An infinite sequence of graphs \( \{G_n\}_{n \geq 0} \) is called recursive if the Tutte polynomials \( T(G_n; x, y) \) satisfy a linear recurrence relation whose coefficients are polynomials in \( x \) and \( y \). In this paper we introduce a general method based on transfer matrices for proving that a family is recursive that covers all examples known to us. As an application we show that, for fixed \( s \), the sequence of complete bipartite graphs \( \{K_{s,n}\} \) is recursive and satisfies a linear recurrence whose degree is the number of partitions of the integer \( s \).

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1. Introduction

In [1] the authors call a family \( \{G_n\}_{n \geq 0} \) of graphs recursive if their Tutte polynomials satisfy a linear recurrence relation with polynomial coefficients, that is,

\[
T(G_{n+r}; x, y) + p_1(x, y)T(G_{n+r-1}; x, y) + \cdots + p_r(x, y)T(G_n; x, y) = 0,
\]

where the \( p_i \) are polynomials in \( x, y \) independent of \( n \). This condition is equivalent to the fact that the ordinary generating function

\[
\sum_{n \geq 0} T(G_n; x, y)z^n
\]
is a rational function in $x$, $y$, and $z$. They show, using the contraction-deletion rule, that several families of graphs are recursive, like cycles, ladders and wheels.

All these families have in common the fact that they can be constructed from an initial graph by the repeated application of a fixed graph operation. For instance, the ladder $L_n$ shown in Fig. 1 is obtained from $L_{n-1}$ by adding two new vertices and three new edges (shown with thick lines in the figure).

The main goal of this paper is to show that any family that can be constructed recursively in this way is in fact recursive. Our method is based on transfer matrices, and is an extension of the one introduced in [3] for computing the Tutte polynomial of a square lattice. In principle, the corresponding linear recurrence can be found explicitly, although the computations usually involve very large matrices. An important novelty in our approach is that we can also delete edges, an operation that, as will be seen, corresponds algebraically to multiply by the inverse of a certain transfer matrix. The deletion of edges allows us to include in this framework families with “cyclic boundary conditions,” like toroidal lattices.

In Section 2, we define the concept of a recursively constructible family of graphs, which formalizes the idea of a family that can be built in a “regular way” from an initial graph. Using transfer matrices we show that every recursively constructible family of graphs is recursive. We also show how several simple operations on graphs give rise to new recursively constructible families of graphs. Our results cover all examples of recursive families known to us.

In Section 3, we apply the previous machinery to the case of complete bipartite graphs. We prove that, if $s$ is a positive integer, the family $\{K_{s,n}\}_{n\geq 0}$ is recursive of degree $p(s)$, where $p(s)$ is the number of partitions of the integer $s$. Moreover, from the particular form of the transfer matrix in this case we deduce that

$$T(K_{s,n}; x, y) = \sum_{\lambda \vdash s} A_\lambda(x, y)L_\lambda(x, y)^n,$$

where the $A_\lambda$ are certain rational functions, the $L_\lambda$ are linear polynomials given by an explicit formula, and the sum runs over all the partitions $\lambda$ of $s$. This result generalizes to families of the form $\{G + N_n\}$, where $G$ is a fixed graph and $N_n$ is the null graph on $n$ vertices (complete bipartite graphs correspond to the case $G = N_s$).

In the last section we conjecture that every recursive family of graphs is recursively constructible, provided we take into account the fact that there exist nonisomorphic graphs having the same Tutte polynomial. We prove the conjecture for families of recursiveness one.
Given a graph $G = (V, E)$, its rank is equal to $r(G) = |V| - k(G)$, where $k(G)$ is the number of components of $G$. The Tutte polynomial is defined as

$$T(G; x, y) = \sum_{A \subseteq E} (x - 1)^{r(E) - r(A)} (y - 1)^{|A| - r(A)},$$

where $r(A)$ is the rank of the spanning subgraph $(V, A)$; see [2,7] for background information on the Tutte polynomial. For our needs, it is often more convenient to work with the rank polynomial

$$R(G; x, y) = T(G; x + 1, y + 1) = \sum_{A \subseteq E} x^{r(E) - r(A)} y^{|A| - r(A)}.$$

2. Recursively constructible families

We say that a sequence of graphs $\{G_n\}_{n \geq 0}$ is a recursively constructible family of graphs if it can be built from a given initial graph by means of a repeated fixed succession of elementary operations involving addition of vertices and edges, and deletion of edges.

Let us formalize this concept. Given a graph $G$ and $U \subseteq V(G)$, let $G[U]$ be the subgraph induced by $U$, and let $N_G(U)$ be the set of vertices in $V(G)$ adjacent to some vertex in $U$. The symbol $\cup$ denotes disjoint union.

