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Modular invariants for group-theoretical modular data. I

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ABSTRACT

We classify indecomposable commutative separable (special Frobenius) algebras and their local modules in (untwisted) group-theoretical modular categories. This gives a description of modular invariants for group-theoretical modular data. As a bi-product we provide an answer to the question when (and in how many ways) two group-theoretical modular categories are equivalent as ribbon categories.

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1. Introduction

An important feature of a rational conformal field theory (RCFT) is a decomposition of its partition function

$$Z(q) = \sum_{i,j} m_{i,j}\chi_i(q)\chi_j(\bar{q}),$$

which reflects a decomposition of the state space into a finite sum of irreducible modules over the left–right chiral algebras. Note that in a given RCFT the left and the right chiral algebras (know in mathematics as vertex operator algebras) can coincide (which is referred to as *non-heterotic* case) or can be different and even can have different categories of modules (*heterotic case*). Modular invariance of the partition function implies that the matrix of non-negative integers $M = (m_{i,j})$ is invariant with respect to the modular group actions on the characters (*modular invariant*). Traditional approach in physics for describing possible full RCFTs with given chiral algebras was to classify all modular invariant matrices, then using some heuristics chose those which can correspond to full RCFTs (so-called *physical* modular invariants). This approach has its flows: sometimes it is not enough heuristics around to separate physical modular invariants from unphysical, sometimes one modular invariant corresponds to several RCFTs. New mathematical approach utilising a special kind of tensor categories (called modular categories) was developed recently. It was known for some time that modules over rational chiral algebras (rational vertex operator algebras) form modular categories [32,20]. Not so long ago it was realised that the state space, considered as an object of the category of representations of the product of the left–right chiral algebras, has a structure of commutative separable algebra [38,21,22]. Thus the problem of classifying modular invariants (or rather full RCFTs) reduces to the classification of certain commutative separable algebras in a modular category (see [34] and the end of Section 2 of this paper).

One of the simplest examples of modular categories are (the categories of representations of) so-called quantum doubles of finite groups [12], also known as (untwisted) group-theoretical modular categories. Appearing in conformal field theory as the modular data of holomorphic orbifolds [13,27], the group-theoretical modular data and corresponding modular invariants were studied extensively (see for example [7,16]). Relatively recently V. Ostrik classified module categories over group-theoretical modular categories [35], which theoretically should give the classification of modular invariants in the case when left and right chiral modular categories coincide (non-heterotic case). The method, used in [35], is based on the theory of Morita equivalences for monoidal categories, developed by M. Müger. Being very elegant it is also quite indirect, which unfortunately makes it very difficult to calculate corresponding modular invariants explicitly.

In this paper we describe modular invariants by classifying commutative separable (special Frobenius) algebras and their local modules in group-theoretical modular categories. Algebras with trivial categories of so-called local modules (*trivialising algebras*) correspond to modular invariants. In particular, we prove that trivialising commutative algebras in the group-theoretical modular category, defined by a group G correspond to pairs (H, γ), consisting of a subgroup $H \subset G$ and a 2-cocycle $\gamma \in Z^2(H, k^*)$ (which is in complete agreement with the results of [35]). We then use the character theory for group-theoretical modular categories to calculate corresponding modular invariants. It turns out that the character of the trivialising algebra, corresponding to a pair (H, γ), has the following simple form:

$$\chi(f, g) = \frac{1}{|H|} \sum_{x\in G,\, xfx^{-1}, xgx^{-1}\in H} \frac{\gamma(xfx^{-1}, xgx^{-1})}{\gamma(xgx^{-1}, xfx^{-1})},$$

where f, g are commuting elements of G. By decomposing the character into a sum of irreducible characters one can get the corresponding modular invariant. We also study trivialising algebras in a product of two group-theoretical modular categories, corresponding to permutation modular invariants. As a result we were able to answer the question when (and in how many ways) two group-theoretical modular categories are equivalent as ribbon categories (see also [33]).

The paper is organised as follows. We start by listing some basic facts from the theory of modular categories, general theory of algebras in modular categories and their relations to modular invariants (Section 2). Then we study commutative separable algebras in group-theoretical modular categories (Section 3). We finish with the description of modular invariants for group-theoretical modular data (Section 4). The case of group-theoretical modular data for the symmetric group S_3 is treated as an example.

2. Commutative algebras in modular categories and modular invariants

Here we summarise some properties of and constructions associated with separable commutative algebras in braided monoidal categories. Then we recall the notions of modular data and modular invariants and their relations to modular categories and commutative algebras.

Throughout the paper k denotes the field of complex numbers (or any other algebraically closed field of characteristic zero). Most of our categories will be k-linear (all Hom sets are finite-dimensional k-vector spaces, compositions are k-bilinear), semi-simple (any object is a sum of simple objects), with finitely many simple objects. In particular, the endomorphism algebra of a simple object is just k. We will denote by $Irr(\mathcal{C})$ the set (of representatives) of isomorphism classes of simple objects in the category $\mathcal{C}$. Functors are also assumed to be k-linear (effects on morphisms being k-linear maps). A *fusion category* is a semi-simple k-linear monoidal rigid category, with k-linear tensor product (i.e. tensor product on morphisms is k-linear). We also assume that the monoidal unit of a fusion category is simple. Since it accommodates well all examples considered in this paper, we assume that our monoidal categories are strict (associative on the nose).

2.1. Modular categories

Slightly changing the definition from [39] we call a fusion category *modular* if it is rigid, braided, ribbon and satisfies the non-degeneracy (*modularity*) condition: for isomorphism classes of simple objects, the traces of double braiding form a non-degenerate matrix

$$\tilde{S} = (\tilde{S}_{X,Y})_{X,Y \in Irr(\mathcal{C})}, \qquad \tilde{S}_{X,Y} = tr(c_{X,Y} c_{Y,X}).$$

Here $c_{X,Y} : X \otimes Y \to Y \otimes X$ is the braiding (see [39,4] for details).

Recall that the Deligne tensor product $\mathcal{C} \boxtimes \mathcal{D}$ of two fusion categories is a fusion category with simple objects $Irr(\mathcal{C} \boxtimes \mathcal{D}) = Irr(\mathcal{C}) \times Irr(\mathcal{D})$ and the tensor product defined by

$$(X \boxtimes Y) \otimes (Z \boxtimes W) = (X \otimes Z) \boxtimes (Y \otimes W).$$

It is straightforward to see that the Deligne tensor product of two modular categories is modular.

Let $\mathcal{C}$ be a ribbon category. Following [39] define $\bar{\mathcal{C}}$ to be just $\mathcal{C}$ as a monoidal category with the new braiding a ribbon twist:

$$\bar{c}_{X,Y} = c_{Y,X}^{-1}, \qquad \bar{\theta}_X = \theta_X^{-1}.$$

Again it is very easy to see that for a modular $\mathcal{C}$, $\bar{\mathcal{C}}$ is also modular.

Examples of modular categories are provided by monoidal centre construction [25]. It was proved in [31] that if a fusion category $\mathcal{S}$ is semi-simple and spherical, then its monoidal centre $\mathcal{Z}(\mathcal{S})$ is modular (see also [6] for more general result).

2.2. Separable algebras and their modules

An (associative, unital) *algebra* in a monoidal category $\mathcal{C}$ is a triple (A, μ, ι) consisting of an object $A \in \mathcal{C}$ together with a *multiplication* $\mu : A \otimes A \to A$ and a *unit* map $\iota : 1 \to A$, satisfying *associativity*

$$\mu(\mu \otimes I) = \mu(I \otimes \mu),$$

and *unit*

$$\mu(\iota \otimes I) = I = \mu(I \otimes \iota)$$

axioms. Where it will not cause confusion we will be talking about an algebra A, suppressing its multiplication and unit maps.

A right *module* over an algebra A is a pair (M, ν), where M is an object of $\mathcal{C}$ and $\nu : M \otimes A \to M$ is a morphism (*action map*), such that

$$\nu(\nu \otimes I) = \nu(I \otimes \mu).$$

A *homomorphism* of right A-modules $M \to N$ is a morphism $f : M \to N$ in $\mathcal{C}$ such that

$$\nu_N(f \otimes I) = f\nu_M.$$

Right modules over an algebra $A \in \mathcal{C}$ together with module homomorphisms form a category $\mathcal{C}_A$. The forgetful functor $\mathcal{C}_A \to \mathcal{C}$ has a right adjoint, which sends an object $X \in \mathcal{C}$ into the *free* A-module $X \otimes A$, with A-module structure defined by

$$X \otimes A \otimes A \xrightarrow{I\mu} X \otimes A.$$

Since the action map $M \otimes A \to M$ is an epimorphism of right A-modules any right A-module is a quotient of a free module.

An algebra (A, μ, ι) in a rigid braided monoidal category $\mathcal{C}$ is called *separable* if the following composition (denoted $e : A \otimes A \to 1$) is a non-degenerate pairing:

$$A \otimes A \xrightarrow{\mu} A \xrightarrow{\epsilon} 1.$$

Here ϵ is the composition

$$A \xrightarrow{I\kappa_A} A \otimes A \otimes A^* \xrightarrow{\mu I} A \otimes A^* \xrightarrow{c_{A,A^*}} A^* \otimes A \xrightarrow{ev_A} 1,$$

where κ_A and ev_A are duality morphisms for A. Non-degeneracy of e means that there is a morphism $\kappa : 1 \to A \otimes A$ such that the composition

$$A \xrightarrow{I\kappa} A^{\otimes 3} \xrightarrow{eI} A$$

is the identity. It also implies that the similar composition

$$A \xrightarrow{\kappa I} A^{\otimes 3} \xrightarrow{Ie} A$$

is also the identity.

Using graphical calculus for morphisms in a (rigid) monoidal category [24] one can represent morphisms between tensor powers of a separable algebra by graphs (one-dimensional CW-complexes), whose end vertices are separated into incoming and outgoing. For example, the multiplication map μ is represented by a trivalent graph with two incoming and one outgoing ends, the duality ϵ is an interval, with both incoming ends etc. It turns out (e.g. it follows from the results of [37]) that separability implies that we can contract loops in connected graphs with at least one end.

For a separable algebra A the adjunction

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left] & \mathcal{C}_A \arrow[l, bend left] \end{tikzcd}$$

splits. Indeed, the splitting of the adjunction map $M \otimes A \to M$ is given by the projector $M \otimes A \to M \otimes A$:

$$M \otimes A \xrightarrow{I\epsilon} M \otimes A^{\otimes 3} \xrightarrow{I\mu I} M \otimes A^{\otimes 2} \xrightarrow{\nu I} M \otimes A.$$

For a separable algebra A the effect on morphisms $\mathcal{C}_A(M,N) \to \mathcal{C}(M,N)$ of the forgetful functor $\mathcal{C}_A \to \mathcal{C}$ has a splitting $P : \mathcal{C}(M,N) \to \mathcal{C}_A(M,N)$. For $f \in \mathcal{C}(M,N)$ the image $P(f)$ is defined as the composition

$$M \xrightarrow{I\epsilon} M \otimes A^{\otimes 2} \xrightarrow{\nu_M I} M \otimes A \xrightarrow{fI} N \otimes A \xrightarrow{\nu_N} N.$$

Moreover, the splitting has the properties

$$P(fg) = fP(g), \qquad P(gh) = P(g)h, \quad f, h \in Mor\,\mathcal{C}_A,\ g \in Mor\,\mathcal{C}.$$

This gives *Maschke's lemma* for separable algebras.

Lemma 2.2.1. *Let A be a separable algebra in a semi-simple rigid monoidal category $\mathcal{C}$. Then the category $\mathcal{C}_A$ of right A-modules in $\mathcal{C}$ is also semi-simple.*

2.3. Local modules over commutative algebras

Now let $\mathcal{C}$ be a braided monoidal category with the braiding $c_{X,Y} : X \otimes Y \to Y \otimes X$ (see [25] for definition). An algebra A in $\mathcal{C}$ is *commutative* if $\mu c_{A,A} = \mu$.

It was shown in [36] that the category ${}_A\mathcal{C}$ of left modules over a commutative algebra A is monoidal with respect to tensor product $M \otimes_A N$ over A, which can be defines as a coequaliser

$$\begin{tikzcd} M \otimes_A N & M \otimes N \arrow[l] \arrow[r, bend left, "1\nu_N"] & M \otimes A \otimes N. \arrow[l, bend left] \arrow[l, bend right, "(\nu_M 1)(c_{M,A} 1)"'] \end{tikzcd}$$

A (right) module (M, ν) over a commutative algebra A is *local* iff the diagram

$$\begin{tikzcd} M \otimes A \arrow[r, "\nu"] \arrow[d, "c_{M,A}"'] & M \\ A \otimes M \arrow[r, "c_{A,M}"] & M \otimes A \arrow[u, "\nu"'] \end{tikzcd}$$

commutes. Denote by $\mathcal{C}_A^{loc}$ the full subcategory of $\mathcal{C}_A$ consisting of local modules. The following result was established in [36].

