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Inequalities for Integral Mean Values

LÁSZLÓ LOSONCZI

*Department of Mathematics, University of Lagos, Lagos, Nigeria, and Department of Mathematics, Kossuth Lajos University, Debrecen, Hungary**

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INTRODUCTION

Let I be a real interval and denote by $P(I)$ the set of all positive functions on I, and by $D(I)$ the set of all real valued functions on I having nonvanishing derivative.

Let (X, S, μ) be a measure space with $0 < \mu(X) < \infty$. We say that (X, S, μ) satisfies the *condition L* if for any natural number n there exists a decomposition $X = A_n \cup B_n$, $A_n, B_n \in S$, $A_n \cap B_n = \phi$ such that $\mu(A_n) > 0$ and

$$\lim_{n\to\infty}(\mu(B_n)/\mu(A_n)) = \infty.$$

The number

$$I_\varphi(x)_f = \varphi^{-1}\left(\frac{\int f(x(z))\,\varphi(x(z))\,d\mu}{\int f(x(z))\,d\mu}\right), \qquad \left(\int = \int_X\right),$$

is called the *integral mean value of* $x \in F_{\varphi f}(I)$ corresponding to the mapping function $\varphi \in D(I)$ and the weight function $f \in P(I)$. Here $F_{\varphi f}(I)$ denotes the set of those measurable functions $x\colon X \to I$ for which $f(x(z)), f(x(z))\,\varphi(x(z))$ are integrable on X.

We remark that $I_\varphi(x)_f$ can be defined under more general circumstances with respect to f, φ, (X, S, μ), but our theorems are proved only under the above-mentioned assumptions.

The aim of this paper is twofold. First, we develop a general theory of these means. The results for the corresponding discrete means ([6; 9; 11]) can be generalized to our case. The second aim is to give a thorough investigation of the homogeneous mean values, generalizing results of [3; 5–9].

The paper consists of five sections. In Section 1 we prove general theorems. In the second section we determine the homogeneous $I_\varphi(x)_f$-mean values. These are

$$\left(\frac{\int x(z)^{a+p}\,d\mu}{\int x(z)^p\,d\mu}\right)^{1/a}, \qquad (a \neq 0) \qquad \text{and} \qquad \exp\left(\frac{\int x(z)^p \ln x(z)\,d\mu}{\int x(z)^p\,d\mu}\right),$$

* Present address.

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where a, p are real constants. For the sake of simplicity we denote these means by $I_a(x)_p$ and $I_0(x)_p$, respectively.

The next section deals with the comparison of two homogeneous means. In Section 4 we study the inequality $I_a(x+t)_p \leqslant I_a(x)_p + t$, where t is a positive scalar. Finally in Section 5 we investigate the sub- and superadditivity of the homogeneous means.

We give necessary and sufficient conditions for each inequality studied. The inequalities dealt with in Sections 3–5 had been investigated in the discrete case [6; 8; 9]. There the variables are in $(0, \infty)$ while in this paper we suppose the range of the "variable" function x to be restricted to an interval $(m, M) \subset (0, \infty)$. This makes the situation more difficult but the validity of the inequalities widens. Because of homogeneity the conditions depend only on the ratio $A = m/M$.

1. General Theorems

Theorem 1. *Suppose that I, I_1, I_2 are arbitrary intervals, $l: I_1 \times I_2 \to I$ is a function differentiable on $I_1 \times I_2$, $\varphi \in D(I)$, $\psi \in D(I_1)$, $\chi \in D(I_2)$; $f \in P(I)$, $g \in P(I_1)$, $h \in P(I_2)$, and the condition L is satisfied.*

The inequality

$$I_\varphi(l(x, y))_f \leqslant l(I_\psi(x)_g, I_\chi(y)_h) \tag{1.1}$$

is true for all $x \in F_{\psi g}(I_1)$, $y \in F_{\chi h}(I_2)$ satisfying $l(x, y) \in F_{\varphi f}(I)$ if and only if

$$\frac{\varphi(l(u, v)) - \varphi(l(t, s))}{\varphi'(l(t, s))} \cdot \frac{f(l(u, v))}{f(l(t, s))} \leqslant \frac{\psi(u) - \psi(t)}{\psi'(t)} \frac{g(u)}{g(t)} \frac{\partial}{\partial t} l(t, s) + \frac{\chi(v) - \chi(s)}{\chi'(s)} \frac{h(v)}{h(s)} \frac{\partial}{\partial s} l(t, s) \tag{1.2}$$

is valid for all $u, t \in I_1$, $v, s \in I_2$.

Equality holds in (1.1) *if and only if*

$$(x(z), y(z), I_\psi(x)_g, I_\chi(y)_h) \in H$$

for almost every $z \in X$, where H is the set of those quadruplets (u, v, t, s) for which the equality sign is valid in (1.2).

Proof. Necessity. Put $x = \tilde{x}_n(z) = u\chi_{A_n}(z) + t\chi_{B_n}(z)$, $y = \tilde{y}_n(z) = v\chi_{A_n}(z) + s\chi_{B_n}(z)$ into (1.1) where $u, t \in I_1$, $v, s \in I_2$, and χ_E denotes the characteristic function of a set E. We obtain

$$\varphi^{-1}\left(\frac{f(l(u, v))\,\varphi(l(u, v)) + \lambda_n f(l(t, s))\,\varphi(l(t, s))}{f(l(u, v)) + \lambda_n f(l(t, s))}\right) \leqslant l\left(\psi^{-1}\left(\frac{g(u)\,\psi(u) + \lambda_n g(t)\,\psi(t)}{g(u) + \lambda_n g(t)}\right), \quad \chi^{-1}\left(\frac{h(v)\,\chi(v) + \lambda_n h(s)\,\chi(s)}{h(v) + \lambda_n h(s)}\right)\right)$$

where $\lambda_n = \mu(B_n)/\mu(A_n)$.

Subtracting $l(t, s)$ from both sides, multiplying by λ_n, and letting $n \to \infty$ we get (1.2) (for a similar limit see [9, Theorem 1]).