Then we require the existence of a positive integer $r$, and a (labeled) graph $M$, such that:

(a) $V(G_0) = W_0$, $E(G_0) = E_0$.
(b) $V(G_n) = V(G_{n-1}) \cup W_n$.
(c) $N_{G_n}(W_n) \subseteq W_0 \cup (\bigcup_{i=0}^{n-1} W_{n-i})$ for $n > r$.
(d) $E(G_n) = (E(G_{n-1}) \setminus S) \cup E_n$, where $S \subseteq \bigcup_{i=1}^{r} E_{n-i}$.
(e) The graph $G_n[W_0 \cup (\bigcup_{i=0}^{n-1} W_{n-i})]$ is equal to $M$ for $n > r$. In particular, $G_n[W_n]$ is always the same graph.

When we say ‘equal’ in condition (e) we mean the following. The vertices in $G_n$ are labeled, the first labels correspond to the initial vertices $W_0$, next labels to $W_1$, and so on. And within a given $W_n$, the labels are also ordered, so that $G_n[W_n]$ is always the same labeled graph. Also, the set $S$ of edges that can be removed in condition (d) is always the same, a fact captured again by condition (e). Since the definition is already complicated enough, we have refrained from giving explicit labels to the vertices.

The elementary operations which compose the step from $G_{n-1}$ to $G_n$ are then of the following three kinds:

1. Addition of a new set $W_n$ of vertices adjacent to vertices only in $W_0 \cup (\bigcup_{i=0}^{n-1} W_{n-i})$.
2. Addition of a fixed new set $E_n$ of edges incident only to vertices in $W_0 \cup (\bigcup_{i=0}^{n-1} W_{n-i})$.
3. Deletion of a fixed set $S$ of edges with one end in $W_0$ and the other end in $\bigcup_{i=1}^{r} W_{n-i}$. 
Operation 1 is the first step for enlarging the graph. Operation 2 specifies the edges between the new and the old vertices, and also within the new vertices. The edges between \( W_n \) and \( W_0 \) are crucial in order to capture families with 'cyclic boundary conditions.' This should be clear if we want to construct cyclic ladders, that is, as in the example of Fig. 1 but with edges between the last two and the first two vertices. Then we also need to remove the corresponding edges added in the previous step, and this is where operation 3 comes into play.

The reader may also wonder about the constant \( r \), which allows to connect vertices in \( W_n \) not only to \( W_{n-1} \) but also to \( W_{n-2} \) up to \( W_{n-r} \). The reason is to be able to capture powers of graphs, as explained later in this section and, more generally, circulant graphs.

We now proceed to prove our main result, namely that any recursively constructible family is recursive in the sense of [1]. The key point is that for every recursively constructible family \( \{G_n\}_{n \geq 0} \) there exists a matrix, the transfer matrix, whose entries are rational functions in \( x \) and \( y \), and such that the rank polynomial of \( G_n \) is given by

\[
R(G_n; x, y) = x^{r(G_n)} X_0 \cdot A^n \cdot 1,
\]

where \( X_0 \) is a vector that depends only on \( G_0 \), and \( 1 \) is the all ones vector.

This is proved in [3] for square grids with one of the dimensions fixed. For the sake of completeness, we reproduce part of the argument here in the case of ladder graphs \( L_n \), which are square grids of size \( 2 \times n \). The graph \( L_n \) is the union of \( L_{n-1} \) and a graph \( P \) consisting of three edges and isomorphic to a path of length three. Consider formula (1) when \( G = L_n \). Each \( A \subseteq E(L_n) \) can be written as

\[
A = B \cup C,
\]

with \( B \subseteq E(L_{n-1}) \), \( C \subseteq E(P) \), and clearly, \(|A| = |B| + |C|\). Let us write

\[
r(B \cup C) = r(B) + \delta(B, C),
\]

where \( \delta(B, C) \) is the variation on the rank of \( B \) produced by the addition of \( C \). Then we rewrite (1) as

\[
R(L_n; x, y) = x^{r(L_n)} \sum_{A=\emptyset}^{C} x^{-r(A)} y^{|A|-r(A)} \sum_{B=\emptyset}^{C} x^{-r(B)} y^{|B|-r(B)} \left( \sum_{C} x^{-\delta(B,C)} y^{|C|-\delta(B,C)} \right).
\]