Proposition 2.3.1. *The category $\mathcal{C}_A^{loc}$ is a full monoidal subcategory of $\mathcal{C}_A$. Moreover, the braiding in $\mathcal{C}$ induces a braiding in $\mathcal{C}_A^{loc}$.*

The following statement was proved in [18].

Proposition 2.3.2. *Let (A, m, i) be a commutative algebra in a braided category $\mathcal{C}$. Let $B = (B, \mu, \iota)$ be an algebra in $\mathcal{C}_A$. Define $\overline{\mu}$ and $\overline{\iota}$ as compositions*

$$B \otimes B \longrightarrow B \otimes_A B \xrightarrow{\mu} B, \qquad 1 \xrightarrow{i} A \xrightarrow{\iota} B.$$

Then $\overline{B} = (B, \overline{\mu}, \overline{\iota})$ is an algebra in $\mathcal{C}$.

The map $\iota : A \to B$ is a homomorphism of algebras in $\mathcal{C}$.

The algebra $\overline{B}$ in $\mathcal{C}$ is separable or commutative if and only if the algebra B in $\mathcal{C}_A$ is such.

The functor $(\mathcal{C}_A^{loc})_B^{loc} \to \mathcal{C}_{\overline{B}}^{loc}$

$$(M, m : B \otimes_A M \longrightarrow M) \longmapsto (M, \overline{m} : B \otimes M \longrightarrow B \otimes_A M \xrightarrow{m} M) \tag{1}$$

is a braided monoidal equivalence.

Remark 2.3.3. The natural map $X \otimes Y \to X \otimes_A Y$ can be seen as a lax monoidal structure on the forgetful functor $\mathcal{C}_A^{loc} \to \mathcal{C}$. The commutative diagram

$$\begin{tikzcd} X \otimes Y \arrow[r, "c_{X,Y}"] \arrow[d] & Y \otimes X \arrow[d] \\ X \otimes_A Y \arrow[r, "c_{X,Y}"] & Y \otimes_A X \end{tikzcd}$$

implies that this lax monoidal structure is braided.

It is known that lax monoidal functors preserve structures of algebras and modules. Braided lax monoidal functors preserve commutative algebras and local modules. This proves a half of Proposition 2.3.2.

We call a separable indecomposable commutative algebra A in a modular category $\mathcal{C}$ *trivialising* if $\mathcal{C}_A^{loc} = \mathcal{V}ect$.

2.4. Full centre

Details of the constructions and proofs of the results of this section can be found in [18,28] (see also [10]).

Let A be an algebra in a braided category $\mathcal{C}$. Its *left centre* $C_l(A)$ is an object in $\mathcal{C}$ with a morphism into A, universal with respect to the following property: for any $C \to A$, such that the diagram

$$\begin{tikzcd}
C \otimes A \arrow[r] \arrow[dd, "c_{C,A}"'] & A \otimes A \arrow[dr, "\mu"] & \\
& & A \\
A \otimes C \arrow[r] & A \otimes A \arrow[ur, "\mu"'] &
\end{tikzcd}$$

commutes, the morphism $C \to A$ factors through a morphism $C \to C_l(A)$. Right centre $C_r(A)$ is defined similarly (by swapping C and A in the left column of the diagram). The universal property implies in particular that $C_l(A)$, $C_r(A)$ are commutative algebras in $\mathcal{C}$. Note that if A is a separable indecomposable algebra then $C_l(A)$, $C_r(A)$ are images of certain idempotents on A (i.e. are direct summands of A).

For the next construction we need to recall the fact that for modular $\mathcal{C}$ the category $\mathcal{C} \boxtimes \overline{\mathcal{C}}$ contains a distinguished separable indecomposable commutative algebra T (as an object $\bigoplus_X X \boxtimes X^\vee$ with the sum over isomorphism classes of simple objects in $\mathcal{C}$). Now the *full centre* of an algebra $A \in \mathcal{C}$ is $Z(A) = C_l((A \boxtimes 1) \otimes T)$ (which also equals $C_r((1 \boxtimes A) \otimes T)$).

Theorem 2.4.1. *For a separable indecomposable algebra A in a modular category $\mathcal{C}$ the full centre $Z(A)$ is a trivialising algebra in $\mathcal{C} \boxtimes \overline{\mathcal{C}}$.*

Moreover, the full centre construction establishes an isomorphism between the set of Morita equivalence classes of separable indecomposable algebras in $\mathcal{C}$ and isomorphism classes of trivialising algebras in $\mathcal{C} \boxtimes \overline{\mathcal{C}}$.

Here two algebras in $\mathcal{C}$ are Morita equivalent if their categories of modules are equivalent as module categories over $\mathcal{C}$. Hence the theorem says that the full centre is an invariant of categories of internal modules in $\mathcal{C}$ (i.e. module categories over $\mathcal{C}$).

2.5. Commutative algebras in products of braided categories and their parents

Let $\mathcal{C} \boxtimes \mathcal{D}$ be the Deligne product of two braided categories. For $X \in \mathcal{C}$ the functor $X \boxtimes - : \mathcal{D} \to \mathcal{C} \boxtimes \mathcal{D}$ has a right adjoint $\mathbf{Hom}_{\mathcal{C}}(X, -) : \mathcal{C} \boxtimes \mathcal{D} \to \mathcal{D}$, which can be defined as the composition

$$\mathcal{C} \boxtimes \mathcal{D} \xrightarrow{\mathcal{C}(X,-) \boxtimes I_{\mathcal{D}}} \mathcal{V}ect \boxtimes \mathcal{D} \longrightarrow \mathcal{D}.$$

By the definition, the value $\mathbf{Hom}_{\mathcal{C}}(X, Y \boxtimes Z)$ of this right adjoint on $Y \boxtimes Z$ coincides with $\mathcal{C}(X, Y)Z$, which is the tensor product over k of the Hom vector space $\mathcal{C}(X, Y)$ and an object $Z \in \mathcal{D}$. This allows us to define a map $X \boxtimes \mathbf{Hom}_{\mathcal{C}}(X, A) \to A$. The object $\mathbf{Hom}_{\mathcal{C}}(X, A)$ can be characterised by a universal property: the pair $(\mathbf{Hom}_{\mathcal{C}}(X, -), X \boxtimes \mathbf{Hom}_{\mathcal{C}}(X, A) \to A)$ is terminal among the pairs $(Y, X \boxtimes Y \to A)$, i.e. for any morphism $X \boxtimes Y \to A$ in $\mathcal{C} \boxtimes \mathcal{D}$ there is a unique morphism $Y \to \mathbf{Hom}_{\mathcal{C}}(X, A)$, which makes the triangle

$$\begin{tikzcd}
X \boxtimes \mathbf{Hom}_{\mathcal{C}}(X, A) \arrow[r] & A \\
X \boxtimes Y \arrow[u] \arrow[ur] &
\end{tikzcd}$$

commute. This, in particular, can be used to define a functorial map

$$\mathbf{Hom}_{\mathcal{C}}(X, A) \otimes \mathbf{Hom}_{\mathcal{C}}(Y, B) \longrightarrow \mathbf{Hom}_{\mathcal{C}}(X \otimes Y, A \otimes B).$$

Similarly for $\mathbf{Hom}_{\mathcal{D}}(Y, -) : \mathcal{C} \boxtimes \mathcal{D} \to \mathcal{C}$ for $Y \in \mathcal{D}$.

In particular, we have braided lax monoidal functors (corresponding to monoidal units in $\mathcal{C}$ and $\mathcal{D}$):

$$\mathcal{C} \xleftarrow{\mathbf{Hom}_{\mathcal{D}}(1,-)} \mathcal{C} \boxtimes \mathcal{D} \xrightarrow{\mathbf{Hom}_{\mathcal{C}}(1,-)} \mathcal{D}.$$

Now, for a commutative algebra C in $\mathcal{C} \boxtimes \mathcal{D}$, the objects $C_l = \mathbf{Hom}_{\mathcal{D}}(1, C) \in \mathcal{C}$, $C_r = \mathbf{Hom}_{\mathcal{C}}(1, C) \in \mathcal{D}$ have the structures of commutative algebras. We call them the *parents* of C. Note that if C is indecomposable or separable, then so are its parents $C_{l,r}$.

The following theorem is a slight generalisation of results from [18], the proof will appear in [11].

Theorem 2.5.1. *Let C be a trivialising algebra in a modular category $\mathcal{C} \boxtimes \mathcal{D}$. Then the functor $\mathbf{Hom}_{\mathcal{C}^{loc}_{C_l}}(-, C)$ induces a braided monoidal equivalence*

$$\left(\mathcal{C}^{loc}_{C_l}\right)^{op} \longrightarrow \mathcal{D}^{loc}_{C_r}$$

of the categories of local modules. Moreover,

$$C = \bigoplus_{M \in Irr(\mathcal{C}^{loc}_{C_l})} M \boxtimes \mathbf{Hom}_{\mathcal{C}}(M, C), \tag{2}$$

where the sum is taken over simple local C_l-modules in $\mathcal{C}$.

Conversely, for any indecomposable separable commutative algebras $A \in \mathcal{C}$, $B \in \mathcal{D}$ and an equivalence of braided monoidal categories $(\mathcal{C}^{loc}_A)^{op} \to \mathcal{D}^{loc}_B$ there exists a maximal indecomposable separable commutative algebra in $C \in \mathcal{C} \boxtimes \mathcal{D}$ such that $C_l = A$, $C_r = B$ and the equivalences $(\mathcal{C}^{loc}_A)^{op} \to \mathcal{D}^{loc}_B$, $(\mathcal{C}^{loc}_{C_l})^{op} \to \mathcal{D}^{loc}_{C_r}$ coincide up to isomorphism of algebras and functors.

In particular, the parents of a full centre $Z(A)$ are (left, right) centres $C_l(A)$, $C_r(A)$.

2.6. Modular data and modular invariants

Let $SL_2(\mathbf{Z})$ be the *modular group*, i.e. the group of determinant 1 integer 2×2-matrices. It is generated by the matrices

$$s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

with the generating system of relations $s^4 = 1$, $(ts)^3 = s^2$.

Let $\mathcal{C}$ be a modular category. Define

$$S = \left(\sqrt{dim(\mathcal{C})}\right)^{-1} \tilde{S}, \qquad T = diag(\theta_X).$$

The pair of matrices S, T is often referred to as *modular data* of $\mathcal{C}$. The proof of the following result can be found in [39].

Theorem 2.6.1. *Let $\mathcal{C}$ be a modular category. Then the operators S and T define a (projective) action of the modular group $SL_2(\mathbf{Z})$ on the complexified Grothendieck group $K_0(\mathcal{C}) \otimes \mathbf{C}$.*

Remark 2.6.2. Projectivity of the above action manifests itself by a scalar multiple appearing in the second defining relation:

$$S^4 = 1, \qquad (TS)^3 = \lambda S^2.$$

Over the complex numbers it is always possible to turn it into a genuine representation, by rescaling T. For the reasons of why one should not do it see [39,4].

An alternative approach to the modular group action was developed in [30] (see also [6]). Recall that the *coend* of a monoidal category $\mathcal{C}$ is an object $C \in \mathcal{C}$ with a natural collection of (action) maps $X \otimes C \to X$, universal in the following sense: for any other object $D \in \mathcal{C}$ together with a natural collection of maps $X \otimes D \to X$ there is a morphism $D \to C$ making the diagram

$$\begin{tikzcd} X \otimes D \arrow[rr] \arrow[dr] & & X \otimes C \arrow[dl] \\ & X & \end{tikzcd}$$

commutative. Alternatively (in the autonomous case) the coend can be defined as a colimit $\int^X X^\vee \otimes X$, which in the case of semi-simple $\mathcal{C}$ coincides with the direct sum $\bigoplus_X X^\vee \otimes X$ over the isomorphism classes of simple objects. The coend has a number of nice properties and structures, e.g. for a braided $\mathcal{C}$ the coend becomes an internal Hopf algebra. In the case of a modular category $\mathcal{C}$ the coend gets equipped with a projective action of a mapping class group of a torus with removed disk. The Hom-space $\mathcal{C}(1, C)$ carries a projective action of the mapping class group of a closed torus, i.e. the modular group action (see [30,6] for details). Note finally, that for a semi-simple modular $\mathcal{C}$, $\mathcal{C}(1, C)$ coincides with $K_0(\mathcal{C}) \otimes \mathbf{C}$ as a module over the modular group. The map $K_0(\mathcal{C}) \otimes \mathbf{C} \to \mathcal{C}(1, C)$ is the composition of the *character map* $K_0(\mathcal{C}) \otimes \mathbf{C} \to End(id_{\mathcal{C}})^\vee$ with the natural identification $End(id_{\mathcal{C}})^\vee \simeq \mathcal{C}(1, C)$. Here $End(id_{\mathcal{C}})$ is the vector space of endomorphisms of the identity functor on $\mathcal{C}$ and the character map sends a class of a simple object X into the function $a \mapsto \lambda$ where $a \in End(id_{\mathcal{C}})$ and $a_X = \lambda I_X$.