Sufficiency. Substitute $u = x(z)$, $v = y(z)$, $t = I_\psi(x)_g$, $s = I_\chi(y)_h$ into (1.2) and integrate over X. The right-hand side disappears and separating the cases $\varphi' > 0$, $\varphi' < 0$ we obtain (1.1).

The statement concerning *equality* can be obtained by observing the proof of sufficiency.

Remark. If f, g, h are positive constant functions, then condition (1.2) can be transformed into a convexity condition using the characterization of differentiable convex functions. See Beck [2] where this convexity condition was obtained in investigating the discrete inequality corresponding to (1.1) with $f = g = h \equiv 1$.

THEOREM 2. *Suppose that the conditions of Theorem* 1 *are satisfied except that now l is differentiable only at (t, s) where t, s are fixed interior points of I_1, I_2, respectively, and in addition f, g, h are continuous on their domains of definition.*

In order that the inequality

$$I_\varphi(l(x, y))_f \leqslant l(t, s) \tag{1.3}$$

holds for all $x \in F_{\psi g}(I_1)$, $y \in F_{\chi h}(I_2)$ satisfying $l(x, y) \in F_{\varphi f}(I)$ and the conditions

$$\begin{aligned} I_\psi(x)_g &= t, \\ I_\chi(y)_h &= s, \end{aligned} \tag{1.4}$$

it is necessary and sufficient that (1.2) *is valid for all $u \in I_1$, $v \in I_2$.*

Equality occurs in (1.3) *exactly when*

$$(x(z), y(z)) \in H_*$$

for almost every $z \in X$ where H_* is the set of all pairs (u, v) for which the equality holds in (1.2).

Proof. Necessity. Substitute $x = x_n^*(z) = u\chi_{A_n}(z) + t_n\chi_{B_n}(z)$, $y = y_n^*(z) = v\chi_{A_n}(z) + s_n\chi_{B_n}(z)$ into (1.3) where $u \in I_1$, $v \in I_2$, and t_n, s_n are chosen such that (1.4) is satisfied. This is possible for n large enough, since

$$I_\psi(x_n^*)_g = t$$

means that

$$-g(u)\,(\psi(t) - \psi(u)) = \lambda_n g(t_n)\,(\psi(t) - \psi(t_n)). \tag{1.5}$$

Let G_n be defined by

$$G_n(T) = \lambda_n g(T)\,(\psi(t) - \psi(T))$$

where T is in a closed bounded interval $J = [t - a, t + a] \subset I_1$. Obviously

$$\lim_{n\to\infty} \inf_{T\in J} G_n(T) = -\infty,$$

$$\lim_{n\to\infty} \sup_{T\in J} G_n(T) = +\infty;$$

hence for $n \geqslant N_1$,

$$\inf_{T\in J} G_n(T) < -g(u)(\psi(t) - \psi(u)) < \sup_{T\in J} G_n(T).$$

G_n is a continuous function on J; therefore it takes on the value $-g(u)\,(\psi(t) - \psi(u))$ at some $T = t_n \in J$. This means that for $n \geqslant N$ there exists $t_n \in I_1$ and similarly $s_n \in I_2$ such that (1.4) is satisfied by $x_n{}^*$, $y_n{}^*$. Moreover,

$$t_n \to t \qquad \text{and} \qquad s_n \to s$$

as $n \to \infty$. For $\{t_n\}$ is bounded; thus it has at least one limit point t. Were $t^* \neq t$ another limit point, then an appropriate subsequence $\{t_{n_k}\}$ would tend to t^*. Taking the limit as $k \to \infty$ of (1.5) (with $n = n_k$) we get a contradiction.

Equation (1.4) and the integrability conditions $x_n{}^* \in F_{\psi g}(I_1)$, etc., being satisfied, we may substitute $x = x_n{}^*$, $y = y_n{}^*$ into (1.3). We get

$$I_\varphi(x_n{}^*, y_n{}^*)_f \leqslant l(t, s)$$

or

$$\varphi^{-1}\left(\frac{f(l(u, v))\,\varphi(l(u, v)) + \lambda_n f(l(t_n, s_n))\,\varphi(l(t_n, s_n))}{f(l(u, v)) + \lambda_n f(l(t_n, s_n))}\right) \leqslant l(t, s).$$

Hence, provided that $\varphi' > 0$, we have

$$f(l(u, v))\,(\varphi(l(u, v)) - \varphi(l(t, s))) \leqslant f(l(t_n, s_n))\,(\varphi(l(t, s) - \varphi(l(t_n, s_n)))\,\lambda_n. \quad (1.6)$$

By the differentiability of l at (t, s) we may write

$$(\varphi(l(t, s)) - \varphi(l(t_n, s_n)))\,\lambda_n = (\varphi'(l(t, s))(\partial/\partial t)\,l(t, s) + \Omega_1)\,\lambda_n(t - t_n)$$
$$+ (\varphi'(l(t, s))(\partial/\partial s)\,l(t, s) + \Omega_2)\,\lambda_n(s - s_n)$$

where Ω_1, $\Omega_2 \to 0$ as $n \to \infty$.

If $u \neq t$, then $t_n \neq t$; thus

$$\lambda_n(t - t_n) = \frac{\lambda_n(\psi(t) - \psi(t_n))}{(\psi(t) - \psi(t_n))/(t - t_n)}$$
$$= \frac{1}{(\psi(t) - \psi(t_n))/(t - t_n)}\,\frac{\lambda_n g(u)(\psi(u) - \psi(t_n))}{g(u) + \lambda_n g(t_n)} \to \frac{\psi(u) - \psi(t)}{\psi'(t)}\,\frac{g(u)}{g(t)},$$

as $n \to \infty$ and the same is true if $u = t$, $t_n = t$. Similarly

$$\lambda_n(s - s_n) \to \frac{\chi(v) - \chi(s)}{\chi'(s)} \frac{h(v)}{h(s)}.$$

Let $n \to \infty$ in (1.6); then after dividing by $f(l(t, s))\, \varphi'(l(t, s))$ we get (1.2). The case $\varphi' < 0$ can be settled similarly.