If we want to use this formulation in a recursive scheme, we must be able to compute the variation \( \delta(B, C) \) without knowledge of the whole edge-set \( B \). Given an edge-set \( B \subseteq E(L_{n-1}) \), we label the two vertices in the \((n-1)\)th column according to the component of the spanning subgraph induced by \( B \) to which they belong; the components are labeled sequentially as they appear. In this way we get a state \( \sigma(B) = (s_1, s_2) \), where the \( s_i \) are the labels of the components. Then the rank of \( B \cup C \), and hence \( \delta(B, C) \), can be computed from the knowledge of the state \( \sigma(B) \) and \( C \).
computations when $\sigma$

the partitions

two possible partitions of $f$
and

to the

$(\sigma,\sigma')$

Initial state

$C$
$|C|$
$\delta(B,C)$

Final state

Contribution to $A$

| Initial state | $C$ | $|C|$ | $\delta(B,C)$ | Final state | Contribution to $A$ |
|---------------|-----|------|----------------|-------------|-------------------|
| (1, 1)        | $\emptyset$ | 0 | 0 | (1, 2) | $1$ |
| (1, 1)        | $\{e\}$ | 1 | 1 | (1, 2) | $(xy)^{-1}y$ |
| (1, 1)        | $\{f\}$ | 1 | 1 | (1, 1) | $(xy)^{-1}y$ |
| (1, 1)        | $\{g\}$ | 1 | 1 | (1, 2) | $(xy)^{-1}y$ |
| (1, 1)        | $\{e, f\}$ | 2 | 2 | (1, 1) | $(xy)^{-2}y^2$ |
| (1, 1)        | $\{f, g\}$ | 2 | 2 | (1, 1) | $(xy)^{-2}y^2$ |
| (1, 1)        | $\{e, g\}$ | 2 | 2 | (1, 1) | $(xy)^{-2}y^2$ |
| (1, 1)        | $\{e, f, g\}$ | 3 | 3 | (1, 1) | $(xy)^{-2}y^3$ |

Next we define a $2 \times 2$ matrix $A$ as follows. The rows and columns are indexed by the two possible partitions of $\{1, 2\}$; they are encoded as $(1, 1)$ and $(1, 2)$, and correspond to the partitions $\{1, 2\}$ and $\{1\} \cup \{2\}$. The matrix $A$ is set initially to $0$. Let $\sigma$ be a partition, and let $C$ be any subset of $E(P)$. Consider $\sigma$ as the state of a subset $B$ of edges in the ladder $L_{n-1}$, add the edge-set $C$, and compute $\delta(B, C)$ and the new state $\sigma'$ of $B \cup C$. Then add the term

$$x^{-\delta(B,C)}y^{|C| - \delta(B,C)}$$

to the $(\sigma, \sigma')$ entry of $A$. In order to illustrate the procedure we show below the computations when $\sigma = (1, 1)$. In Table 1, $e$ and $g$ are the two horizontal edges of $P$, and $f$ is the vertical edge (see Fig. 1).

Similar computations when $\sigma = (1, 2)$ give the final expression

$$A = \begin{pmatrix} x^{-1} + 3x^{-2} + yx^{-2} & 1 + 2x^{-1} \\ x^{-1} + 2x^{-2} + x^{-3} & 1 + 2x^{-1} + x^{-2} \end{pmatrix}.$$  

Finally, we define the vector $X_0 = (x^{-1}, 1)^t$, corresponding to the contributions of the states $(1, 1)$ and $(1, 2)$ to the initial polynomial $R(L_0; x, y)$, where $L_0$, the initial graph in the family of ladders, is just an edge. Then, as shown in [3], we have that

$$R(L_n; x, y) = x^{2n-1}(x^{-1}, 1)^t \cdot A^{n-1} \cdot 1,$$

where $1 = (1, 1)^t$. Using this formula, the reader can check, for example, that

$$T(L_3; x, y) = R(L_3; x - 1, y - 1) = 2x^2 + x + 2xy + y + y^2 + 3x^3 + 2x^2y + 2x^4 + x^5.$$  

The method in this example can be extended to any recursively constructible family of graphs $\{G_n\}_{n \geq 0}$ satisfying the conditions (a) through (e) of the definition. Assume for simplicity that $r = 1$. Then the states in the step $n$ are the partitions of $W_0 \cup W_n$. The operation of adding edges in condition (d) has a certain associated matrix. To handle the deletion of the set of edges $S$, we consider the deletion of $S$ as the inverse operation of adding $S$. If $A$ is the matrix associated to the addition of $S$, then the inverse matrix $A^{-1}$ is the matrix associated to the deletion of $S$. 