Now we explain the relation between modular invariants and trivialising algebras. The next theorem is Theorem 4.5 from [29].

Theorem 2.6.3. *Let A be an indecomposable separable commutative algebra in a modular category $\mathcal{C}$ with $\theta_A = 1$. Then $\mathcal{C}_A^{loc}$ is a modular category and the map $K_0(\mathcal{C}_A^{loc}) \otimes_{\mathbf{Z}} \mathbf{C} \to K_0(\mathcal{C}) \otimes_{\mathbf{Z}} \mathbf{C}$, induced by the forgetful functor $\mathcal{C}_A^{loc} \to \mathcal{C}$ is $SL_2(\mathbf{Z})$-equivariant.*

Corollary 2.6.4. *Let Z be a trivialising algebra in a modular category $\mathcal{C}$. Then its class $[A]$ in the Grothendieck ring $K_0(\mathcal{C}) \otimes_{\mathbf{Z}} \mathbf{C}$ is a modular invariant element.*

Proof. Since A is a trivialising algebra, the Grothendieck group $K_0(\mathcal{C}_A^{loc})$ is isomorphic to $\mathbf{Z}$ and the homomorphism $K_0(\mathcal{C}_A^{loc}) \to K_0(\mathcal{C})$ sends an integer n into $n[A]$. Modular invariance of the complexification of this homomorphism implies that $[A]$ is a modular invariant element. $\square$

It was shown in [19,21], that rational conformal field theories correspond to trivialising algebras in $\mathcal{C}_l \boxtimes \overline{\mathcal{C}}_r$. Here $\mathcal{C}_{l,r}$ are *chiral* modular categories of the theory (representation categories of chiral vertex operator algebras). In particular, the coefficients of the decomposition of the partition function of the theory into the sum of chiral irreducible characters are the decomposition coefficients of the trivialising algebra in the basis of simple objects in $K_0(\mathcal{C}_l \boxtimes \overline{\mathcal{C}}_r) = K_0(\mathcal{C}_l) \otimes K_0(\mathcal{C}_r)$. Traditionally [32] elements in $K_0(\mathcal{C}_l) \otimes K_0(\mathcal{C}_r)$, invariant with respect to the (anti-)diagonal modular group action, are called *modular invariants*. A modular invariant is *physical* if it corresponds to a rational conformal field theory, i.e. is the class of a trivialising algebra. Simplest examples in the case when $\mathcal{C}_l = \mathcal{C}_r$ (*non-heterotic case*) are the *diagonal modular invariant* $\bigoplus_X [X] \otimes [X^\vee]$ and the *conjugation modular invariant*

$\bigoplus_X [X] \otimes [X]$. Here sums are over isomorphism classes of simple objects of $\mathcal{C} = \mathcal{C}_l = \mathcal{C}_r$. While the diagonal modular invariant is always physical (is the class of the full centre $Z(1_{\mathcal{C}}) \in \mathcal{Z}(\mathcal{C}) \simeq \mathcal{C} \boxtimes \bar{\mathcal{C}}$) the conjugation modular invariant can be non-physical.

3. Commutative algebras in group-theoretical modular categories

3.1. Group-theoretical modular categories

Here we describe the monoidal centre $\mathcal{Z}(\mathcal{C}(G))$ of the fusion category $\mathcal{C}(G)$ of G-graded finite-dimensional vector spaces. The results of this section are mostly well known.

A *compatible* G-action on a G-graded vector space $V = \bigoplus_{g \in G} V_g$ is a collection of automorphisms $f : V \to V$ for each $f \in G$ such that $f(V_g) = V_{fgf^{-1}}$ and $(fg)(v) = f(g(v))$.

Proposition 3.1.1. *The monoidal centre $\mathcal{Z}(\mathcal{C}(G))$ is isomorphic, as braided monoidal category, to the category $\mathcal{Z}(G)$, whose objects are G-graded vector spaces $X = \bigoplus_{g \in G} X_g$ together with a compatible G-action and morphisms are graded and action preserving homomorphisms of vector spaces. The tensor product in $\mathcal{Z}(G)$ is the tensor product of G-graded vector spaces with the G-action defined by*

$$f(x \otimes y) = f(x) \otimes f(y), \quad x \in X,\ y \in Y. \tag{3}$$

The monoidal unit is $1 = 1_e = k$ with trivial G-action.

The braiding is given by

$$c_{X,Y}(x \otimes y) = f(y) \otimes x, \quad x \in X_f,\ y \in Y. \tag{4}$$

The category $\mathcal{Z}(G)$ is rigid, with dual objects $X^\vee = \bigoplus_f (X^\vee)_f$ given by

$$\left(X^\vee\right)_f = (X_{f^{-1}})^\vee = \mathbf{Hom}(X_{f^{-1}}, k),$$

with the action

$$g(l)(x) = l\left(g^{-1}(x)\right), \quad l \in \mathbf{Hom}(X_{f^{-1}}, k),\ x \in X_{gf^{-1}g^{-1}}.$$

The category $\mathcal{Z}(G)$ is unitarisable with the ribbon twist

$$\theta_X(x) = f^{-1}(x), \quad x \in X_f.$$

The (unitary) trace of an endomorphism $a : X \to X$ can be written in terms of ordinary traces on vector spaces X_g:

$$tr(a) = \sum_{g \in G} tr_{X_g}(a_g),$$

and the (unitary) dimension of an object $X \in \mathcal{Z}(G)$ is the dimension of its underlying (graded) vector space

$$dim(X) = \sum_{g \in G} dim(X_g).$$

Proof. For an object (X, x) of the centre $\mathcal{Z}(\mathcal{C}(G))$ the natural isomorphism

$$x_V : V \otimes X \to X \otimes V, \quad V \in \mathcal{C}(G)$$

is defined by its evaluations on one-dimensional graded vector spaces. Denote by $k(f)$ such a one-dimensional graded vector space, sitting in degree f. Then the isomorphism $x_{k(f)}$ can be seen as an automorphism $f : X \to X$. The fact, that $x_{k(f)}$ preserves grading, amounts to the condition $f(X_g) = X_{fgf^{-1}}$:

$$X_g = \big(k(f) \otimes X\big)_{fg} \xrightarrow{\chi_{k(f)}} \big(X \otimes k(f)\big)_{fg} = X_{fgf^{-1}}.$$

The coherence condition for x is equivalent to the action axioms. The diagram, defining the second component $\chi|\psi$ of the tensor product $(X, x) \otimes (Y, y) = (X \otimes Y, x|y)$, is equivalent to the tensor product of actions (3).

The description of the monoidal unit in a monoidal centre corresponds to the answer for the monoidal unit in $\mathcal{Z}(G)$.

Clearly, the braiding $c_{(X,x),(Y,y)} = x_Y$ in the centre $\mathcal{Z}(\mathcal{C}(G))$ corresponds to (4).

The answer for the dual object in $\mathcal{Z}(G)$ follows from the general construction of dual objects in monoidal centres of spherical categories (see [31]). In our concrete case it can also be verified directly. Indeed, the evaluation map $ev_X : X^\vee \otimes X \to 1$ pairs $(X^\vee)_f$ with $X_{f^{-1}}$ via $ev_X(l \otimes x) = l(x)$. Its G-invariance follows form the definition of the G-action on $X^\vee$:

$$ev_X\big(g(l \otimes x)\big) = ev_X\big(g(l) \otimes g(x)\big) = g(l)\big(g(x)\big) = l\big(g^{-1}\big(g(x)\big)\big) = l(x).$$

The coevaluation map $\kappa_X : 1 \to X \otimes X^\vee$ is defined as follows: projected to $X_g \otimes (X^\vee)_{g^{-1}} = X_g \otimes X_g^*$ it coincides with coevaluation κ_{X_g}. The duality axioms are straightforward.

Note that the inverse to θ has the form $\theta^{-1}(x) = f(x)$. Indeed,

$$\theta^{-1}\theta(x) = \theta^{-1}\big(f^{-1}(x)\big) = f\big(f^{-1}(x)\big) = x.$$

The balancing axiom for θ can be checked directly. Indeed, the effect of the double braiding on $x \otimes y \in X_f \otimes Y_g$ is

$$x \otimes y \longmapsto f(y) \otimes x \longmapsto fgf^{-1}(x) \otimes f(y),$$

while $\theta_{X\otimes Y}^{-1}(\theta_X \otimes \theta_Y)$ acts as

$$\begin{aligned} x \otimes y &\longmapsto f^{-1}(x) \otimes g^{-1}(y) \longmapsto (fg)\big(f^{-1}(x) \otimes g^{-1}(y)\big) \\ &= (fg)\big(f^{-1}(x)\big) \otimes (fg)\big(g^{-1}(y)\big) = \big(fgf^{-1}\big)(x) \otimes f(y). \end{aligned}$$

The self-duality for the ribbon twist $\theta_{X^\vee} = (\theta_X^{-1})^\vee$ is straightforward.

The formula for the trace follows from the fact that the duality structure in $\mathcal{Z}(G)$ is the same as in the category of finite-dimensional (G-graded) vector spaces. □

Remark 3.1.2. Note that $\mathcal{Z}(G)$ coincides with the category of modules over Drinfeld double $D(G) = D(k[G])$ of the group G. Thus Proposition 3.1.1 follows from the statement $\mathcal{Z}(H - Mod) \cong D(H) - Mod$, which is true for any finite-dimensional Hopf algebra H (see [26] for details).

In the next statement we describe simple objects and the S-matrix of the category $\mathcal{Z}(G)$ (see also [1,7]).

Proposition 3.1.3. *Simple objects of $\mathcal{Z}(G)$ are parametrised by pairs (g, U), where $g \in G$ and U is a simple module over the group algebra $k[C_G(g)]$.*

The dimension of the category $\mathcal{Z}(G)$ is $|G|^2$.

The category $\mathcal{Z}(G)$ is modular with the S- and T-matrices:

$$S_{(f,\psi),(g,\xi)} = \frac{1}{|G|} \sum_{u\in f^G,\, v\in g^G,\, uv=vu} \psi\left(xv^{-1}x^{-1}\right)\xi\left(yu^{-1}y^{-1}\right),$$

where $u = x^{-1}fx$, $v = y^{-1}gy$, and

$$T_{(f,\psi),(f,\psi)} = \frac{\psi(f)}{\psi(e)}. \tag{5}$$

Proof. Clearly the support of a simple object V in $\mathcal{Z}(G)$ should be an indecomposable G-subset in G (with conjugation action), i.e. a conjugacy class of G. Let g be an element of the support. The axioms of the action imply that V is induced from the $k[C_G(g)]$-module V_g. Finally, for V to be simple, the $k[C_G(g)]$-module V_g must be simple as well.

For $g \in G$ the sum $\sum_U dim(U)^2$ over isomorphism classes of irreducible $k[C_G(g)]$-modules is equal to $|C_G(g)|$. Since

$$dim(g, U) = dim\left(Ind^G_{C_G(g)}(U)\right) = \left[G : C_G(g)\right] dim(U)$$

we have

$$\begin{aligned} dim\left(\mathcal{Z}(G)\right) &= \sum_{g,U} \left[G : C_G(g)\right]^2 dim(U)^2 = \sum_g \left[G : C_G(g)\right]^2 \left|C_G(g)\right| \\ &= |G|^2 \sum_g \left|C_G(g)\right|^{-1}, \end{aligned}$$

where g runs through representatives of conjugacy classes of G. It is well known in group theory that the last sum is equal to $|G|^2$.

The formula for the S-matrix can be obtained by calculating the trace of the double braiding $c_{(g,\xi),(f,\psi)}c_{(f,\psi),(g,\xi)}$ in the category $\mathcal{Z}(G)$. $\square$

The next result describes Deligne products and mirrors of group-theoretical modular categories.