Sufficiency. Suppose $x \in F_{\psi g}(I_1)$, $y \in F_{\chi h}(I_2)$ satisfy $l(x, y) \in F_{\varphi f}(I)$ and (1.4). Substitute $u = x(z)$, $v = y(z)$ into (1.2) and integrate over X. The right-hand side disappears and after multiplying by $\varphi'(l(t, s))\, f(l(t, s))$ we get (1.3).

The statement concerning *equality* can be proved by scrutinizing the proof of sufficiency.

Remark. For discrete mean values a similar theorem can be found in [11, Theorem 3].

THEOREM 3. *Let I be an open interval, let $f \in P(I)$ be differentiable on I, let $\varphi \in D(I)$ be twice differentiable on I, and suppose that condition L is satisfied. Denote by I_x the set of those t's for which $t + x(z) \in I$ for all $z \in X$.*

The inequality

$$I_\varphi(t + x(z))_f \leqslant t + I_\varphi(x(z))_f \tag{1.7}$$

holds for all $x \in F_{\varphi f}(I)$ and $t \in I_x \cap [0, \infty)$ with $t + x(z) \in F_{\varphi f}(I)$ if and only if for all $u, v \in I$

$$\frac{\varphi'(u) - \varphi'(v)}{\varphi'(v)} + \frac{\varphi(u) - \varphi(v)}{\varphi'(v)} \left[\frac{f'(u)}{f(u)} - \frac{f'(v)}{f(v)} - \frac{\varphi''(v)}{\varphi'(v)}\right] \leqslant 0. \tag{1.8}$$

If in (1.8) *strict inequality is valid for $u \neq v$, then in* (1.7) *too strict inequality holds except when $t = 0$ and $x(z) =$ a constant almost everywhere in X.*

Proof. To prove the necessity let $u, v \in I$, and $\tilde{x}_n(z) = u\chi_{A_n}(z) + v\chi_{B_n}(z)$. $t + \tilde{x}_n(z) \in I$ for all $z \in X$ if $t + u$, $t + v \in I$. Therefore substituting $x = \tilde{x}_n$, $t \geqslant 0$ such that $t + u$, $t + v \in I$ into (1.7) we get

$$\varphi^{-1}\left(\frac{f(u + t)\, \varphi(u + t) + \lambda_n f(u + t)\, \varphi(v + t)}{f(u + t) + \lambda_n f(v + t)}\right)$$

$$\leqslant t + \varphi^{-1}\left(\frac{f(u)\, \varphi(u) + \lambda_n f(v)\, \varphi(v)}{f(u) + \lambda_n f(v)}\right).$$

Substracting $v + t$, multiplying by λ_n, and letting $n \to \infty$ we have

$$G(u + t, v + t) \leqslant G(u, v) \qquad (u, v, t + u, t + v \in I, t \geqslant 0) \tag{1.9}$$

where $G(u, v) = ((\varphi(u) - \varphi(v))/\varphi'(v)\,(f(u)/f(v))$. This is equivalent to

$$\frac{\partial G(u, v)}{\partial u} + \frac{\partial G(u, v)}{\partial v} \leqslant 0 \qquad (u, v \in I) \tag{1.10}$$

according to a remark following [12, Theorem 4]. An easy calculation shows that (1.10) is identical to (1.8).

Sufficiency. Substitute $u = x(z) \in F_{\varphi f}(I)$, $v = I_\varphi(x)_f$, and $t \in I_x \cap [0, \infty)$ into (1.9) (which is a consequence of (1.10), i.e., (1.8)) and integrate the inequality obtained. We get

$$\frac{\int f(x(z) + t)\, \varphi(x(z) + t)\, d\mu - \varphi(v + t) \int f(x(z) + t)\, d\mu}{\varphi'(v + t)\, f(v + t)} \leqslant 0, \tag{1.11}$$

which implies (1.7).

If in (1.8), i.e., in (1.10) *strict inequality* holds for $u \neq v$ then using Taylor's formula it can be seen that strict inequality holds also in (1.9) for $t \neq 0$, $u \neq v$, hence also in (1.11) provided that $t \neq 0$ and $x(z) \neq I_\varphi(x)_f$ a.e. on X.

In case $f = e \equiv 1$ we get from (1.8),

$$[\varphi'(u) - \varphi'(v)]\, \varphi'(v) \leqslant [\varphi(u) - \varphi(v)]\, \varphi''(v), \qquad u, v \in I,$$

which can be reformulated as

$$\varphi'(\varphi^{-1}(t)) - \varphi'(\varphi^{-1}(s)) \leqslant (t - s)\, [\varphi'(\varphi^{-1}(s))]', \qquad t, s \in \varphi(I), \tag{1.12}$$

provided that $\varphi' > 0$. Equation (1.12) is exactly the criterion for $\varphi'(\varphi^{-1}(t))$ to be concave on $\varphi(I)$. Taking into consideration the case when $\varphi' < 0$, we obtain

COROLLARY 1. *Let I be an open interval, let $\varphi \in D(I)$ be twice differentiable on I, and suppose that the condition L is satisfied. In case $f = e \equiv 1$, inequality* (1.7) *is valid if and only if either $\varphi'(u) > 0$ $(u \in I)$ and $\varphi'(\varphi^{-1}(s)$ is concave on $\varphi(I)$ or $\varphi'(u) < 0$ $(u \in I)$ and $\varphi'(\varphi^{-1}(s))$ is convex on $\varphi(I)$.*

THEOREM 4. *Let the conditions of Theorem* 3 *be satisfied and denote by $\bar{I}_x$ the set of all t's satisfying $tx(z) \in I$ for $z \in X$. In order that the inequality*

$$I_\varphi(tx(z))_f \leqslant tI_\varphi(x(z))_f \tag{1.13}$$

be true for all $x \in F_{\varphi f}(I)$ and $t \in \bar{I}_x \cap [1, \infty)$ with $tx(z) \in F_{\varphi f}(I)$ it is necessary and sufficient that

$$\frac{u\varphi'(u) - v\varphi'(v)}{\varphi'(v)} + \frac{\varphi(u) - \varphi(v)}{\varphi'(v)} \left[\frac{uf'(u)}{f(u)} - \frac{vf'(v)}{f(v)} - \frac{\varphi'(v) + v\varphi''(v)}{\varphi'(v)}\right] \leqslant 0, \tag{1.14}$$

for all $u, v \in I$.