Table 1
In general, the step from \( G_{n-1} \) to \( G_n \) is decomposed into an ordered sequence of elementary operations \( \alpha_1, \ldots, \alpha_k \), having associated matrices \( A_1, \ldots, A_k \). The transfer matrix of the family is then obtained as \( \Lambda = A_1 \ldots A_k \), and the order of the factors cannot be altered. This is summarized in the following lemma.

**Lemma 1.** Let \( A_\alpha \) be the matrix associated to an elementary operation \( \alpha \) and let \( A_1, \ldots, A_k \) be the matrices associated to the \( k \) elementary operations \( \alpha_1, \ldots, \alpha_k \) which compose the step from \( G_{n-1} \) to \( G_n \). If \( \Lambda \) is the transfer matrix of the family, then

1. \( A_{\alpha^{-1}} = (A_\alpha)^{-1} \).
2. \( \Lambda = A_1 \ldots A_k \) (noncommutative product).

To ensure that the product in (ii) works correctly, the final vertices in the encoding of \( \alpha_i \) must agree with the initial vertices in the encoding of \( \alpha_{i+1} \), and the states must be arranged according to the same order in the corresponding matrices. This is because \( \alpha_i \) and \( \alpha_{i+1} \) are concatenated in this order.

The regularity condition (e) in the definition guarantees that Eq. (2) holds, and this is what we need for proving the main result in this section.

**Theorem 1.** Every recursively constructible family of graphs is recursive.

**Proof.** Let \( \{G_n\}_{n \geq 0} \) be a recursively constructible family of graphs. We prove that the generating function \( \sum_{n \geq 0} T(G_n; x, y)z^n \) is a rational function.

If \( \Lambda \) is the transfer-matrix of the family, we know that \( R(G_n; x, y) = x^{r(G_n)}X_0^t \cdot A^n \cdot 1 \), where \( X_0^t \) is the transpose of the initial rank vector, and \( 1 \) is the all ones vector. Hence

\[
\sum_{n \geq 0} R(G_n; x, y)z^n = \sum_{n \geq 0} (x^{r(G_n)}X_0^t \cdot A^n \cdot 1)z^n = \sum_{n \geq 0} \left( x^{an+b}X_0^t \cdot A^n \cdot 1 \right)z^n,
\]

where \( a = |W_n| \) (for \( n > 0 \)) and \( b = |W_0| - 1 \). Rearranging the terms we get

\[
\sum_{n \geq 0} R(G_n; x, y)z^n = X_0^t \cdot x^b \left( \sum_{n \geq 0} (x^a z A)^n \right) \cdot 1 = X_0^t \cdot x^b \left( I - x^a z A \right)^{-1} \cdot 1.
\]

Since the entries of \( \Lambda \) are rational functions in \( x \) and \( y \), the entries of the inverse matrix \( (I - x^a z A)^{-1} \) are rational in \( x \), \( y \), and \( z \). This proves that the generating function is rational; this also applies to the generating function of the Tutte polynomials, since \( T(G_n; x, y) = R(G_n; x - 1, y - 1) \).

**An example.** In order to show how the scheme works in the case of deletion of edges, we consider the family of wheels \( \{W_n\}_{n \geq 0} \). Wheels are indeed a recursively constructible family: the step from \( W_n \) to \( W_{n+1} \) is composed of two elementary operations \( \alpha_1 \) and \( \alpha_2 \), represented in Fig. 2. The first one consists of the deletion of the edge \( \{1, n\} \), and the second one of the addition of a new vertex \( n + 1 \) adjacent to 0, 1, and \( n \). We obtain the transfer-matrix \( \Lambda \) as the product of the matrices associated to \( \alpha_1 \) and \( \alpha_2 \).
We cannot compute the matrix of $\alpha_1$ directly, but we can compute the matrix associated to the addition of the edge $\{1, n\}$; its inverse is the matrix associated to the deletion of this edge. Hence, the matrices are

$$A_{\alpha_1} = \begin{pmatrix} 1 + y & 0 & 0 & 0 & 0 \\ 0 & 1 + y & 0 & 0 & 0 \\ x^{-1} & 0 & 1 & 0 & 0 \\ x^{-1} & 0 & 0 & 1 & 0 \\ 0 & x^{-1} & 0 & 0 & 1 \end{pmatrix}^{-1}$$

and

$$A_{\alpha_2} = \begin{pmatrix} 3x^{-1} + 3yx^{-1} + y^2x^{-1} & 0 & 1 & 0 & 0 \\ 2x^{-2} + yx^{-2} & 2x^{-1} + yx^{-1} & 0 & x^{-1} & 1 \\ 2x^{-1} + yx^{-1} + 2x^{-2} + yx^{-2} & 0 & 1 + x^{-1} & 0 & 0 \\ 2x^{-2} + yx^{-2} & x^{-1} & 0 & 2x^{-1} + yx^{-1} & 1 \\ x^{-2} + x^{-3} & x^{-1} + x^{-2} & 0 & x^{-1} + x^{-2} & 1 + x^{-1} \end{pmatrix}.$$
from which we obtain

\[
\sum_{n \geq 0} T(W_n; x, y)z^n = \frac{(xy + x + x^2 + y^2 + y) - (2x^2y + 2xy^2)z + x^2y^2z^2}{1 - (x + y + 2)z + (xy + y + x + 1)z^2 - xyz^3}.
\]