Proposition 3.1.4.

$$\mathcal{Z}(G_1) \boxtimes \mathcal{Z}(G_2) \simeq \mathcal{Z}(G_1 \times G_2), \qquad \overline{\mathcal{Z}(G)} \simeq \mathcal{Z}(G).$$

Proof. Follows from the straightforward equivalences:

$$\mathcal{Z}(\mathcal{C}) \boxtimes \mathcal{Z}(\mathcal{D}) \simeq \mathcal{Z}(\mathcal{C} \boxtimes \mathcal{D}), \qquad \overline{\mathcal{Z}}(\mathcal{C}) \simeq \mathcal{Z}\left(\mathcal{C}^{op}\right). \qquad \square$$

For the case $\mathcal{C} = \mathcal{Z}(G)$ the space of characters has the following description (see [3,2]). It is the space of k-valued functions on

$$C^2(G) = \left\{(f, g) \in G^{\times 2},\ fg = gf\right\}.$$

In this realisation the $SL_2(\mathbf{Z})$-action is given by

$$S(\chi)(f, g) = \chi\left(g, f^{-1}\right), \qquad T(\chi)(f, g) = \chi(f, fg).$$

For an object X the character map $K_0(\mathcal{Z}(G)) \otimes_{\mathbf{Z}} \mathbf{C} \to \mathbf{Hom}(C^2(G), k)$ sends the class $[X]$ into the function (the *character*):

$$\chi_X(f, g) = tr_{X_f}(g).$$

In particular, the character of the dual object (the *dual character*) has a form:

$$\chi_{X^\vee}(f, g) = \chi_X\big(f^{-1}, g^{-1}\big).$$

As in the ordinary character theory, the space of characters of $\mathcal{Z}(G)$ comes equipped with a scalar product (see [3])

$$(\chi, \psi) = \frac{1}{|G|} \sum_{f,g \in G} \chi(f, g)\overline{\psi(f, g)},$$

which calculates dimensions of corresponding Hom-spaces in $\mathcal{Z}(G)$:

$$(\chi_X, \chi_Y) = dim\big(\mathcal{Z}(G)(X, Y)\big).$$

In particular, for irreducible X, Y, $(\chi_X, \chi_Y) = 1$ iff $X = Y$ and zero otherwise.

3.2. Algebras in group-theoretical modular categories

We start by expanding the structure of an algebra in the category $\mathcal{Z}(G)$ in plain algebraic terms. Recall that a G-graded vector space $A = \bigoplus_{g \in G} A_g$ is a *G-graded algebra* if the multiplication preserves grading $A_f A_g \subset A_{fg}$.

Proposition 3.2.1. *An algebra in the category $\mathcal{Z}(G)$ is a G-graded associative algebra together with a G-action such that*

$$f(ab) = f(a)f(b), \quad a, b \in A. \tag{6}$$

An algebra A in the category $\mathcal{Z}(G)$ is commutative iff

$$ab = f(b)a, \quad \forall a \in A_f,\ b \in A. \tag{7}$$

The twist θ_A is trivial iff

$$f(a) = a, \quad a \in A_f.$$

Proof. Being a morphism in the category $\mathcal{Z}(G)$ the multiplication of an algebra in $\mathcal{Z}(G)$ preserves grading and G-action (hence the property (6)). Associativity of multiplication in $\mathcal{Z}(G)$ is equivalent to ordinary associativity.

The formula (4) for the braiding in $\mathcal{Z}(G)$ implies that commutativity for an algebra A in the category $\mathcal{Z}(G)$ is equivalent to the condition (7). □

By *G-algebra* we mean an algebra (in $\mathcal{V}ect$) with an action of G by algebra homomorphisms (i.e. an algebra in the category $\mathcal{R}ep(G)$ of G-modules). Note that the degree e part A_e of an algebra A in the category $\mathcal{Z}(G)$ is a G-algebra and A is a module over A_e. Moreover the algebra A_e is commutative if A is a commutative algebra in the category $\mathcal{Z}(G)$.

Proposition 3.2.2. *An algebra A in the category $\mathcal{Z}(G)$ is separable iff*

$$A_f \otimes A_{f^{-1}} \xrightarrow{\mu} A_e \xrightarrow{\epsilon} k$$

defines a non-degenerate bilinear pairing for any $f \in G$. In particular, the algebra A_e is separable if A is a separable algebra in the category $\mathcal{Z}(G)$.

Proof. Being a graded homomorphism the separability map $A \to 1$ is zero on A_f for $f \neq e$. Hence the separability bilinear form is zero on $A_f \otimes A_g$ unless $fg = e$. In particular, the restriction of ϵ to A_e makes it a separable algebra in the category of vector spaces. □

3.3. Commutative separable algebras in trivial degree and their local modules

We start with a well-known (see for example [29]) description of indecomposable commutative separable G-algebras. We give (a sketch of) the proof for completeness.

Lemma 3.3.1. *Commutative separable G-algebras are function algebras on G-sets. Indecomposable G-algebras correspond to transitive G-sets.*

Proof. A separable commutative algebra over an algebraically closed field is a function algebra $k(X)$ on a finite set X (with elements of X corresponding to minimal idempotents of the algebra). The G-action on the algebra amounts to a G-action on the set X. Obviously, the algebra of functions $k(X \cup Y)$ on the disjoint union of G-sets is the direct sum of G-algebras $k(X) \oplus k(Y)$ and any direct sum decomposition of G-algebras appears in that way. □

Let $k(X)$ be an indecomposable G-algebra. By choosing a minimal idempotent $p \in X$, we can identify the G-set X with the set G/H of cosets modulo the stabiliser subgroup $H = St_G(p)$.

Theorem 3.3.2. *The category $\mathcal{Z}(G)^{loc}_{k(G/H)}$, of local left $k(G/H)$-modules in $\mathcal{Z}(G)$, is equivalent, as a ribbon category, to $\mathcal{Z}(H)$.*

Proof. For a right $k(G/H)$-module M the product Mp with a chosen idempotent is a G-graded vector space with H-action. For a local M the support of Mp (elements of G, whose graded components are non-zero) is a subset of H. Indeed, for $m \in M_f$ the locality condition implies that $mp = mf(p)$ and $mp = mp^2 = mpf(p)$. Thus if $mp \neq 0$ the product $pf(p)$ is also non-zero and $f(p) = p$. Hence for a local H the subspace Mp is an object of $\mathcal{Z}(H)$, which defines a functor

$$\mathcal{Z}(G)^{loc}_{k(G/H)} \longrightarrow \mathcal{Z}(H), \quad M \longmapsto Mp.$$

The functor is obviously monoidal $(M \otimes_A N)p = Mp \otimes Np$, braided and balanced.

Now let $U \in \mathcal{Z}(H)$. The tensor product $k(G) \otimes_H U$ (which is spanned by $p_g \otimes u$, modulo $p_{gh} \otimes u = p_g \otimes h(u)$) is naturally equipped with the G-grading

$$|p_g \otimes u| = g|u|g^{-1}$$

and the G-action $f(p_g \otimes u) = p_{fg} \otimes u$, making it an object of $\mathcal{Z}(G)$. The homomorphism of algebras $k(G/H) \to k(G)$ (induced by the quotient map $G \to G/H$) makes $k(G) \otimes_H U$ a right $k(G/H)$-module. Explicitly, for a coset $x \in G/H$

$$(p_g \otimes u)p_x = \delta_{g,x} p_g \otimes u.$$

Here $\delta_{g,x}$ is the δ-function, which is equal to 1, if g belongs to x, and zero otherwise. Moreover, $k(G) \otimes_H U$ is a local left $k(G/H)$-module: the value of the product map on $p_x \otimes (p_g \otimes u)$ coincides with the value on

$$(c \circ c)\big(p_x \otimes (p_g \otimes u)\big) = g|u|g^{-1}(p_x) \otimes (p_g \otimes u) = p_{g|u|g^{-1}x} \otimes (p_g \otimes u).$$

Indeed, g belongs to x (i.e. $x = gH$) iff g belongs to $g|u|g^{-1}x = g|u|g^{-1}gH = g|u|H = gH$. Thus we have a functor

$$\mathcal{Z}(H) \longrightarrow \mathcal{Z}(G)^{loc}_{k(G/H)}, \quad U \longmapsto k(G) \otimes_H U.$$

Finally, the maps

$$U \longrightarrow \big(k(G) \otimes_H U\big)p, \quad u \longmapsto p_e \otimes u,$$
$$k(G) \otimes_H M \longrightarrow M, \quad p_g \otimes mp \longmapsto g(mp)$$

are isomorphisms. □

Remark 3.3.3. It follows from the proof of Theorem 3.3.2 that the category $\mathcal{Z}(G)_{k(G/H)}$ of right $k(G/H)$ modules can be identified with the category of G-graded vector spaces equipped with H-actions.

Remark 3.3.4. Theorem 3.3.2 in combination with Proposition 2.3.2 gives an interpretation of the *transfer*, defined in [40]. The transfer turns an algebra from $\mathcal{Z}(H)$ into an algebra in $\mathcal{Z}(G)$. Indeed, by Theorem 3.3.2 an algebra from $\mathcal{Z}(H)$ is an algebra in $\mathcal{Z}(G)^{loc}_{k(G/H)}$, which by Proposition 2.3.2 gives an algebra in $\mathcal{Z}(G)$.

Corollary 3.3.5. *For an indecomposable separable algebra A in $\mathcal{Z}(G)$ there is a subgroup $H \subset G$ such that A is the transfer of an indecomposable separable algebra B in $\mathcal{Z}(H)$ with $B_e = k$.*

Proof. The subalgebra A_e is an indecomposable commutative G-algebra. By Lemma 3.3.1 it is isomorphic to $k(X)$ for some transitive G-set X. By Proposition 2.3.2, A is a commutative algebra in $\mathcal{Z}(G)^{loc}_{A_e}$. Thus, by Theorem 3.3.2, A is the transfer of the indecomposable separable algebra $B = pA$ from $\mathcal{Z}(H)$ (here p is the minimal idempotent of A_e, corresponding to an element of X, with the stabiliser $H = St_G(p)$). Finally, $B_e = pA_e = k$ by minimality of p. □

3.4. Commutative separable algebras trivial in trivial degree and their local modules

Here we describe indecomposable commutative separable algebras B in $\mathcal{Z}(H)$ with $B_e = k$.

Lemma 3.4.1. *Let B be a separable algebra in $\mathcal{Z}(H)$ such that $B_e = k$. Then*

$$dim(B_h) \leqslant 1, \quad \forall h \in H.$$

Moreover the support of B

$$F = \{f \in H \mid B_f \neq 0\}$$

is a normal subgroup of H.

Proof. By Proposition 3.2.2 an algebra B, such that $B_e = k$, is separable iff the multiplication defines the non-degenerate pairing $m : B_g \otimes B_{g^{-1}} \to A_e = k$. Thus, associativity of multiplication implies that, for any $a, c \in B_g$ and $b \in B_{g^{-1}}$, $m(a,b)c = am(b,c)$. For non-zero a, c, choosing b such that $m(a,b), m(b,c) \neq 0$, we get that a and c are proportional.

Now, it follows from the non-degeneracy of $m : B_g \otimes B_{g^{-1}} \to A_e = k$, that a generator of a non-zero B_f is invertible. Thus, for non-zero components B_f, B_g the product $B_f B_g$ is also non-zero. □

Let $F \lhd H$ be a normal subgroup and $\gamma \in Z^2(F, k^*)$ be a normalised 2-cocycle, i.e. $\gamma(e,g) = \gamma(f,e) = 1$ and

$$\gamma(f,g)\gamma(fg,h) = \gamma(g,h)\gamma(f,gh).$$

Note that for a 2-cocycle $\gamma \in Z^2(G, k^*)$ the expression

$$\gamma_f(g) = \gamma(f,g)\gamma(g,f)^{-1}$$

defines a multiplicative map (a character) $\gamma_f : C_G(f) \to k^*$ of the centraliser $C_G(f)$.

Denote by $k[F,\gamma]$ an H-graded associative algebra with the basis e_f, $f \in F$, graded as $|e_f| = f$, and with multiplication defined by $e_f e_g = \gamma(f,g) e_{fg}$.

Proposition 3.4.2. *An indecomposable commutative separable algebra B in $\mathcal{Z}(H)$ with $B_e = k$ has a form $k[F,\gamma]$ with the H-action given by*

$$h(e_f) = \varepsilon_h(f) e_{hfh^{-1}},$$

for some $\varepsilon : H \times F \to k^*$ satisfying

$$\varepsilon_{gh}(f) = \varepsilon_g\big(hfh^{-1}\big)\varepsilon_h(f), \quad g,h \in H,\ f \in F, \tag{8}$$

$$\gamma(f,g)\varepsilon_h(fg) = \varepsilon_h(f)\varepsilon_h(g)\gamma\big(hfh^{-1}hgh^{-1}\big), \quad h \in H,\ f,g \in F, \tag{9}$$

$$\gamma(f,g) = \varepsilon_f(g)\gamma\big(fgf^{-1}, f\big), \quad f,g \in F. \tag{10}$$

Proof. Indeed, action axiom requires that $(gh)(e_f) = \varepsilon_{gh}(f) e_{ghfh^{-1}g^{-1}}$ should coincide with

$$g\big(h(e_f)\big) = \varepsilon_h(f)\varepsilon_g\big(hfh^{-1}\big) e_{ghfh^{-1}g^{-1}},$$

which gives the first identity. Multiplicativity of the action amounts to the equality between

$$h(e_f e_g) = \gamma(f,g)\varepsilon_h(fg) e_{hfgh^{-1}}$$

and

$$h(e_f)h(e_g) = \varepsilon_h(f)\varepsilon_h(g)\gamma\big(hfh^{-1}, hgh^{-1}\big) e_{hfgh^{-1}},$$

which gives the second identity. Finally, commutativity implies that $e_f e_g = \gamma(f,g) e_{fg}$ is equal to

$$f(e_g)e_f = \varepsilon_f(g) e_{fgf^{-1}} e_f = \varepsilon_f(g)\gamma\big(fgf^{-1}, f\big) e_{fg}. \qquad \square \tag{11}$$

Denote by $k[F,\gamma,\varepsilon]$ an indecomposable commutative separable algebra in $\mathcal{Z}(H)$, defined in Proposition 3.4.2.