The proof is similar to that of Theorem 3. Putting $\tilde{x}_n(z) = u\chi_{A_n}(z) + v\chi_{B_n}(z)$ into (1.13) and letting $n \to \infty$ we get

$$G(tu, tv) \leqslant tG(u, v) \qquad (u, v, tu, tv \in I, t \geqslant 1)$$

where G is the same function as in the proof of the former theorem. This is equivalent to

$$u \frac{\partial G}{\partial u} + v \frac{\partial G}{\partial v} \leqslant G(u, v) \tag{1.15}$$

(see the remark following [12, Theorem 4]). A simple calculation shows that (1.14) is identical to (1.15). The sufficiency of (1.14) can be proved by integrating the inequality obtained from (1.14) upon the substitutions $u = x(z)$, $v = I_\varphi(x)_f$.

In case $f = e \equiv 1$ (1.14) goes over into convexity conditions and we have

COROLLARY 2. *Let the conditions of Theorem* 4 *be satisfied* (*except for conditions concerning* f). *For* $f = e \equiv 1$ *the inequality* (1.13) *holds if and only if either* $\varphi'(u) > 0$ $(u \in I)$ *and* $\varphi^{-1}(s)\,\varphi'(\varphi^{-1}(s))$ $(s \in \varphi(I))$ *is concave or* $\varphi'(u) < 0$ $(u \in I)$ *and* $\varphi^{-1}(s)\,\varphi'(\varphi^{-1}(s))$, $(s \in \varphi(I))$ *is convex.*

Remark. Theorems 1–4 remain valid if we replace $\geqslant$ by $\leqslant$ in (1.1)–(1.3), (1.7), (1.8), (1.13), and (1.14).

2. HOMOGENEOUS $I_\varphi(x)_f$ MEAN VALUES

The mean value $I_\varphi(x)_f$ is called *homogeneous* if

$$I_\varphi(tx)_f = tI_\varphi(x)_f \tag{2.1}$$

holds for all $x \in F_{\varphi f}(R_+)$, $t \in R_+ = (0, \infty)$, with $tx \in F_{\varphi f}(R_+)$. Let $t > 0$ be fixed and introduce the notation $\psi(u) = \varphi(tu)$, $g(u) = f(tu)$; then (2.1) can be written as

$$I_\psi(x)_g = I_\varphi(x)_f, \qquad x \in F_{\varphi f}(R_+) \cap F_{\psi g}(R_+). \tag{2.2}$$

According to Theorem 1 a necessary and sufficient condition for (2.2) is

$$\frac{\psi(v) - \psi(s)}{\psi'(s)} \frac{g(v)}{g(s)} = \frac{\varphi(v) - \varphi(s)}{\varphi'(s)} \frac{f(v)}{f(s)}, \qquad (v, s \in R_+). \tag{2.3}$$

Similar reasoning shows that a criterion for the homogeneity of the discrete mean

$$M_\varphi(x)_f = \varphi^{-1}\left(\frac{\sum_{i=1}^n f(x_i)\,\varphi(x_i)}{\sum_{i=1}^n f(x_i)}\right), \qquad x = (x_1, \ldots, x_n) \in R_+^n$$

is again (2.3) (we have to apply [9, Theorem 1]. Thus (2.1) is equivalent to the functional equation

$$M_\varphi(tx)_f = tM_\varphi(x)_f \qquad (x \in R_+{}^n,\, t \in R_+,\, n = 1, 2,\ldots). \tag{2.4}$$

Assuming that f is continuous, the solutions of (2.4) or (2.1) are ([10, Theorem 4] or [1, Theorem 3])

$$\varphi(u) = \alpha \ln u + \beta, \qquad f(u) = \rho u^p \tag{2.5}$$

and

$$\varphi(u) = (\gamma u^a + \delta)/(\epsilon u^a + 1 - \epsilon), \qquad f(u) = \rho u^p(\epsilon u^a + 1 - \epsilon), \tag{2.6}$$

where the constants a, p, α, β, γ, δ, ϵ, ρ satisfy the conditions

$$a\alpha \neq 0; \qquad \rho, \epsilon > 0; \qquad \gamma(1 - \epsilon) - \epsilon\delta \neq 0.$$

Making use of solution (2.5) we get the mean value

$$\exp\left(\int x(z)^p \ln x(z)\, d\mu \Big/ \int x(z)^p\, d\mu\right), \tag{2.7}$$

while with (2.6) we obtain

$$\left(\int x(z)^{a+p}\, d\mu \Big/ \int x(z)^p\, d\mu\right)^{1/a} \qquad (a \neq 0). \tag{2.8}$$

These means are clearly homogeneous; thus we have proved

THEOREM 5. *Suppose that $f \in P(R_+)$ is a continuous function on R_+, $\varphi \in D(R_+)$, and the condition L holds. Then the mean value $I_\varphi(x)_f$ is homogeneous if and only if it has one of the forms* (2.7), (2.8).

For simplicity the means (2.7), (2.8) are denoted by $I_0(x)_p$, $I_a(x)_p$, respectively, while in cases $\varphi = \ln t, f = t^p$ and $\varphi = t^a, f = t^p$ we use the notations $F_{0p}(I)$ and $F_{ap}(I)$ for $F_{\varphi f}(I)$ $(I \subset R_+!)$.

It is clear that

$$I_{-a}(x)_p \equiv I_a(x)_{-a+p} \tag{2.9}$$

and

$$\left(\int x(z)^a\, d\mu\right)^{1/a} \equiv \mu(X)^{1/a}\, I_a(x)_0\,. \tag{2.10}$$

The latter identity shows the connection between our homogeneous means and the norms in the spaces $L_a(X, S, \mu)$.