Expanding the series we can obtain explicitly the Tutte polynomial of any wheel. For example,

\[
T(W_4; x, y) = 3x + 3y + 6y^2 + 6x^2 + 4x^3 + 4y^3 + 9yx + 4yx^2 + x^4 + y^4 + 4xy^2.
\]

The denominator of this rational function gives us explicitly the linear recurrence equation satisfied by the Tutte polynomials of wheels, namely

\[
W_{n+3} - (x + y + 2)W_{n+2} + (xy + x + y + 1)W_{n+1} - xyW_n = 0.
\]

To conclude this section we show how certain operations applied to a recursively constructible family preserve the recursive character. For this purpose we recall the following definitions. Given two graphs \(G\) and \(H\) with disjoint vertex sets, the join graph \(G + H\) is the graph with

\[
V(G + H) = V(G) \cup V(H),
\]

\[
E(G + H) = E(G) \cup E(H) \cup \\{(u, v) \mid u \in E(G), v \in E(H)\}.
\]

The Cartesian product \(G \square H\), the categorical product \(G \times H\), and the strong product \(G \boxtimes H\) all have \(V(G) \times V(H)\) as set of vertices, and as set of edges

\[
E(G \square H) = \\{(u, v), (u', v') \mid u = u', \{v, v'\} \in E(H) \text{ or } v = v', \{u, u'\} \in E(G)\};
\]

\[
E(G \times H) = \\{(u, v), (u', v') \mid \{u, u'\} \in E(G), \{v, v'\} \in E(H)\};
\]

\[
E(G \boxtimes H) = E(G \square H) \cup E(G \times H).
\]

Finally, given a positive integer \(k\), the \(k\)th power of \(G\), denoted by \(G^k\), has the same vertices as \(G\), and two of them are adjacent if their distance in \(G\) is at most \(k\).

**Theorem 2.** Let \(\{G_n\}_{n \geq 0}\) be a recursively constructible family of graphs. Then, for any fixed graph \(H\) and integer \(k\), the following families are also recursively constructible:

\[
\{G_n + H\}_n, \quad \{G_n \square H\}_n, \quad \{G_n \times H\}_n, \quad \{G_n \boxtimes H\}_n, \quad \{G_n^k\}_n.
\]

**Proof.** We prove the claim for the first case; the proof for the remaining cases is very similar and we omit it in order to avoid repetition. Since \(\{G_n\}_{n \geq 0}\) is a recursively constructible family, it satisfies conditions (a) through (e) of the definition. Let \(W_n, E_n, \text{ and } S\) have the same meaning as there.

The family \(J_n = \{G_n + H\}_{n \geq 0}\) can be realized as a recursively constructible family as follows:
For instance, power of cycles $C_k$ account that the number of spanning trees of a graph fixed parameters satisfy linear recurrence equations with constant coefficients. Taking into subsets of $(i) V(J_0) = W_0^J = W_0 \cup V(H)$, $E(G_0) = E_0^J = E_0 \cup E(H)$; 
(ii) $V(J_n) = V(J_{n-1}) \cup W_n^J$, where $W_n^J = W_n \cup V(H)$; 
(iii) $E(J_n) = (E(J_{n-1}) \setminus S) \cup E_n^J$, where $E_n^J = E_n \cup \{|x, y| x \in W_n, y \in H\}$.

Then it is straightforward to check the remaining properties of the definition, given the corresponding properties satisfied by the family $\{G_n\}$. 

The results obtained so far show that many well-known families of graphs are recursive. For instance, power of cycles $C_k^n$ with $k$ fixed or, more generally, circulant graphs $C_{k_1,\ldots,k_r}(n)$. In [8] it is proved that the number of spanning trees in circulant graphs with fixed parameters satisfy linear recurrence equations with constant coefficients. Taking into account that the number of spanning trees of a graph $G$ is equal to $T(G; 1, 1)$, this is consistent with our results. Also, regular lattices, such as square lattices, form recursive families if all the dimensions but one are fixed; boundaries can be identified or not in any of the dimensions. An example is the family $\{C_m \sqcap C_n\}_n$ of toroidal lattices, so-called because they have a natural embedding on the torus as a grid of parallels and meridians. These families play an important role in statistical mechanics; see for example [5].

