is a k -box.*

Proof. Let A be an $n \times n$ $(0, 1)$ -matrix with total support which corresponds to \mathcal{F} . By Corollary 5.4, $v \leq 2^k$. If \mathcal{F} is a k -box, then $v = 2^k$. Now assume $v = 2^k$. By Corollary 5.4 we may assume $A = L_1 \oplus \cdots \oplus L_k \oplus I$, where L_i is of the form (5.2) for $i = 1, \dots, k$. Let P_i be the permutation matrix such that $L_i = I + P_i$ for $i = 1, \dots, k$. For each $i = 1, \dots, k$, let Q_i be the matrix obtained from A by replacing L_i by P_i and all other L_j by I . Denote the line segment joining I and Q_i by S_i for $i = 1, \dots, k$. Then S_1, \dots, S_k are mutually orthogonal line segments with a common point and \mathcal{F} is their vector sum. Hence \mathcal{F} is a k -box.

Let \mathcal{P} be a polytope in Euclidean d -space R^d and x a point in R^d not in the affine hull of \mathcal{P} . Then the pyramid [7, p. 54] with basis \mathcal{P} and apex x is defined to be the convex hull \mathcal{Q} of $\mathcal{P} \cup \{x\}$. Observe that \mathcal{Q} has exactly one more vertex than \mathcal{P} . Conversely, if \mathcal{P} and \mathcal{Q} are polytopes such that \mathcal{P} is a face of \mathcal{Q} and \mathcal{Q} has exactly one more vertex than \mathcal{P} , then \mathcal{Q} is a pyramid with basis \mathcal{P} . Hence we have the following.

THEOREM 5.6. *Let A and B be $n \times n$ $(0, 1)$ -matrices with $B \leq A$. Then $\mathcal{F}(A)$ is a pyramid with basis $\mathcal{F}(B)$ if and only if $\text{per } A = \text{per } B + 1$.*

THEOREM 5.7. *Let A be an $n \times n$ $(0, 1)$ -matrix with total support such that $\mathcal{F}(A)$ is a pyramid with basis some face of Ω_n . If A is not a permutation matrix, then A has exactly one nontrivial fully indecomposable component.*

Proof. Suppose A is not a permutation matrix. Then A has at least one nontrivial fully indecomposable component. Let $B \leq A$ be a $(0, 1)$ -matrix such that $\mathcal{F}(A)$ is a pyramid with basis $\mathcal{F}(B)$. Without loss in generality we may assume that

$$A = A_1 \oplus \cdots \oplus A_t \oplus I, \quad B = B_1 \oplus \cdots \oplus B_t \oplus I,$$

where $B_1 \neq A_1$ while A_i is a fully indecomposable matrix of order at least 2 and $B_i \leq A_i$ for $i = 1, \dots, t$. Suppose $t \geq 2$. Since $B_1 \leq A_1$, $B_1 \neq A_1$,

there exist r and s such that the (r, s) -entry of A_1 is 1 while the (r, s) -entry of B_1 is 0. Since A_1 is fully indecomposable, $\text{per } A_1(r; s) \geq 1$. Moreover, since A_2 is a fully indecomposable $(0, 1)$ -matrix of order at least 2, $\text{per } A_2 \geq 2$. Hence $\text{per } A(r; s) \geq 2$. It follows that

$$\text{per } A \geq \text{per } B + \text{per } A(r; s) \geq \text{per } B + 2.$$

Therefore by Theorem 5.6, $t = 1$.

We now characterize $(0, 1)$ -matrices A and B with total support such that $\mathcal{F}(A)$ is a pyramid with basis $\mathcal{F}(B)$. By Theorem 5.7 it suffices to assume that A is fully indecomposable. We adopt the convention that if A is a matrix of order one, then $\text{per } A(1; 1) = 1$.