The notation $I_0(x)_p$ is justified by the relation

$$\lim_{a\to 0} I_a(x)_p = I_0(x)_p \qquad (x \in F_{ap}(R_+) \text{ for } |a| < \delta). \tag{2.11}$$

To prove (2.11) let us observe that the function $a \to (x^{a+p} - x^p)/a$ is increasing, hence by Levi's theorem,

$$\begin{aligned}\left[\frac{d}{da}\int x(z)^{a+p}\, d\mu\right]_{a=0} &= \lim_{a\to 0}\int \frac{x(z)^a - 1}{a}\, x(z)^p\, d\mu \\ &= \int \left(x(z)^p \lim_{a\to 0}\frac{x(z)^a - 1}{a}\right) d\mu \\ &= \int x(z)^p \ln x(z)\, d\mu;\end{aligned}$$

thus

$$\lim_{a\to 0} I_a(x)_p = \exp\left[\frac{d}{da}\ln\int x(z)^{a+p}\, d\mu\right]_{a=0} = I_0(x)_p\,.$$

In the following sections of the paper we deal exclusively with the means $I_a(x)_p$. The function

$$\begin{aligned} G_{ap}(u, v) &= ((u^a - v^a)/av^{a-1})(u^p/v^p), \qquad &\text{if } a \neq 0 \\ &= ((\ln u - \ln v)/v^{-1})(u^p/v^p), \qquad &\text{if } a = 0 \end{aligned} \tag{2.12}$$

plays a significant role. From the homogeneity,

$$G_{ap}(u, v) = vj_{ap}(u/v),$$

where

$$\begin{aligned} j_{ap}(z) &= (z^{a+p} - z^p)/a \qquad &(a \neq 0) \\ &= z^p \ln z \qquad &(a = 0). \end{aligned} \tag{2.13}$$

The mean values $I_a(x)_p$ are investigated for functions of the class $F_{ap}((m, M))$ where $0 \leqslant m < M \leqslant +\infty$. To simplify the notation we omit one pair of parentheses and write simply $F_{ap}(m, M)$. Thus $F_{ap}(m, M)$ is the set of all functions $x: X \to (m, M)$ measurable on X such that x^p and x^{a+p} ($x^p \ln x$ in case $a = 0$) are integrable over X. The fact that the interval (m, M) is open substantially simplifies the cases of equality in our inequalities.

Let us agree that

$$0/+\infty = 0, \qquad +\infty/0 = +\infty \qquad \text{and} \qquad q/0 = +\infty, \qquad q/+\infty = 0,$$

for $0 < q < \infty$. If G_{ap} is defined for $u, v \in (m, M)$, then j_{ap} is defined on $(A, 1/A)$ where $A = m/M$. We see later that our conditions concerning the parameters involve only the number A.

If g is a function on $(0, \infty)$ then $g(0)$, $g(+\infty)$ means $\lim_{z\to 0+} g(z)$, $\lim_{z\to\infty} g(z)$, respectively, provided that these exist.

Let the function g_a be defined by

$$\begin{aligned} g_a(z) &= -a/(1 - z^{-a}), && \text{if} \quad a \neq 0, \quad z \in (0, 1) \cup (1, \infty) \\ &= -1/(\ln z), && \text{if} \quad a = 0, \quad z \in (0, 1) \cup (1, \infty) \\ &= -a/2, && \text{if} \quad z = 1, \quad a \ \text{arbitrary}. \end{aligned} \tag{2.14}$$

It can be seen that g_a is continuous on $(0, 1) \cup (1, \infty)$ and

$$g_a(0) = (|a| - a)/2. \tag{2.15}$$

By the identity $g_a(1/z) + g_{-a}(z) \equiv 0$ we get

$$g_a(+\infty) = -(|a| + a)/2. \tag{2.16}$$

The following result plays a key role in our investigations.

THEOREM 6. *The function h_{ab} defined by*

$$h_{ab}(z) = g_b(z) - g_a(z) \qquad (z \in R_+,\ a, b \in R = (-\infty, \infty)) \tag{2.17}$$

is continuous on R_+ (even at $z = 1$!) for all $a, b \in R$ and is

strictly increasing,	*if*	$\lvert a\rvert > \lvert b\rvert$,
constant,	*if*	$\lvert a\rvert = \lvert b\rvert$,
strictly decreasing,	*if*	$\lvert a\rvert < \lvert b\rvert$.

Proof. The continuity of h_{ab} at $z = 1$ can be proved by L'Hospital's rule. *If* $a\,b \neq 0$, $z \neq 1$ *then*

$$h'_{ab}(z) = \frac{[z^{b/2}(z^a - 1)/a + z^{a/2}(z^b - 1)/b]\; H_{ab}(z)}{[((z^a - 1)/a)\,(z^b - 1)/b]^2}$$

where

$$H_{ab}(z) = [(z^a - 1)/a - z^{(a-b)/2}(z^b - 1)/b]\; z^{b/2-1}.$$

The first factor of the numerator is $\gtrless 0$ for $z \gtrless 1$; in fact both its summands possess that property. For the second factor of the numerator we have

$$H'_{ab}(z) = z^{((a-b)/2)-1}[z^{(a+b)/2} - 1 - ((a - b)/2)((z^{-b} - 1)/-b)]. \tag{2.18}$$