3. Complete bipartite graphs

Before stating the main result in this section, we recall some basic definitions. A partition of a set $S$ is a collection $\pi = \{B_1, B_2, \ldots, B_k\}$ of pairwise disjoint nonempty subsets of $S$, the blocks of $\pi$, with $\bigcup B_i = S$. A partition of a positive integer $n$ is a sequence of positive integers $\lambda = (\lambda_1, \ldots, \lambda_k)$, the parts of $\lambda$, with $\sum \lambda_i = n$ and $\lambda_1 \geq \cdots \geq \lambda_k \geq 1$. If $\lambda$ is a partition of $n$ we write $\lambda \vdash n$. We also use the notation $\lambda = (1^{a_1}, \ldots, n^{a_n})$, where $a_i$ is the number of parts in $\lambda$ equal to $i$, and $\sum a_i = n$. The type of a partition $\pi$ of a set of size $n$ is the partition of the integer $n$ induced by the sizes of the blocks in $\pi$.

We write a partition of $[n] = \{1, 2, \ldots, n\}$ with $k$ blocks as a word $a_1a_2\ldots a_n$ in the alphabet $\{1, 2, \ldots, k\}$, where $a_i = j$ if $i$ is in the $j$th block, and blocks are ordered in increasing order of their smallest element. For instance, the partition $\{(1, 4, 6), (2, 7), (3), (5, 8)\}$ is represented as $12314124$.

Finally, we recall that the number of partitions of an $n$-set is the $n$th Bell number $B(n)$, and the number of partitions of the integer $n$ is the partition number $p(n)$. We denote by $P(n)$ be the set of all partitions of the integer $n$, so that $|P(n)| = p(n)$.

Now consider, for fixed $s \geq 1$, the family of complete bipartite graphs $\{K_{s,n}\}_{n \geq 0}$. One can think of $K_{s,n}$ as the join $N_s + N_n$, where $N_s$ and $N_n$ are the null graphs with $s$ and $n$ vertices, respectively. The family $\{K_{s,n}\}_{n \geq 0}$ is a recursively constructible family of graphs, where the step from $K_{s,n}$ to $K_{s,n+1}$ is composed of a unique elementary operation, which consists of the addition of vertex $n + 1$ adjacent to the $s$ vertices of $N_s$. 

$\square$
The very special shape of the transfer-matrix of this family allows us to prove the following result.

**Theorem 3.** For fixed $s$, the family $\{K_{s,n}\}_{n \geq 0}$ is recursive, with

$$
\sum_{n \geq 0} T(K_{s,n}; x, y) z^n = \frac{P(x, y, z)}{Q(x, y, z)},
$$

where $Q(x, y, z) = \prod_{\lambda \in P(s)} L_\lambda(x, y, z)$ and $L_\lambda$ is a linear factor in $z$ computable from

$$
L_\lambda(x, y, z) = 1 - \left( x - 1 + \sum_{i=1}^{j} a_i (1 + \cdots + y^{i-1}) \right) z.
$$

**Proof.** Since $\{K_{s,n}\}_{n \geq 0}$ is a recursively constructible family of graphs, we know by Theorem 1 that

$$
\sum_{n \geq 0} R(K_{s,n}; x, y) z^n = X_0^{x_b} b \left( I - x a z A \right)^{-1} \cdot 1,
$$

where $A$ is the transfer-matrix of the family, $X_0$ is the initial rank vector, $b = s - 1$, and $a = 1$ is the number of vertices added in the step from $K_{s,n}$ to $K_{s,n+1}$.

In order to compute $A$, we codify the $s$ vertices of $N_s$ both in the initial and the final state. Two vertices of $N_s$ are encoded with the same number if they belong to the same connected component. Thus, the number of states is the number $B(s)$ of partitions of $[s]$, and $A$ is a $B(s) \times B(s)$ matrix. If we order the states according to the lexicographic order of the type of the partitions, then $A$ satisfies two properties:

(a) it is a lower triangular matrix, and
(b) it has a block diagonal structure.

To prove (a), observe that in every step we are always encoding the same $s$ vertices and only adding edges. It is impossible from a given state to reach a state lexicographically larger, since adding edges the connected components cannot be disconnected.