THEOREM 5.8. *Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $n \times n$ $(0, 1)$ -matrices, where A is fully indecomposable and B has total support. Then $\mathcal{F}(A)$ is a pyramid with basis $\mathcal{F}(B)$ if and only if one of the following holds.*

(i) *B is fully indecomposable and there exist r and s such that $\text{per } A(r; s) = 1$, $a_{rs} = 1$, and B is obtained from A by replacing a_{rs} by 0.*

(ii) *B is not fully indecomposable and there exist permutation matrices P and Q such that*

$$PAQ = \begin{bmatrix} A_1 & 0 & \cdots & 0 & E_k \\ E_1 & A_2 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & A_{k-1} & 0 \\ 0 & 0 & \cdots & E_{k-1} & A_{k-1} \end{bmatrix}, \quad PBQ = A_1 \oplus \cdots \oplus A_k, \quad (5.3)$$

where for $i = 1, \dots, k$, A_i is fully indecomposable, $\sigma(E_i) = 1$, the $(1, 1)$ -entry of E_i is 1, and $\text{per } A_i(1; 1) = 1$.

Proof. First, suppose (i) holds. Then $\text{per } A = \text{per } B + \text{per } A(r; s) = \text{per } B + 1$. By Theorem 5.6, $\mathcal{F}(A)$ is a pyramid with basis $\mathcal{F}(B)$. Now suppose (ii) holds. Then

$$\text{per } A = \text{per } B + \prod_{i=1}^k \text{per } A_i(1; 1) = \text{per } B + 1,$$

and it follows from Theorem 5.6 that $\mathcal{F}(A)$ is a pyramid with basis $\mathcal{F}(B)$.

Conversely, suppose that $\mathcal{F}(A)$ is a pyramid with basis $\mathcal{F}(B)$. Then $\mathcal{F}(B)$ is a facet of $\mathcal{F}(A)$. Suppose B is fully indecomposable. By Corollary 2.11 there exists r and s such that $a_{rs} = 1$ and B is obtained from A by replacing a_{rs} by 0. Since $\text{per } A = \text{per } B + \text{per } A(r; s)$, it follows from Theorem 5.6 that $\text{per } A(r; s) = 1$. Now suppose that B is not fully indecomposable. It follows from Corollary 2.11 that there exist permutation matrices P and Q

such that (5.3) holds, where for $i = 1, \dots, k$, A_i is fully indecomposable, $\sigma(E_i) = 1$, and the $(1, 1)$ -entry of E_i is 1. Since $\text{per } A = \text{per } B + \prod_{i=1}^k \text{per } A_i(1; 1)$, it follows from Theorem 5.6 that $\text{per } A_i(1; 1) = 1$ for $i = 1, \dots, k$. Hence (i) or (ii) holds.

As a consequence of Theorems 5.3, 5.5, 5.8, and Corollary 5.4 we have the following.

THEOREM 5.9. *Let \mathcal{F} be a face of Ω_n with dimension d and v vertices where $v \geq 2^{d-1} + 1$. Then either \mathcal{F} is the orthogonal vector sum of an m -box and a pyramid, with basis a $(d - m - 1)$ -box for some m with $0 \leq m \leq d - 1$, or \mathcal{F} is the orthogonal vector sum of a $(d - 4)$ -box and two triangles.*

To conclude this part of the paper we describe the three-dimensional faces of Ω_n . For $n \leq 5$ some of these do not occur. Let \mathcal{F} be a three-dimensional face of Ω_n with v vertices, and let B be a $(0, 1)$ -matrix with total support such that $\mathcal{F} = \mathcal{F}(B)$. Then $v \geq 4$. Let $v = 4$. Then \mathcal{F} is a tetrahedron and B has exactly one nontrivial fully indecomposable component A , where A satisfies (ii) of Theorem 3.4 with $\sigma(A_3) = 4$. Now suppose $v \geq 5$. Then it follows from Theorem 5.3 that $v \in \{5, 6, 8\}$ and A satisfies (a) of Theorem 5.3 for $m = 0, 1$, or 2 . From Theorem 5.9 we conclude the following. If $v = 5$, then \mathcal{F} is a pyramid with basis a rectangle. If $v = 6$, then \mathcal{F} is a prism with basis a triangle and sides orthogonal to the basis. If $v = 8$, then \mathcal{F} is a 3-box. Therefore, if $n \geq 6$, the three-dimensional faces of Ω_n are of four different combinatorial types. From our description of these faces we conclude that if \mathcal{F}_1 is a three-dimensional face of Ω_m and \mathcal{F}_2 is a three-dimensional face of Ω_n such that \mathcal{F}_1 and \mathcal{F}_2 have the same combinatorial type, then \mathcal{F}_1 and \mathcal{F}_2 are affinely equivalent.

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