We show that $H'_{ab}(z) \gtreqless 0$ for $|a/b| \gtreqless 1$. By Taylor's formula for $u > 0$

$$u^{(c+1)/2} - 1 - ((c + 1)/2)\,(u - 1) = ((c^2 - 1)/8)\; \xi^{(c-3)/2}(u - 1)^2 \tag{2.19}$$

where ξ lies between u and 1. Hence for $u \neq 1$ the left-hand side of (2.19) is $\gtreqless 0$ for $|c| \gtreqless 1$. Putting $u = z^{-b}$, $c = -(a/b)$ here we obtain that for $z \neq 1$, $z > 0$,

$$z^{(a-b)/2} - 1 - ((a - b)/2)\,((z^{-b} - 1)/-b) \gtreqless 0 \qquad \text{for } |a/b| \gtreqless 1,$$

which implies our proposition concerning $H'_{ab}(z)$. As

$$H_{ab}(z) = \int_1^z H'_{ab}(s)\, ds,$$

thus

$$\text{if} \quad |a/b| > 1 \quad \text{then} \quad H_{ab}(z) \gtreqless 0, \quad \text{for } z \gtrless 1,$$
$$\text{if} \quad |a/b| = 1 \quad \text{then} \quad H_{ab}(z) = 0, \quad \text{for } z \in (0, \infty),$$

while

$$\text{if} \quad |a/b| < 1 \quad \text{then} \quad H_{ab}(z) \gtrless 0, \quad \text{for } 1 \gtrless z.$$

Hence

$$h'_{ab}(z) \gtreqless 0, \quad \text{for} \quad z \in (0, 1) \cup (1, \infty) \quad \text{and} \quad |a/b| \gtreqless 1,$$

and h_{ab} is continuous on $(0, \infty)$ which proves our theorem for $ab \neq 0$.

If $ab = 0$ the proof is similar.

3. The Inequality $I_a(x)_p \leqslant I_b(x)_q$

Throughout this section we exclude the cases $a = b$, $p = q$ and $-a = b$, $a + p = q$ when $I_a(x)_p \equiv I_b(x)_q$. Our aim is to prove

THEOREM 7. *Suppose that* $(a - b)^2 + (p - q)^2 > 0$, $(a + b)^2 + (a + p - q)^2 > 0$, *and the condition L is satisfied.*

The inequality

$$I_a(x)_p \leqslant I_b(x)_q \tag{3.1}$$

is valid for all $x \in F_{ap}(m, M) \cap F_{bq}(m, M)$ *if and only if*

$$|a| \geqslant |b|, \quad q - p > (a - b)/2,$$
$$\text{and} \quad q - p \geqslant -(1/\log A) \log(g_b(A)/g_a(A)), \tag{3.2}$$

or

$$|a| \leqslant |b|, \quad q - p > (a - b)/2,$$
$$\text{and} \quad q - p \geqslant a - b + (1/\log A) \log(g_b(A)/g_a(A)), \tag{3.3}$$

where $A = (m/M)$ *and* g_a *is the function defined in* (2.14).

Equality occurs in (3.1) *exactly when*

$$x(z) = \text{constant} \tag{3.4}$$

almost everywhere in X.

Proof. According to Theorem 1, (3.1) is equivalent to

$$G_{ap}(u, t) \leqslant G_{bq}(u, t), \qquad u, t \in (m, M),$$

or to (3.5)

$$j_{ap}(z) \leqslant j_{bq}(z), \qquad z \in (A, 1/A)$$

where G_{ap}, j_{ap} have been defined in Section 2. Equality occurs in (3.1) if and only if

$$(x(z), I_b(x)_q) \in \{(u, t) \mid j_{ap}(u/t) = j_{bq}(u/t);\ u, t \in (m, M)\}$$

almost everywhere in X.

Set

$$f(z) = f_{abpq}(z) = j_{bq}(z)/j_{ap}(z) \quad \text{if} \quad z \in (0, 1) \cup (1, \infty),$$
$$= 1 \quad \text{if} \quad z = 1.$$

A simple calculation shows that f is continuously differentiable on $(0, \infty)$ and

$$f'(z) = (f(z)/z)\,(q - p - h(z)), \tag{3.6}$$

where $h(z) = h_{ab}(z)$ is the function appearing in Theorem 6. Observing that $j_{ap}(z), j_{bq}(z) \gtrless 0$ according to $z \gtrless 1$ (hence $f(z) > 0$ on $(0, \infty)$), (3.5) can be written as

$$f(z) \leqslant 1, \quad \text{for } z \in (A, 1), \qquad \text{and} \qquad 1 \leqslant f(z) \quad \text{for } z \in (1, 1/A) \tag{3.7}$$

(at $z = 1$, (3.5) is always true). Equation (3.7) shows that f goes through the point $z = 1$ nondecreasingly, i.e.,

$$f'(1) \geqslant 0. \tag{3.8}$$

In what follows we shall distinguish three cases.

Case I. $|a| > |b|$. By Theorem 6 h is strictly increasing, hence its range on $(A, 1/A)$ is the interval $(h(A), h(1/A))$.

Ia. If $q - p \notin (h(A), h(1/A))$ then from (3.6), $f'(z) \neq 0$ on $(A, 1/A)$. Therefore (3.7) holds if and only if $f'(z) > 0$ for $z \in (A, 1/A)$, that is, if

$$h(1/A) \leqslant q - p. \tag{3.9}$$

Ib. If $q - p \in (h(A), h(1/A))$ then $f'(z)$ vanishes at some $z_1 \in (A, 1/A)$ and $f'(z) \gtrless 0$ for $z_1 \gtrless z$. By (3.8) inequality (3.7) can be satisfied only if $z_1 \in (1, 1/A)$, i.e., if

$$q - p \in (h(1), h(1/A)).$$

It is easy to see that, if we add to this the condition $1 \leqslant f(1/A)$, we get a necessary and sufficient condition for (3.7):

$$\begin{aligned} h(1) < q - p < h(1/A), \\ 1 \leqslant f(1/A). \end{aligned} \tag{3.10}$$

Case II. $|a| = |b|$. If $a = b$ then $h(z) \equiv 0$, $f'(z) = f(z)/z)(q - p)$, and since now $q - p \neq 0$, (3.7) holds if and only if

$$h(1) = 0 < q - p. \tag{3.11}$$

In case $-a = b$, $h(z) \equiv a$; thus $f'(z) = (f(z)/z)(q - p - a)$. From (3.8) we get that (3.7) can only be satisfied if $q - p - a > 0$, and one sees immediately that this is sufficient too. Therefore if $-a = b$ then the criterion for (3.7) is

$$h(1) = a < q - p. \tag{3.12}$$

Case III. $|a| < |b|$. Now h is strictly decreasing and its range on $(A, 1/A)$ is the interval $(h(1/A), h(A))$.