To prove (b), we group the rows and columns of $A$ according to the type of the partitions, and within each group the order does not matter. It is impossible from an initial state to reach a different final state of the same type by joining connected components. This means that each group gives raise to a diagonal submatrix with constant entries, depending only on the type the partitions in the block. (See Table 2 for an example.)

Therefore, the matrix $(I - x z A)$ is also lower triangular and has a block diagonal structure. This implies that the denominator of $(I - x z A)^{-1}$ is a product of linear factors, one for every block, that is, one for every partition of the integer $s$. Thus we have

$$
Q(x, y, z) = \prod_{\lambda \in P(s)} L_\lambda(x, y, z),
$$

where $L_\lambda$ is a linear factor that depends on $\lambda$. It only remains to show that $L_\lambda$ is as claimed.

Let $\lambda = (1^{a_1}, \ldots, s^{a_s})$. According to the previous discussion, the diagonal term of $A$ of the submatrix corresponding to $\lambda$ is the contribution to the rank polynomial of the edges subsets that do not modify the state of the vertices when they are added. Hence, they are the submatrix corresponding to $\lambda$ subset contributes with 1). Hence, given an initial state of type $\lambda$, we are allowed to add edge subsets incident to at most one connected component.

For a component of size $i$, $1 \leq i \leq s$, we have $\binom{s}{i}$ subsets of size $k$, $1 \leq k \leq i$, each one with a contribution to the rank polynomial of $x^{-1}y^{k-1}$, since the rank increases by one (we add one vertex $n + 1$ and the number of components does not change) and the increase in the size is $k$. We proceed in this way for the $\sum_{i=1}^{s} a_i$ parts of $\lambda$ (notice that the empty subset contributes with 1). Hence, given an initial state of type $\lambda$, the total contribution to the rank polynomial when reaching the same final state is

$$1 + \sum_{i=1}^{s} a_i \sum_{k=1}^{i} \binom{i}{k} x^{-1}y^{k-1} = 1 + \sum_{i=1}^{s} a_i (xy)^{-1} \sum_{k=1}^{i} \binom{i}{k} y^k$$