Lemma 3.4.3. *Two algebras $k[F,\gamma,\varepsilon]$ and $k[F',\gamma',\varepsilon']$ in the category $\mathcal{Z}(H)$ are isomorphic iff there is a cochain $c: F \to k^*$ such that*

$$c(fg)\gamma(f,g) = \gamma'(f,g)c(f)c(g), \qquad \varepsilon_h(f)c\big(hfh^{-1}\big) = c(f)\varepsilon'_h(g).$$

Proof. Isomorphic algebras in $\mathcal{Z}(H)$ have to have the same supports. Thus $F = F'$. Since the components of both $k[F,\gamma,\varepsilon]$ and $k[F',\gamma',\varepsilon']$ are all one-dimensional, an isomorphism $k[F,\gamma,\varepsilon] \to k[F',\gamma',\varepsilon']$ has a form $e_f \mapsto c(f)e_f$ for some $c(f) \in k^*$. Finally, multiplicativity of this mapping is equivalent to the first condition, while H-equivariance is equivalent to the second. □

For the sake of keeping it short we will not give complete description of the category of local modules over the algebra $k[F,\gamma,\varepsilon]$ (which will be given in a subsequent paper). Instead we characterise those algebras which have trivial category of local modules (i.e. trivialising algebras).

Theorem 3.4.4. *The algebra $k[F,\gamma,\varepsilon]$ in the category $\mathcal{Z}(H)$ is trivialising iff $F = H$.*

Proof. The structure of a right $k[F,\gamma,\varepsilon]$-module on an object $M = \bigoplus_{h\in H} M_h$ of $\mathcal{Z}(H)$ amounts to a collection of isomorphisms $e_f : M_h \to M_{hf}$ (right multiplication by $e_f \in k[F,\gamma,\varepsilon]$) such that

$$e_e = I, \qquad e_f e_{f'} = \gamma\big(f,f'\big)e_{f'f}, \qquad he_f h^{-1} = \varepsilon_h(f)e_{hfh^{-1}}, \quad f,f' \in F,\ h \in H.$$

Here $h: M_{h'} \to M_{hh'h^{-1}}$ is the projective H-action on M. The $k[F,\gamma,\varepsilon]$-module M is local iff $e_f = \varepsilon_h(f)hfh^{-1}e_{hfh^{-1}}$ on M_h. Indeed, the double braiding in $\mathcal{Z}(H)$ transforms an element $m \otimes e_f \in M \otimes A$ (with $m \in M_h$) as follows

$$m \otimes e_f \longmapsto h(e_f) \otimes m = \varepsilon_h(f)e_{hfh^{-1}} \otimes m \longmapsto \varepsilon_h(f)hfh^{-1}(m) \otimes e_{hfh^{-1}}.$$

An equivalent way of expressing the locality condition is:

$$f = \varepsilon_h\big(h^{-1}fh\big)^{-1}\gamma\big(h^{-1}fh, f^{-1}\big)\gamma\big(f,f^{-1}\big)^{-1}e_{h^{-1}fhf^{-1}} = \epsilon_h(f)e_{[h^{-1},f]}.$$

In particular, F acts trivially on M_e.

Now if $F = H$ the action map $M_e \otimes A \to M$ is an isomorphism, i.e. any local module is free. Conversely for $F \neq H$ take a non-trivial H/F-representation U and define an H-action on the H-graded vector space

$$M = V \otimes A = \bigoplus_{f\in F} M_f, \quad M_f = V \otimes e_f$$

by

$$h(v \otimes e_f) = \varepsilon_h(f)h(v) \otimes e_{hfh^{-1}}.$$

Then M is a right $k[F,\gamma,\varepsilon]$-module

$$v \otimes e_f \otimes e_{f'} \longmapsto \gamma\big(f,f'\big)v \otimes e_{ff'},$$

which is local and non-free. □

3.5. Commutative separable algebras and their local modules

In this section we combine the previous results on commutative separable algebras in group-theoretical modular categories and on their local modules.

Define $A(H, F, \gamma, \varepsilon)$ as a vector space, spanned by $a_{g,f}$, with $g \in G$, $f \in F$, modulo the relations

$$a_{gh,f} = \varepsilon_h(f) a_{g,hfh^{-1}}, \quad \forall h \in H,$$

with the G-grading, given by $|a_{g,f}| = gfg^{-1}$, the G-action $g'(a_{g,f}) = a_{g'g,f}$ and the multiplication

$$a_{g,f} a_{g',f'} = \delta_{g,g'} \gamma\big(f, f'\big) a_{g,ff'}.$$

Theorem 3.5.1. *Any indecomposable separable commutative algebra in $\mathcal{Z}(G)$ has a form $A(H, F, \gamma, \varepsilon)$. Thus indecomposable separable commutative algebras in $\mathcal{Z}(G)$ correspond to quadruples $(H, F, \gamma, \varepsilon)$, where $H \subset G$ is a subgroup, $F \lhd H$ is a normal subgroup, $\gamma \in Z^2(F, k^*)$ is a cocycle and $\varepsilon : H \times F \to k^*$ satisfies the conditions* (8)–(10).

Proof. Follows from Corollary 3.3.5 and Propositions 3.4.2 and 2.3.2. □

Remark 3.5.2. Note that the twist θ_A is always trivial on the algebra $A = A(H, F, \gamma, \varepsilon)$. Indeed,

$$\theta_A^{-1}(a_{g,f}) = \big(gfg^{-1}\big)(a_{g,f}) = a_{gfg^{-1}g,f} = a_{gf,f} = \varepsilon_f(f) a_{g,f}$$

with $\varepsilon_f(f) = \gamma(f, f)\gamma(f, f)^{-1} = 1$ by (10).

Theorem 3.5.3. *The algebra $A(H, F, \gamma, \varepsilon)$ in the category $\mathcal{Z}(G)$ is trivialising iff $F = H$.*

Proof. Follows from Theorems 3.3.2, 3.4.4 and Proposition 2.3.2. □

Note that when $F = H$ the map ε is completely determined by γ. Thus trivialising algebras in $\mathcal{Z}(G)$ correspond to pairs (H, γ), where $H \subset G$ is a subgroup and $\gamma \in Z^2(H, k^*)$ is a 2-cocycle.

Remark 3.5.4. It follows from the theorem that trivialising algebras in $\mathcal{Z}(G) \boxtimes \mathcal{Z}(G) \simeq \mathcal{Z}(G \times G)$ correspond to pairs (U, γ), where $U \subset G \times G$ is a subgroup and $\gamma \in Z^2(U, k^*)$ is a 2-cocycle. This coincides with the parametrisation of module categories obtained in [34], which illustrates the fact (formulated in Section 2.4) that the full centre defines a bijection between equivalence classes of indecomposable module categories over $\mathcal{Z}(G)$ and maximal indecomposable separable commutative algebras in $\mathcal{Z}(G) \boxtimes \mathcal{Z}(G)$.

3.6. Trivialising algebras in products of group-theoretical module categories and equivalences between group-theoretical module categories

In this section we describe the parents of maximal indecomposable commutative separable algebras in $\mathcal{Z}(G) \boxtimes \mathcal{Z}(Q)$ and use this description to analyse braided monoidal equivalences between $\mathcal{Z}(G)$ and $\mathcal{Z}(Q)$.

Let G, Q be finite groups. It is straightforward to see that the functor, defined in Section 2.5,

$$\mathbf{Hom}_{\mathcal{Z}(G)}(1, \ ) : \mathcal{Z}(G \times Q) \simeq \mathcal{Z}(G) \boxtimes \mathcal{Z}(Q) \longrightarrow \mathcal{Z}(Q)$$

sends X into subspace of invariants $(\bigoplus_{q \in Q} X_{(e,q)})^{G \times \{e\}}$. Let $U \subset G \times Q$ be a subgroup and $\gamma \in Z^2(G \times Q, k^*)$ be a normalised 2-cocycle. The pair (U, γ) defines a maximal indecomposable commutative separable algebra $A(U, \gamma)$ in $\mathcal{Z}(G \times Q) \simeq \mathcal{Z}(G) \boxtimes \mathcal{Z}(Q)$.

Theorem 3.6.1. *The parent* $\mathbf{Hom}_{\mathcal{Z}(G)}(1, A) \in \mathcal{Z}(Q)$ *of the maximal indecomposable commutative separable algebra* $A(U, \gamma)$ *in* $\mathcal{Z}(G) \boxtimes \mathcal{Z}(Q)$ *is isomorphic to* $A(pr_2(U), K, \gamma|_K, \varepsilon)$*, where* $pr_2(U) \subset Q$ *is the projection of* $U \subset G \times Q$ *onto the second factor,* K *is the kernel of the homomorphism*

$$\overline{\gamma} : U \cap \big(\{e\} \times Q\big) \longrightarrow U \cap \widehat{\big(G \times \{e\}\big)}, \quad (e, q) \longmapsto \gamma_{(e,q)},$$

and $\varepsilon : pr_2(U) \times K \to k^*$ *is given by*

$$\varepsilon_q(v) = \gamma\big((g, q)|v\big) = \gamma\big((g, q), v\big)\gamma\big((g, q)v(g, q)^{-1}, (g, q)\big), \quad q \in pr_2(U),\ v \in K.$$

Here $U \cap \widehat{(G \times \{e\})}$ *is the character group of* $U \cap (G \times \{e\})$.

Proof. As was noted in Section 2.5, the algebra $B = \mathbf{Hom}_{\mathcal{Z}(G)}(1, A)$ is an indecomposable commutative separable algebra in $\mathcal{Z}(Q)$. Thus, by Theorem 3.5.1, it should have a form $A(H, F, \gamma, \varepsilon)$ for some $F \lhd H \subset Q$. To find H we need to look at the trivial degree component B_e. Since

$$B_e = A_{(e,e)}^{G \times \{e\}} = k(G \times Q/U)^{G \times \{e\}} = k\big(\big(G \times \{e\}\big) \setminus G \times Q/U\big),$$

H can be defined as the stabiliser of (e, e) with respect to the (transitive) Q-action on $(G \times \{e\}) \setminus G \times Q/U$, which coincides with

$$\big\{q \in Q \ \big|\ \exists g \in G\colon\ (g, q) \in U\big\} = pr_2(U).$$

Thus as a Q-algebra $B_e = k(Q/pr_2(U))$. To determine the rest of the defining data for B we need to look at pB, where p is a minimal idempotent of B_e. Let $p \in B_e$ be the minimal idempotent, corresponding to the unit element $e \in Q$. As an element of $A_{(e,e)}$ it has the following decomposition $p = \sum_{g \in G} g(\tilde{p})$, where $\tilde{p}$ is the minimal idempotent in $A_{(e,e)}$ corresponding to (e, e). Hence

$$pB = \Big(\sum_{g \in G} g(\tilde{p})\Big)\Big(\bigoplus_{q \in Q} A_{(e,q)}\Big)^{G \times \{e\}} = \Big(\bigoplus_{q \in Q} \tilde{p} A_{(e,q)}\Big)^{(G \times \{e\}) \cap U}.$$

Now, since $\tilde{p}A = k[U, \gamma]$, we have that $\bigoplus_{q \in Q} \tilde{p} A_{(e,q)} = k[U \cap (\{e\} \times Q), \gamma]$. The conjugation action of $U \cap (G \times \{e\})$ on $U \cap (\{e\} \times Q)$ is trivial, so the only non-triviality comes from γ: for $u \in U \cap (G \times \{e\})$, $v \in U \cap (\{e\} \times Q)$

$$u(e_v) = e_u e_v e_u^{-1} = \gamma(u|v)e_v, \qquad \gamma(u|v) = \gamma(u, v)\gamma(v, u)^{-1}.$$

Note that, restricted to $(U \cap (G \times \{e\})) \times (U \cap (\{e\} \times Q))$, $\gamma(\ |\ )$ is a bi-multiplicative pairing. Hence e_v is an invariant iff $\gamma(\ |v)$ is trivial. Thus

$$pB = k\big[U \cap \big(\{e\} \times Q\big), \gamma\big]^{(G \times \{e\}) \cap U} = k[K, \gamma|_K].$$

Finally, to determine $\varepsilon : pr_2(U) \times K \to k^*$ we need to write the conjugation action of $pr_2(U)$ on $k[K, \gamma|_K]$ in the form $(g, q)(e_v) = \varepsilon_q(v)e_{qvq^{-1}}$ for $q \in pr_2(U)$, $v \in K$. Since

$$(g, q)(e_v) = e_{(g,q)} e_v e_{(g,q)}^{-1} = \gamma\big((g, q), v\big)\gamma\big((g, q)v(g, q)^{-1}, (g, q)\big)e_{qvq^{-1}}$$

we have the description for ε. Note that for $q \in pr_2(U)$, $v \in K$ the value of $\gamma((g, q)|v)$ does not depend on the choice of g. $\square$

Remark 3.6.2. Similarly, the parent $\mathbf{Hom}_{\mathcal{Z}(Q)}(1, A) \in \mathcal{Z}(G)$ of $A(U, U, \gamma) \in \mathcal{Z}(G) \boxtimes \mathcal{Z}(Q)$ is isomorphic to $A(pr_1(U), K, \gamma|_K, \varepsilon)$, where $pr_1(U) \subset G$ is the projection of $U \subset G \times Q$ onto the first factor, K is the kernel of the homomorphism $\overline{\gamma} : U \cap (G \times \{e\}) \to U \cap \widehat{(\{e\} \times Q)}$, induced by $\gamma(\ |\ )$, and $\varepsilon : pr_1(U) \times K \to k^*$ is given by

$$\varepsilon_g(v) = \gamma\big((g,q)|v\big) = \gamma\big((g,q), v\big)\gamma\big((g,q)v(g,q)^{-1}, (g,q)\big), \quad g \in pr_1(U),\ v \in K.$$

Ribbon equivalences of group-theoretic modular categories were recently described in [33]. Here, as a bi-product of our classification of trivialisng algebras, we give an alternative description. We plan to compare two descriptions in a subsequent paper.