IIIa. If $q - p \notin (h(1/A), h(A))$ then $f'(z)$ has no zero in $(A, 1/A)$ and (3.7) holds if and only if $f'(z) > 0$, that is, if

$$h(A) \leqslant q - p. \tag{3.13}$$

IIIb. If $q - p \in (h(1/A), h(A))$ then $f'(z)$ has a zero $z_2 \in (A, 1/A)$ and $f'(z) \gtrless 0$ for $z \gtrless z_2$. Equation (3.8) implies that z_2 should be in $(A, 1)$, i.e., $q - p \in (h(1), h(A))$. Adding the condition $f(A) \leqslant 1$ we get a necessary and sufficient condition for (3.7):

$$\begin{aligned} h(1) < q - p < h(A) \\ f(A) \leqslant 1. \end{aligned} \tag{3.14}$$

We remark that in all cases equality occurs in (3.7) only at $z = 1$.

Since all cases have been listed, (3.7) is valid if and only if one of the following systems of inequalities is satisfied:

$$|a| > |b| \quad \text{and} \quad h(1/A) \leqslant q - p, \tag{3.15}$$

$$|a| > |b| \quad \text{and} \quad h(1) < q - p < h(1/A), \quad \text{and} \quad 1 \leqslant f(1/A), \tag{3.16}$$

$$|a| = |b| \quad \text{and} \quad h(1) < q - p, \tag{3.17}$$

$$|a| < |b| \quad \text{and} \quad h(A) \leqslant q - p, \tag{3.18}$$

$$|a| < |b| \quad \text{and} \quad h(1) < q - p < h(A), \quad \text{and} \quad f(A) \leqslant 1. \tag{3.19}$$

Since from (3.7) with $z \to 1/A-$, $z \to A+$ the inequalities $1 \leqslant f(1/A)$, $f(A) \leqslant 1$ can be obtained, we may add any of these to our conditions. We may therefore unite (3.15) and (3.16) and also (3.18) and (3.19) and may add (3.17) to any of the resulting inequalities. Thus we get

$$|a| \geqslant |b|, \qquad h(1) < q - p, \qquad \text{and} \qquad 1 \leqslant f(1/A) \tag{3.20}$$

$$|a| \leqslant |b|, \qquad h(1) < q - p, \qquad \text{and} \qquad f(A) \leqslant 1, \tag{3.21}$$

as a necessary and sufficient condition for the fulfillment of (3.7).

Observing that for $z \neq 1$,

$$f(z) = z^{q-p+b-a}(g_a(z)/g_b(z)),$$

the condition $f(A) \leqslant 1$ can be written as

$$q - p \geqslant a - b + (1/\log A) \log(g_b(A)/g_a(A)). \tag{3.22}$$

Making use of the identity $g_a(1/z) = -z^{-a}g_a(z)$ $(z \neq 1)$ we get

$$f(1/z) = z^{p-q}(g_a(z)/g_b(z));$$

hence the inequality $1 \leqslant f(1/A)$ can be written in the form

$$q - p \geqslant -(1/\log A) \log(g_b(A)/g_a(A)). \tag{3.23}$$

By (3.22), (3.23), and $h(1) = (a - b)/2$ we get from (3.20), (3.21) exactly the necessary and sufficient conditions (3.2), (3.3) stated in our theorem.

Equality holds in (3.7) only for $z = 1$; thus equality is valid in (3.1) if and only if $x(z) = I_b(x)_q$ a.e. in X, that is, if (3.4) is true. This completes the proof of Theorem 7.

The conditions are significantly simpler *in the case of* $A = 0$, i.e., if (m, M) is $(0, 1)$, $(1, \infty)$, or $(0, \infty)$. Suppose first that $a < 0$, $b < 0$; then

$$\lim_{A\to 0+} \frac{\log(g_b(A)/g_a(A))}{\log A} = \lim_{A\to 0+} \frac{(\log(b/a))((1 - A^{-a})/(1 - A^{-b}))}{\log A} = 0,$$

and the same is true for $a \leqslant 0$, $b \leqslant 0$. Using the identity

$$g_{-a}(z) = g_a(z)\, z^{-a} \qquad (z \neq 1)$$

we obtain

$$\lim_{A\to 0+} \frac{\log(g_b(A)/g_a(A))}{\log A} = -(((a - b)/2) + ((|a| - |b|)/2)),$$

for all a, b.

COROLLARY 3. *Suppose that condition L is satisfied. In order that inequality*

$$I_a(x)_p \leqslant I_b(x)_q \tag{3.24}$$

holds for all $x \in F_{ap}(I) \cap F_{bq}(I)$ *where* I *is* $(0, 1)$, $(1, \infty)$, *or* $(0, \infty)$, *it is necessary and sufficient that*

$$q - p \geqslant ((a - b)/2) + |(|a| - |b|)/2| . \tag{3.25}$$

Equality occurs in (3.24) *if and only if* $a = b$, $q = p$ *or* $-a = b$, $a + p = q$, *or if* $x(z) =$ *constant almost everywhere in* X.

Proof. Using the above limit, (3.2) and (3.3) can be united into one system

$$q - p > (a - b)/2, \qquad \text{and} \qquad q - p \geqslant (a - b)/2 + |(|a| - |b|)/2| .$$

The second inequality implies the first except in the cases $a = b$, $p = q$ and $-a = b$, $a + p = q$ which were excluded in Theorem 7. But now they are included, so we may omit the first inequality and get (3.25).