$$= 1 + \sum_{i=1}^{s} a_i (xy)^{-1} ((1 + y)^i - 1).$$

Table 2
The transfer-matrix $A_4$

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\begin{align*}
r_{11} &= 1 + x^{-1}(4 + 6y + 4y^2 + y^3), \\
r_{21} &= 1 + x^{-1}(3 + 3y + y^2), \\
r_{22} &= x^{-1}(4 + 3y + y^2), \\
r_{31} &= x^{-2}(3 + 3y + y^2), \\
r_{33} &= x^{-1}(4 + 4y + y^2),
\end{align*}
```
Therefore, the linear factor in the denominator of \((I - xzA)^{-1}\) corresponding to \(\lambda\) is
\[
1 - xz \left( 1 + \sum_{i=1}^{s} \alpha_i(x,y)^{-1}((1 + y)^i - 1) \right).
\]
Replacing \(x, y\) by \(x - 1, y - 1\) we get the desired result.

The denominator \(Q(x, y, z)\) of the Tutte generating function is
\[
\prod_{\lambda \in P(s)} L_\lambda(x, y, z),
\]
since the lower triangular and block diagonal structure of \((I - xzA)\) implies that each factor \(L_\lambda(x, y, z)\) appears only once in the denominator of \((I - xzA)^{-1}\).

Remark. The above result provides an explicit linear recurrence equation of degree \(p(s)\) satisfied by the Tutte polynomials \(T(K_s, n; x, y)\). We conjecture that there is no equation with smaller degree, which is equivalent to saying that \(P(x, y, z)\) and \(Q(x, y, z)\) are relatively prime. We have checked this fact for \(s \leq 4\); the problem in general is that \(P(x, y, z)\) is not easy to compute, as it depends on a large number of initial graphs in the family.

As an illustration, we show the transfer-matrix of the family \(\{K_4, n\}\) in Table 2, where the \(r_{ij} = r_{ij}(x, y)\) are rational functions. Since \(B(4) = 15\), it is a \(15 \times 15\) matrix. Observe that \(\Lambda_4\) is lower triangular and has \(p(4) = 5\) diagonal submatrices. Replacing \(x, x - 1\) by \(y, y - 1\) we obtain the generating function of the Tutte polynomials:
\[
\sum_{n \geq 0} T(K_4, n; x, y)z^n = \frac{P(x, y, z)}{Q(x, y, z)},
\]
where the numerator \(P(x, y, z)\) is a polynomial of degree at most 4 in \(z\), depending on the initial graphs, and the denominator
\[
Q(x, y, z) = (1 - (x + 3)z)(1 - (x + y + y^2 + y^3)z)(1 - (x + 2 + y)z)
\]
\[
\times (1 - (x + 1 + 2y)z)(1 - (x + 1 + y + y^2)z)
\]
provides explicitly the recurrence.

Finally, exactly the same proof as in the previous theorem gives the following generalization.

**Theorem 4.** Let \(G\) be a graph with \(s\) vertices. Then, \(\{N_n + G\}_{n \geq 0}\) is a recursive family of graphs and
\[
\sum_{n \geq 0} T(N_n + G; x, y)z^n = \frac{P(x, y, z)}{Q(x, y, z)},
\]
where \( Q(x, y, z) = \prod_{\lambda \in P(s)} L_\lambda(x, y, z) \) and \( L_\lambda \) is a linear factor in \( z \) computable from \( \lambda = (1^{\alpha_1}, \ldots, s^{\alpha_s}) \) as

\[
L_\lambda(x, y, z) = 1 - \left( x - 1 + \sum_{i=1}^{s} \alpha_i (1 + \cdots + y^{i-1}) \right) z.
\]

4. A conjecture

We have shown that every recursively constructible family of graphs is recursive. It is natural to ask whether the converse statement is true, that is: if the Tutte polynomials of a family of graphs \( \{G_n\} \) satisfy a linear recurrence, is it true that the \( G_n \) can be constructed in a recursive way in the sense of our definition?

The answer is plainly no if we do not take into account the fact that nonisomorphic graphs can have the same Tutte polynomial (examples are easy to produce for non-3-connected graphs; for 3-connected graphs the first examples were given by Tutte [6]). Suppose the family \( \{G_n\} \) is recursive and take for every \( n \) a graph \( H_n \) nonisomorphic to \( G_n \) with \( T(H_n; x, y) = T(G_n; x, y) \). Then obviously the family \( \{H_n\} \) is also recursive, but if we choose the \( H_n \) not in a ‘regular way,’ then \( \{H_n\} \) can be quite far from being a recursively constructible family.

So the right question seems the following. Let us say that two families \( \{G_n\} \) and \( \{H_n\} \) are Tutte equivalent if \( T(G_n; x, y) = T(H_n; x, y) \) for every \( n \). Is it true then that any recursive family is Tutte equivalent to a recursively constructible family? We conjecture that this is the case, that is, that our definition of a recursively constructible family captures (up to Tutte equivalence) all the families that can give rise to linear recurrence equations.

We believe this is an interesting problem and probably not easy to settle. We prove it here for equations of degree one, something which is very modest but already requires the use of a nontrivial result. In [4] the following is proved, solving a conjecture of Brylawski:

The Tutte polynomial of a 2-connected graph is irreducible over the integers.

It is known and easy to establish that the Tutte polynomial of a graph is the product of the Tutte polynomials of its 2-connected components (blocks). Then, if \( B_1, \ldots, B_k \) are the blocks of a graph \( G \), the previous result says that

\[
T(G; x, y) = T(B_1; x, y) \cdots T(B_k; x, y)
\]

is precisely the factorization of \( T(G; x, y) \) into irreducible factors in \( Z[x, y] \).

Assume now that \( \{G_n\} \) is a recursive family of degree one, that is,

\[
T(G_{n+1}; x, y) = A(x, y)T(G_n; x, y), \quad n \geq 0,
\]

where \( A(x, y) \) is a polynomial in \( Z[x, y] \). Then \( T(G_n; x, y) = A(x, y)^n T(G_0; x, y) \). For the sake of brevity, we treat only the case where \( G_0 \) is 2-connected and \( A(x, y) \) is irreducible; the general case is proved analogously by considering the blocks of \( G_0 \), and
the irreducible factors of $A(x, y)$. Then, by the result just quoted, $G_n$ has a block $H_n$ with $T(H_n; x, y) = T(G_0; x, y)$. Let $M_n$ be the graph obtained by removing $H_n$ from $G_n$ (but cut-vertices are not removed). Then

$$T(M_n; x, y)T(H_n; x, y) = T(G_n; x, y) = A(x, y)^nT(G_0; x, y),$$

so that $T(M_n; x, y) = A(x, y)^n$. Since we are assuming $A(x, y)$ is irreducible, $M_n$ must have $n$ blocks, each of them with the same Tutte polynomial $A(x, y)$. Let $J$ be a graph with $T(J; x, y) = A(x, y)$. Then $\{G_n\}$ is Tutte equivalent to the following recursively constructible family. The initial graph is $G_0$ and for every $n > 0$, $G_n$ is the graph obtained by identifying a vertex of $G_{n-1}$ with a vertex of $J$.

References