Corollary 3.6.3. *An equivalence between $\mathcal{Z}(G)$ and $\mathcal{Z}(Q)$, as ribbon categories, corresponds to a subgroup $U \subset G \times Q$, such that $pr_1(U) = G$, $pr_2(U) = Q$, together with a 2-cocycle $\gamma \in Z^2(U, k^*)$, such that $\gamma(\ |\ )$ induces a non-degenerate pairing*

$$\big(U \cap \big(G \times \{e\}\big)\big) \times \big(U \cap \big(\{e\} \times Q\big)\big) \longrightarrow k^*. \tag{12}$$

Proof. As was mentioned before (Section 2.5) ribbon equivalences between $\mathcal{Z}(G)$ and $\mathcal{Z}(Q)$ correspond to algebras $A(U, U, \gamma)$ in $\mathcal{Z}(G) \boxtimes \mathcal{Z}(Q)$ with trivial parents. By applying Theorem 3.6.1 we get the conditions of the corollary. □

The next auxiliary result, describing subgroups of direct products, will be used to get a different presentation for ribbon equivalences.

Lemma 3.6.4. *Subgroups in $G \times Q$ correspond to diagrams of groups*

$$\begin{tikzcd} G & & P & & Q \\ & M \arrow[ul, hook] \arrow[ur, "i"] & & N \arrow[ul, "j"'] \arrow[ur, hook] & \end{tikzcd} \tag{13}$$

The diagram, corresponding to a subgroup $U \subset G \times Q$ has a form:

$$\begin{tikzcd} G & & P & & Q \\ & pr_1(U) \arrow[ul, hook] \arrow[ur, "i"] & & pr_2(U) \arrow[ul, "j"'] \arrow[ur, hook] & \end{tikzcd} \tag{14}$$

Conversely, the subgroup, corresponding to a diagram (13)*, is*

$$U = M \times_P N = \big\{(g,q) \in M \times N \,\big|\, i(g) = j(q)\big\} \subset G \times Q.$$

Proof. The group P and the surjections in the diagram (14) are defined as follows. First note that, as a subgroup of G, $U \cap (G \times \{e\})$ is a normal subgroup of $pr_1(U)$. This, indeed, follows from the fact that for $(g,q), (f,e) \in U$

$$(g,q)(f,e)(g,q)^{-1} = \big(gfg^{-1}, e\big)$$

lies in U. Similarly, $U \cap (\{e\} \times Q)$ is a normal subgroup of $pr_2(U)$. Moreover, there is an isomorphism of quotient groups

$$pr_1(U)/U \cap \big(G \times \{e\}\big) \longrightarrow pr_2(U)/U \cap \big(\{e\} \times Q\big), \tag{15}$$

given by the assignment on cosets $g(U \cap (G \times \{e\})) \mapsto q(U \cap (\{e\} \times Q))$ each time (g, q) belongs to U. Thus, in the diagram (14), we can set $P = pr_1(U)/U \cap (G \times \{e\})$ with i being the quotient map and j being the composition of the quotient map with the inverse of (15).

The fact that the constructions, described in the lemma, are mutually inverse can be verified directly. □

Remark 3.6.5. With the help of Lemma 3.6.4, the statement of Corollary 3.6.3 can be reformulated as follows. Equivalences between $\mathcal{Z}(G)$ and $\mathcal{Z}(Q)$, as ribbon categories, correspond to diagrams

$$\begin{tikzcd}
 & P & \\
G \arrow[ur] & & Q \arrow[ul] \\
 & S \arrow[ul, hook, "i"] \arrow[ur, hook, "j"'] &
\end{tikzcd}$$

with abelian S, where the inclusions are normal and such that the actions of P on S, induced by the extensions, coincide; together with the coboundary $\gamma \in C^2(U, k^*)$ on $U = G \times_P Q$

$$d(\gamma) = 1,$$

such that $\gamma(\ |\ )$ induces a non-degenerate pairing $i(S) \times j(S) \to k^*$.

Here we defined S to be $U \cap (G \times \{e\})$ with the obvious inclusion i and with j defined by a choice of isomorphism $S \to \hat{S}$ followed by the map $\hat{S} \to U \cap (\{e\} \times Q)$, induced by the pairing (12).

Remark 3.6.6. It is well known that the category $\mathcal{Z}(G)$ is equivalent to the monoidal centre $\mathcal{Z}(\mathcal{R}ep(G))$ of the category $\mathcal{R}ep(G)$ of (finite-dimensional) representations of G. In particular, a monoidal equivalence $\mathcal{R}ep(G) \to \mathcal{R}ep(Q)$ gives rise to an equivalence of ribbon categories $\mathcal{Z}(G) \to \mathcal{Z}(Q)$. Monoidal equivalences between categories of representations of finite groups were described in [8,9] (see also [15]). According to [9], monoidal equivalences $\mathcal{R}ep(G) \to \mathcal{R}ep(Q)$ correspond to the following data: a diagram of groups

$$\begin{tikzcd}
 & P & \\
G \arrow[ur] & & Q \arrow[ul] \\
 & S \arrow[ul, hook, "i"] \arrow[ur, hook, "j"'] &
\end{tikzcd}$$

with abelian S (such that the actions of P on S, induced by the extensions, coincide); together with a P-invariant cohomology class $\gamma \in H^2(S, k^*)^P$ and a homomorphism

$$G \times_P Q \longrightarrow N(P, S, \gamma) = \big\{(p, \pi) \in P \times C^1\big(S, k^*\big) \mid p(\overline{\gamma})\overline{\gamma}^{-1} = d(\pi)\big\}$$

(here $\overline{\gamma} \in Z^2(S, k^*)$ is a representative of the class γ) fitting into the commutative diagram with exact rows and columns:

$$\begin{array}{ccccc} & & P & = & P \\ & & \uparrow & & \uparrow \\ S & \to & G \times_P Q & \to & N(P, S, \gamma) \\ \| & & \uparrow & & \uparrow \\ S & \to & S \times S & \to & \hat{S}, \end{array}$$

where $S \to S \times S$ is the diagonal embedding and $S \times S \to \hat{S}$ is a skew-diagonal projection given by the pairing $\overline{\gamma}(\ |\ ) : S \to \hat{S}$.

The group-theoretical data of the corresponding ribbon equivalence $\mathcal{Z}(G) \to \mathcal{Z}(Q)$ is given by the 2-class $\tilde{\gamma} \in H^2(G \times_P Q, k^*)$, which can be constructed using the short exact sequence $S \to G \times_P Q \to N(P, S, \gamma)$. The details will appear elsewhere.

Remark 3.6.7. Since the category $\mathcal{Z}(G)$ is isomorphic to the monoidal centre $\mathcal{Z}(\mathcal{C}(G))$, any monoidal equivalence $\mathcal{C}(G) \to \mathcal{C}(Q)$ gives a ribbon equivalence $\mathcal{Z}(G) \to \mathcal{Z}(Q)$. Monoidal equivalences $\mathcal{C}(G) \to \mathcal{C}(Q)$ correspond to isomorphisms $\phi : G \to Q$ together with a cocycle $\gamma \in C^2(G, k^*)$, $d(\gamma) = 1$. It is straightforward to see, that the corresponding subgroup $U \subset G \times_P Q$ is the graph of ϕ, $U = \{(g, \phi(g)) \mid g \in G\}$ and that the coboundary $\tilde{\gamma} \in C^2(U, k^*)$ is given by $\tilde{\gamma}((f, \phi(f)), (g, \phi(g))) = \gamma(f, g)$.

4. Modular invariants for group-theoretical modular data

In this section we calculate characters (and thus classes in the Grothendieck ring) of indecomposable separable commutative algebras in (untwisted) group-theoretic modular categories. As was explained in Section 2.6, characters of trivialising algebras are invariant with respect to the modular group action on the Grothendieck ring. Thus here we give the description of physical modular invariants for (untwisted) group-theoretical modular data.

4.1. Characters of commutative algebras and their local modules

Proposition 4.1.1. *The map $K_0(\mathcal{Z}(H)) \cong K_0(\mathcal{Z}(G)_{k(H)}) \to K_0(\mathcal{Z}(G))$, induced by the transfer $\mathcal{Z}(H) \cong \mathcal{Z}(G)_{k(H)} \to \mathcal{Z}(G)$, sends a character $\chi \in K_0(\mathcal{Z}(H))$ into*

$$\overline{\chi}(f, g) = \frac{1}{|H|} \sum_{x \in G,\, xfx^{-1}, xgx^{-1} \in H} \chi\left(xfx^{-1}, xgx^{-1}\right).$$

Proof. The proof is completely analogous to the proof of the induction formula in character theory (see for example [23]). By the definition, the character $\overline{\chi}(f, g)$ is the trace $tr_{(k(G) \otimes_H U)_f}(g)$ of g acting on the graded component $(k(G) \otimes_H U)_f$. By the definition of the transfer, the graded component $(k(G) \otimes_H U)_f$ coincides with $\bigoplus_x p_x \otimes U_{x^{-1}fx}$, where the sum is taken over cosets $\{x\colon x^{-1}fx \in H\}/H$ (with respect to the H-action on the set $\{x\colon x^{-1}fx \in H\}$ by left multiplications). So that

$$tr_{(k(G) \otimes_H U)_f}(g) = \sum_x tr_{p_x \otimes U_{x^{-1}fx}}(g).$$

Note that g preserves $p_x \otimes U_{x^{-1}fx}$ iff $x^{-1}gx$ is in H:

$$g(p_x \otimes U_{x^{-1}fx}) = p_{gx} \otimes U_{x^{-1}fx} = p_{xx^{-1}gx} \otimes U_{x^{-1}fx} = p_x \otimes x^{-1}gx(U_{x^{-1}fx}),$$

and that, in this case, the restriction of g to $p_x \otimes U_{x^{-1}fx}$ coincides with $p_x \otimes x^{-1}gx$. Thus

$$tr_{p_x \otimes U_{x^{-1}fx}}(g) = \chi\left(xfx^{-1}, xgx^{-1}\right). \qquad \square$$

Corollary 4.1.2. *The character of a trivialising algebra $A(H,\gamma)$ has the form*:

$$\chi_{A(H,\gamma)}(f,g) = \frac{1}{|H|} \sum_{x \in G,\, xfx^{-1}, xgx^{-1} \in H} \gamma\left(xfx^{-1} \middle| xgx^{-1}\right). \tag{16}$$

Proof. By the definition the algebra $A(H,\gamma)$ is the image of the algebra $k[H,\gamma]$ under the transfer $\mathcal{Z}(H) \cong \mathcal{Z}(G)_{k(H)} \to \mathcal{Z}(G)$. Thus the corollary follows from Proposition 4.1.1 and the fact that the character of $k[F,\gamma]$ is $\chi(x,y) = \gamma(x|y)$, which can be checked directly. Indeed, x-graded component of $k[F,\gamma]$ is spanned (over k) by e_x, with the action of y on it

$$y(e_x) = \varepsilon_x(y) e_{yxy^{-1}} = \frac{\gamma(x,y)}{\gamma(xyx^{-1},x)} e_x.$$

Thus, for commuting x, y, we have that

$$tr_{k[F,\gamma]_x}(y) = \frac{\gamma(x,y)}{\gamma(xyx^{-1},x)} = \gamma(x|y). \qquad \square$$

We finish this section with examples of trivialising algebras in $\mathcal{Z}(G)$ with the same character (the same class in $K_0(\mathcal{Z}(G))$). By the formula (16), to construct such example it is enough to have a finite group H with a non-trivial 2-class $\gamma \in H^2(H,k^*)$, such that γ_h is a trivial character of $C_H(h)$ for any $h \in H$. We will use the well-known correspondence between second cohomology classes and central extensions (see [5], for example) to give a group-theoretic conditions, which guarantee existence of such class.

Lemma 4.1.3. *Let $\tilde{H}$ be a finite group with a central cyclic subgroup $Z \subset Z(\tilde{H})$, which does not contain commutators, and such that the extension $Z \cap [\tilde{H},\tilde{H}] \to [\tilde{H},\tilde{H}] \to [\tilde{H},\tilde{H}]/Z \cap [\tilde{H},\tilde{H}]$ is non-trivial. Let k be an algebraically closed field of characteristic zero. Then the class $\gamma \in H^2(H,k^*)$, extended from the extension class $\overline{\gamma} \in H^2(H,Z)$ with respect to an embedding $Z \to k^*$, is non-trivial and such that γ_h is a trivial character of $C_H(h)$ for any $h \in H$. Here $H = \tilde{H}/Z$.*

Proof. We begin by showing that the absence of commutators in Z implies that γ_h is a trivial character of $C_H(h)$ for any $h \in H$. Indeed, it is straightforward to see that for any commuting $x, h \in H$ the commutator $[\tilde{h},\tilde{x}]$ of (any of) their preimages in $\tilde{H}$ lies in Z and coincides with $\overline{\gamma}_h(x)$. So if Z does not contain commutators, then $\overline{\gamma}_h$ is trivial for any $h \in H$, which implies the triviality of γ_h.

Next we show non-triviality of the extended class $\gamma \in H^2(H,k^*)$. We identify Z with the n-torsion subgroup of k^* ($n = |Z|$). It follows from the long exact sequence, corresponding to the coefficient

extension $Z \to k^* \xrightarrow{\times n} k^*$, that the kernel of the coefficient extension $H^2(H,Z) \to H^2(H,k^*)$ coincides with the image of the connecting map $\partial : H^1(H,k^*) \to H^2(H,Z)$, which fits into a diagram

$$\begin{tikzcd}
 & & H^2([\tilde{H},\tilde{H}],Z) & & \\
\arrow[r] & H^1(H,k^*) \arrow[r, "\partial"] & H^2(H,Z) \arrow[u] \arrow[r] & H^2(H,k^*) \arrow[r] & \\
\arrow[r] & H^1(H^{ab},k^*) \arrow[u] \arrow[r] & H^2(H^{ab},Z) \arrow[u] \arrow[r] & &
\end{tikzcd}$$

Here $H^{ab} = H/[H,H]$. The commutative square implies that the image of ∂ coincides with the image of $H^2(H^{ab},Z) \to H^2(H,Z)$. So if the image of $\gamma \in H^2(H,Z)$ in $H^2([\tilde{H},\tilde{H}],Z)$ is non-trivial, then γ cannot be in the image of $H^2(H^{ab},Z) \to H^2(H,Z)$ and thus cannot be killed by $H^2(H,Z) \to H^2(H,k^*)$. $\square$

Example 4.1.4. Let $p \geqslant 5$ be a prime and $\tilde{H}$ be the free meta-abelian group of period p^2, generated by x_1, x_2, x_3, x_4. Let Z be the central subgroup, generated by $([x_1,x_2][x_3,x_4])^p$. Note that $V = \tilde{H}^{ab}$ is the free abelian group of period p^2, with four generators e_1, e_2, e_3, e_4, and $[\tilde{H},\tilde{H}]$ can be identified with the exterior square $\Lambda^2 V$. In this presentation the commutator pairing corresponds to the wedge product $V \times V \to \Lambda^2 V$ so the set of commutators in $[\tilde{H},\tilde{H}]$ corresponds to the Plücker quadric $\{x \in \Lambda^2 V,\ x \wedge x = 0\} \subset \Lambda^2 V$. The element $v = p(e_1 \wedge e_2 + e_3 \wedge e_4)$ is not on the quadric, which shows that Z does not contain (non-trivial) commutators. The inclusion $\langle v \rangle \to \Lambda^2 V$ does not split, which implies that the extension $Z \cap [\tilde{H},\tilde{H}] \to [\tilde{H},\tilde{H}] \to [\tilde{H},\tilde{H}]/Z \cap [\tilde{H},\tilde{H}]$ is non-trivial.

Applying the lemma we get a desired example.

4.2. S_3 modular data and modular invariants

Here we describe modular invariants for modular data of S_3 (the symmetric group acting on 3 elements). Below A_3 will denote the alternating group (the group of even permutations) and C_2 will denote (the conjugacy class of) a subgroup, generated by a transposition.

Recall that $H^2(H,k^*)$ is trivial for any subgroup H of S_3 (including S_3 itself). The classes of simple objects are labelled by

$$(e,\xi_0), \qquad (e,\xi_1), \qquad (e,\xi_2), \qquad \big((123),\pi_0\big), \qquad \big((123),\pi_1\big),$$
$$\big((123),\pi_0\big), \qquad \big((12),\psi_0\big), \qquad \big((12),\psi_1\big).$$

Here $\xi_i \in Irr(S_3)$, $\pi_i \in Irr(C_3)$, $\psi_i \in Irr(C_2)$.

The S- and T-matrices have the following form:

$$S = \frac{1}{6}\begin{pmatrix} 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 \\ 1 & 1 & 2 & 2 & 2 & 2 & -3 & -3 \\ 2 & 2 & 4 & -2 & -2 & -2 & 0 & 0 \\ 2 & 2 & -2 & 4 & -2 & -2 & 0 & 0 \\ 2 & 2 & -2 & -2 & -2 & 4 & 0 & 0 \\ 2 & 2 & -2 & -2 & 4 & -2 & 0 & 0 \\ 3 & -3 & 0 & 0 & 0 & 0 & 3 & -3 \\ 3 & -3 & 0 & 0 & 0 & 0 & -3 & 3 \end{pmatrix},$$

$$T = diag\big(1,1,1,1,\omega,\omega^{-1},1,-1\big), \quad 1+\omega+\omega^2 = 0.$$

Table 1

$H \rhd F$	H/F	$\mathcal{Z}(S_3)^{loc}_{A(H,F)}$
$S_3 \rhd S_3$	$\{e\}$	$\chi_0 + \chi_3 + \chi_6$
$A_3 \rhd A_3$	$\{e\}$	$\chi_0 + \chi_1 + 2\chi_3$
$C_2 \rhd C_2$	$\{e\}$	$\chi_0 + \chi_2 + \chi_6$
$\{e\} \rhd \{e\}$	$\{e\}$	$\chi_0 + \chi_1 + 2\chi_2$
$S_3 \rhd A_3$	C_2	$\chi_0 + \chi_3,\ \chi_1 + \chi_3,\ \chi_6,\ \chi_7$
$C_2 \rhd \{e\}$	C_2	$\chi_0 + \chi_2,\ \chi_1 + \chi_2,\ \chi_6,\ \chi_7$
$A_3 \rhd \{e\}$	A_3	$\chi_0 + \chi_1,\ \chi_2,\ \chi_3,\ \chi_4,\ \chi_5$
$S_3 \rhd \{e\}$	S_3	$\chi_0,\ \chi_1,\ \chi_2,\ \chi_3,\ \chi_4,\ \chi_5,\ \chi_6,\ \chi_7$

In Table 1 we list all indecomposable commutative separable algebras in $\mathcal{Z}(S_3)$ together with the characters of their simple local modules (the first character is the character of the algebra itself):

Note, that in the case $A_3 \rhd \{e\}$, for each $i = 2, \dots, 5$ there are two different simple local modules with the character χ_i.

According to Lemma 3.6.4, there are 22 conjugacy classes of subgroups in $S_3 \times S_3$. Sixteen of them have a form $A \times B$ for $A, B \subset S_3$ and correspond to diagrams (we omit the embeddings into S_3)

$$\begin{tikzcd} & \{e\} & \\ A \arrow[ur] & & B \arrow[ul] \end{tikzcd};$$

four have the form

$$\delta(C_2), \qquad \delta(C_2)\big(\{e\} \times A_3\big), \qquad \delta(C_2)\big(A_3 \times \{e\}\big), \qquad \delta(C_2)(A_3 \times A_3)$$

and correspond to diagrams

$$\begin{tikzcd} & C_2 & \\ A \arrow[ur] & & B \arrow[ul] \end{tikzcd}$$

with $A, B = C_2$ or S_3; and two remaining are $\delta(A_3)$, $\delta(S_3)$, corresponding to the diagrams:

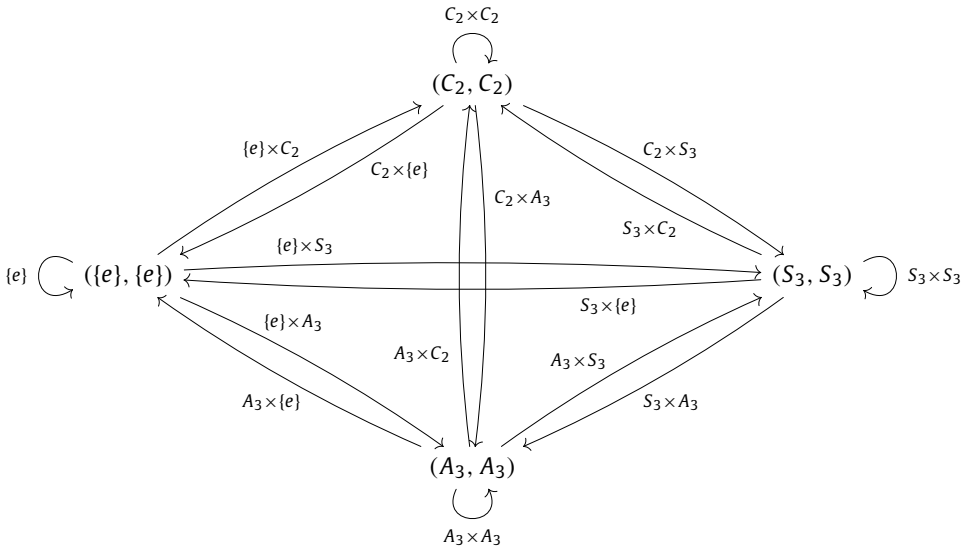
$$\begin{tikzcd} & A_3 & \\ A_3 \arrow[ur] & & A_3, \arrow[ul] \end{tikzcd} \qquad \begin{tikzcd} & S_3 & \\ S_3 \arrow[ur] & & S_3 \arrow[ul] \end{tikzcd}$$

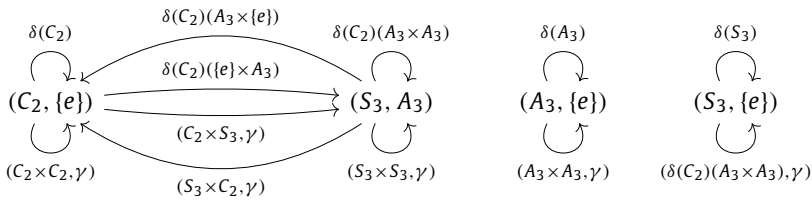
respectively. Here $\delta : G \to G \times G$ stand for the diagonal embedding. Only six of them:

$$C_2 \times C_2, \qquad S_3 \times S_3, \qquad C_2 \times S_3, \qquad S_3 \times C_2, \qquad A_3 \times A_3, \qquad \delta(C_2)(A_3 \times A_3)$$

have non-trivial cohomology $H^2(U, k^*)$. For the first four $H^2(U, k^*)$ is cyclic of order 2 and for two remaining it is cyclic of order 3. In the last two cases the conjugation action of the normaliser in $S_3 \times S_3$ permutes two non-trivial cohomology classes. Thus, in all cases, there is just one (up to conjugation) non-trivial cohomology class, which, somewhat loosely, will be denoted γ.

Maximal commutative algebras in $\mathcal{Z}(S_3) \boxtimes \mathcal{Z}(S_3) \simeq \mathcal{Z}(S_3 \times S_3)$ are depicted as edges of the following four graphs:





Vertices are labelled by the (conjugacy classes of) pairs of subgroups $F \lhd H \subset S_3$, which correspond to indecomposable commutative separable algebras in $\mathcal{Z}(S_3)$; edges are labelled by (conjugacy classes of) (H, γ), where $H \subset S_3 \times S_3$ and $\gamma \in H^2(H, k^*)$ is a cohomology class (omitted if trivial), which correspond to maximal indecomposable commutative separable algebras in $\mathcal{Z}(S_3 \times S_3)$. An edge goes from A to B if for the corresponding algebra $C \in \mathcal{Z}(S_3 \times S_3)$

$$\mathbf{Hom}_{1 \boxtimes \mathcal{Z}(S_3)}(1, C) = A, \qquad \mathbf{Hom}_{\mathcal{Z}(S_3) \boxtimes 1}(1, C) = B.$$

Table 2 contains characters of maximal indecomposable commutative separable algebras in $\mathcal{Z}(S_3 \times S_3)$, written in the (traditional) form of a partition function, ordered by the rank of the corresponding modular invariant.

5. Concluding remarks

It is known that modular categories give rise to 3-dimensional topological field theories (see [39,4]). In particular, group-theoretical modular categories correspond to Dijkgraaf–Witten topological field theories [14,17]. Equivalent modular categories give rise to equivalent topological field theories and, in particular, to the same invariants of closed 3-manifolds. It follows from the results of [14,17] that the invariant of a 3-manifold M, defined by the modular category $\mathcal{Z}(G)$, has the form

$$Z_G(M) = \frac{|\mathbf{Hom}(\pi_1(M), G)|}{|G|}. \tag{17}$$

Table 2

$A(H,\gamma)$	Z
$A(S_3 \times S_3)$	$\lvert\chi_0+\chi_3+\chi_6\rvert^2$
$A(S_3 \times A_3)$	$(\chi_0+\chi_3+\chi_6)(\chi_0+\chi_1+2\chi_3)^*$
$A(S_3 \times C_2)$	$(\chi_0+\chi_3+\chi_6)(\chi_0+\chi_2+\chi_6)^*$
$A(S_3 \times \{e\})$	$(\chi_0+\chi_3+\chi_6)(\chi_0+\chi_1+2\chi_2)^*$
$A(A_3 \times A_3)$	$\lvert\chi_0+\chi_1+2\chi_3\rvert^2$
$A(A_3 \times S_3)$	$(\chi_0+\chi_1+2\chi_3)(\chi_0+\chi_3+\chi_6)^*$
$A(A_3 \times C_2)$	$(\chi_0+\chi_1+2\chi_3)(\chi_0+\chi_2+\chi_6)^*$
$A(A_3 \times \{e\})$	$(\chi_0+\chi_1+2\chi_3)(\chi_0+\chi_1+2\chi_2)^*$
$A(C_2 \times C_2)$	$\lvert\chi_0+\chi_2+\chi_6\rvert^2$
$A(C_2 \times S_3)$	$(\chi_0+\chi_2+\chi_6)(\chi_0+\chi_3+\chi_6)^*$
$A(C_2 \times A_3)$	$(\chi_0+\chi_2+\chi_6)(\chi_0+\chi_1+2\chi_3)^*$
$A(C_2 \times \{e\})$	$(\chi_0+\chi_2+\chi_6)(\chi_0+\chi_1+2\chi_2)^*$
$A(\{e\} \times \{e\})$	$\lvert\chi_0+\chi_1+2\chi_2\rvert^2$
$A(\{e\} \times S_3)$	$(\chi_0+\chi_1+2\chi_2)(\chi_0+\chi_3+\chi_6)^*$
$A(\{e\} \times A_3)$	$(\chi_0+\chi_1+2\chi_2)(\chi_0+\chi_1+2\chi_3)^*$
$A(\{e\} \times C_2)$	$(\chi_0+\chi_1+2\chi_2)(\chi_0+\chi_2+\chi_6)^*$
$A(\delta(C_2)(A_3 \times A_3))$	$\lvert\chi_0+\chi_3\rvert^2+\lvert\chi_1+\chi_3\rvert^2+\lvert\chi_6\rvert^2+\lvert\chi_7\rvert^2$
$A(S_3 \times S_3,\gamma)$	$\lvert\chi_0+\chi_3\rvert^2+(\chi_1+\chi_3)\chi_6^*+\chi_6(\chi_1+\chi_3)^*+\lvert\chi_7\rvert^2$
$A(\delta(C_2)(A_3 \times \{e\}))$	$(\chi_0+\chi_3)(\chi_0+\chi_2)^*+(\chi_1+\chi_3)(\chi_1+\chi_2)^*+\lvert\chi_6\rvert^2+\lvert\chi_7\rvert^2$
$A(S_3 \times C_2,\gamma)$	$(\chi_0+\chi_3)(\chi_0+\chi_2)^*+(\chi_1+\chi_3)\chi_6^*+\chi_6(\chi_1+\chi_2)^*+\lvert\chi_7\rvert^2$
$A(\delta(C_2))$	$\lvert\chi_0+\chi_2\rvert^2+\lvert\chi_1+\chi_2\rvert^2+\lvert\chi_6\rvert^2+\lvert\chi_7\rvert^2$
$A(S_3 \times S_3,\gamma)$	$\lvert\chi_0+\chi_2\rvert^2+(\chi_1+\chi_2)\chi_6^*+\chi_6(\chi_1+\chi_2)^*+\lvert\chi_7\rvert^2$
$A(\delta(C_2)(\{e\} \times A_3))$	$(\chi_0+\chi_2)(\chi_0+\chi_3)^*+(\chi_1+\chi_2)(\chi_1+\chi_3)^*+\lvert\chi_6\rvert^2+\lvert\chi_7\rvert^2$
$A(C_2 \times S_3,\gamma)$	$(\chi_0+\chi_2)(\chi_0+\chi_3)^*+(\chi_1+\chi_2)\chi_6^*+\chi_6(\chi_1+\chi_3)^*+\lvert\chi_7\rvert^2$
$A(\delta(A_3))$	$\lvert\chi_0+\chi_1\rvert^2+2\lvert\chi_2\rvert^2+2\lvert\chi_3\rvert^2+2\lvert\chi_4\rvert^2+2\lvert\chi_5\rvert^2$
$A(A_3 \times A_3,\gamma)$	$\lvert\chi_0+\chi_1\rvert^2+2\chi_2\chi_3^*+2\chi_3\chi_2^*+\lvert\chi_4\rvert^2+\lvert\chi_5\rvert^2$
$A(\delta(S_3))$	$\lvert\chi_0\rvert^2+\lvert\chi_1\rvert^2+\lvert\chi_2\rvert^2+\lvert\chi_3\rvert^2+\lvert\chi_4\rvert^2+\lvert\chi_5\rvert^2+\lvert\chi_6\rvert^2+\lvert\chi_7\rvert^2$
$A(\delta(C_2)(A_3 \times A_3),\gamma)$	$\lvert\chi_0\rvert^2+\lvert\chi_1\rvert^2+\chi_2\chi_3^*+\chi_3\chi_2^*+\lvert\chi_4\rvert^2+\lvert\chi_5\rvert^2+\lvert\chi_6\rvert^2+\lvert\chi_7\rvert^2$

It follows from Corollary 3.6.3 that for G and Q, satisfying the conditions of the corollary, the invariants coincide

$$Z_G(M) = Z_Q(M)$$

for all closed 3-manifolds M. In particular, the number of homomorphisms $\pi_1(M) \to G$ is equal to the number of homomorphisms $\pi_1(M) \to Q$, for G and Q satisfying the conditions of Corollary 3.6.3. In other words, fundamental groups of 3-manifolds *do not detect the difference* between such G and Q.

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References

[1] D. Altschuler, A. Coste, Quasi-quantum groups, knots, three-manifolds, and topological field theory, Comm. Math. Phys. 150 (1) (1992) 83–107.
[2] P. Bántay, Orbifolds, Hopf algebras, and the moonshine, Lett. Math. Phys. 22 (3) (1991) 187–194.
[3] P. Bántay, Algebraic aspects of orbifold models, arXiv:hep-th/9303009.
[4] B. Bakalov, A. Kirillov Jr., Lectures on Tensor Categories and Modular Functors, Univ. Lecture Ser., vol. 21, American Mathematical Society, Providence, RI, 2001, 221 pp.
[5] K. Brown, Cohomology of Groups, Grad. Texts in Math., vol. 87, Springer-Verlag, New York, 1982.

[6] A. Bruguières, A. Verelizier, The double of a Hopf monad, http://arxiv.org/abs/0812.2443.
[7] A. Coste, T. Gannon, P. Ruelle, Finite group modular data, Nuclear Phys. B 581 (3) (2000) 679–717.
[8] A. Davydov, Finite groups with the same character tables, Drinfel'd algebras and Galois algebras, in: Algebra, Moscow, 1998, de Gruyter, Berlin, 2000, pp. 99–111, arXiv:math/0001119.
[9] A. Davydov, Galois algebras and monoidal functors between categories of representations of finite groups, J. Algebra 244 (1) (2001) 273–301.
[10] A. Davydov, Centre of an algebra, arXiv:0908.1250.
[11] A. Davydov, M. Müger, D. Nikshych, V. Ostrik, in preparation.
[12] R. Dijkgraaf, V. Pasquier, P. Roche, Quasi Hopf algebras, group cohomology and orbifold models, Nuclear Phys. B Proc. Suppl. 18 (1990) 60–72 (1991).
[13] R. Dijkgraaf, C. Vafa, E. Verlinde, H. Verlinde, The operator algebra of orbifold models, Comm. Math. Phys. 123 (3) (1989) 485–526.
[14] R. Dijkgraaf, E. Witten, Topological gauge theories and group cohomology, Comm. Math. Phys. 129 (2) (1990) 393–429.
[15] P. Etingof, S. Gelaki, Isocategorical groups, Int. Math. Res. Not. IMRN 2 (2001) 59–76.
[16] D. Evans, P. Pinto, Subfactor realisation of modular invariants, Comm. Math. Phys. 237 (2003) 309–363.
[17] D. Freed, Higher algebraic structures and quantization, Comm. Math. Phys. 159 (2) (1994) 343–398.
[18] J. Fröhlich, J. Fuchs, I. Runkel, C. Schweigert, Correspondences of ribbon categories, Adv. Math. 199 (1) (2006) 192–329.
[19] J. Fröhlich, J. Fuchs, I. Runkel, C. Schweigert, TFT construction of RCFT correlators. V: Proof of modular invariance and factorisation, Theory Appl. Categ. 16 (2006) 342–433, hep-th/0503194.
[20] Y.-Z. Huang, Vertex operator algebras, fusion rules and modular transformations, in: Contemp. Math., vol. 391, Amer. Math. Soc., 2005, pp. 135–148.
[21] Y.-Z. Huang, L. Kong, Open-string vertex algebras, tensor categories and operads, Comm. Math. Phys. 250 (3) (2004) 433–471.
[22] Y.-Z. Huang, L. Kong, Full field algebras, Comm. Math. Phys. 272 (2) (2007) 345–396.
[23] M. Isaacs, Character Theory of Finite Groups, Academic Press, New York, 1976, 303 pp.
[24] A. Joyal, R. Street, The geometry of tensor calculus. I, Adv. Math. 88 (1) (1991) 55–112.
[25] A. Joyal, R. Street, Braided tensor categories, Adv. Math. 102 (1) (1993) 20–78.
[26] C. Kassel, Quantum Groups, Grad. Texts in Math., vol. 155, Springer-Verlag, New York, 1995.
[27] A. Kirillov Jr., Modular categories and orbifold models, Comm. Math. Phys. 229 (2) (2002) 309–335.
[28] L. Kong, I. Runkel, Morita classes of algebras in modular tensor categories, Adv. Math. 219 (2008) 1548–1576.
[29] A. Kirillov Jr., V. Ostrik, On a q-analogue of the McKay correspondence and the ADE classification of sl_2 conformal field theories, Adv. Math. 171 (2) (2002) 183–227.
[30] V. Lyubashenko, Modular transformations for tensor categories, J. Pure Appl. Algebra 98 (3) (1995) 279–327.
[31] M. Müger, From subfactors to categories and topology. II. The quantum double of tensor categories and subfactors, J. Pure Appl. Algebra 180 (1-2) (2003) 159–219.
[32] G. Moore, N. Seiberg, Polynomial equations for rational conformal field theories, Phys. Lett. B 212 (4) (1988) 451–460.
[33] D. Naidu, D. Nikshych, Lagrangian subcategories and braided tensor equivalences of twisted quantum doubles of finite groups, Comm. Math. Phys. 279 (3) (2008) 845–872.
[34] V. Ostrik, Module categories, weak Hopf algebras and modular invariants, Transform. Groups 8 (2) (2003) 177–206.
[35] V. Ostrik, Module categories over the Drinfeld double of a finite group, Int. Math. Res. Not. IMRN 27 (2003) 1507–1520.
[36] B. Pareigis, On braiding and dyslexia, J. Algebra 171 (2) (1995) 413–425.
[37] R. Rosebrugh, N. Sabadini, R. Walters, Generic commutative separable algebras and cospans of graphs, Theory Appl. Categ. 15 (6) (2005/06) 164–177.
[38] C. Schweigert, J. Fuchs, I. Runkel, Categorification and correlation functions in conformal field theory, in: International Congress of Mathematicians, vol. III, Eur. Math. Soc., Zürich, 2006, pp. 443–458.
[39] V. Turaev, Quantum Invariants of Knots and 3-Manifolds, de Gruyter Stud. Math., vol. 18, Walter de Gruyter, Berlin, 1994, 588 pp.
[40] V. Turaev, Homotopy field theory in dimension 2 and group-algebras, arXiv:math/9910010.