Let us remark that the discrete inequality corresponding to (3.24) has been investigated and in [6] the following criterion was given for (3.24)

$$\operatorname{sgn} a = \operatorname{sgn} b, \qquad \text{and} \qquad q - p \geqslant \max\{a - b, 0\},$$
$$\operatorname{sgn} a \neq \operatorname{sgn} b, \qquad \text{and} \qquad q - p \geqslant \max\{a, -b\}.$$

This is clearly equivalent to (3.25). From Corollary 3 it can be seen that $I_a(x)_p$ is an increasing function of both a and p. With a suitable choice of the parameters we get

COROLLARY 4. *Assume that our measure space satisfies the condition L. Then the inequality*

$$\| x \|_{L_p} \leqslant I_b(x)_q \, \mu(X)^{1/p} \qquad (p \neq 0) \tag{3.26}$$

is valid for all $x \in L_p(X, S, \mu) \cap F_{bq}(R_+)$ *if and only if*

$$q \geqslant ((p - b)/2) + |(|p| - |b|)/2| ,$$

while the inequality opposite to (3.26) *holds if and only if*

$$q \leqslant ((p - b)/2) - |(|p| - |b|/2| .$$

Equality occurs in (3.26) *or in the reverse inequality exactly when* $b = q$, $q = 0$ *or* $b = -q$, $q = p$, *or if* $x(z) =$ *constant almost everywhere in* X.

4. THE INEQUALITY $I_a(x + t)_p \leqslant I_a(x)_p + t$

THEOREM 8. *Assume that the condition L holds. In order that the inequality*

$$I_a(x + t)_p \leqslant I_a(x)_p + t \tag{4.1}$$

be valid for all $x \in F_{ap}(m, M)$ and for all constants $t \in [0, \infty)$ it is necessary and sufficient that either

$$|a| \geqslant 1, \quad \text{and} \quad p \geqslant g_a(A) - g_1(A) \tag{4.2}$$

or

$$|a| < 1, \quad \text{and} \quad p \geqslant g_a(1/A) - g_1(1/A). \tag{4.3}$$

The reverse inequality

$$I_a(x + t)_p \geqslant I_a(x)_p + t \qquad (x \in F_{ap}(m, M), t \in [0, \infty)) \tag{4.4}$$

is true if and only if either

$$|a| \geqslant 1, \quad \text{and} \quad p \leqslant g_a(1/A) - g_1(1/A) \tag{4.5}$$

or

$$|a| < 1, \quad \text{and} \quad p \leqslant g_a(A) - g_1(A). \tag{4.6}$$

Apart from the cases $a = 1$, $p = 0$, and $a = -1$, $p = 1$ equality holds in (4.1) *or in* (4.4) *exactly if $t = 0$ or if $x(z) =$ constant almost everywhere in X.*

Before proceeding with the proof we prove

LEMMA 1. *$F_{ap}(0, \infty)$ is closed under addition provided that the measure space (X, S, μ) is finite.*

Proof. First we remark that for an arbitrary function φ increasing on (a, ∞),

$$\varphi(t + s) \leqslant \max\{\varphi(2t), \varphi(2s)\} \leqslant |\varphi(2t)| + |\varphi(2s)|, \qquad t, s \in (a, \infty). \tag{4.7}$$

Suppose that $x, y: X \to (0, \infty)$ are measurable functions.

If $a \neq 0$ we have to show that the integrability of x^p, x^{a+p}, y^p, y^{a+p} imply that of $(x + y)^p$, $(x + y)^{a+p}$. This is a consequence of the inequality

$$\begin{aligned}(x + y)^q &\leqslant x^q && \text{for} \quad q < 0\\ &\leqslant 2^q x^q + 2^q y^q && \text{for} \quad q \geqslant 0,\end{aligned} \tag{4.8}$$

where for $q \geqslant 0$ we use (4.7).

If $a = 0$ we should prove that $(x + y)^p$, $(x + y)^p \ln(x + y)$ are integrable provided that x^p, $x^p \ln x$, y^p, $y^p \ln y$ are. The integrability of $(x + y)^p$ follows from (4.8) while the integrability of $(x + y)^p \ln(x + y)$ can be proved by separating the cases $p \gtreqless 0$.

For $p > 0$ the function $\varphi(t) = |t^p \ln t|$ is bounded on $(0, 1]$, positive, and increasing on $(1, \infty)$. Therefore by (4.7)

$$\begin{aligned}&|(x + y)^p \ln(x + y)|\\ &\quad \leqslant \max\{B_1, |(2x)^p \ln 2x| + |(2y)^p \ln 2y|\}\\ &\quad \leqslant \max\{B_1, 2^p |x^p \ln x| + 2^p x^p \ln 2 + 2^p |y^p \ln y| + 2^p y^p \ln 2\}\end{aligned}$$

with a constant B_1. Since all functions in the curly brackets are integrable, so is $(x+y)^p \ln(x+y)$.

If $p = 0$ then by (4.7),

$$\ln x \leqslant \ln(x+y) \leqslant |\ln 2x| + |\ln 2y| \leqslant |\ln x| + |\ln y| + 2\ln 2$$

which implies the integrability of $\ln(x+y)$.

For $p < 0$ the function $\varphi(t) = |t^p \ln t|$ is decreasing on $(0, 1]$ and is bounded on $(1, \infty)$; thus with an appropriate constant B_2,

$$|(x+y)^p \ln(x+y)| \leqslant \max\{B_2, |x^p \ln x|\},$$

which shows that $(x+y)^p \ln(x+y)$ is integrable and completes the proof.

Proof of Theorem 8. Let us first remark that the conditions $t \in I_x$ (i.e., $t + x(z) \in I$) and $t + x \in F_{\varphi f}(I)$ in Theorem 3 were assumed only to ensure that $I_\varphi(x+t)_f$ is meaningful. Here $I = (m, M)$ and in view of Lemma 1 the condition $t \geqslant 0$ ensures that $x + t \in F_{ap}(0, \infty)$, i.e., $I_a(x+t)_p$ is meaningful. Thus we may apply Theorem 3 and obtain that (4.1) is equivalent to

$$\frac{\partial G_{ap}(u, v)}{\partial u} + \frac{\partial G_{ap}(u, v)}{\partial v} \leqslant 0, \qquad u, v \in (m, M). \tag{4.9}$$

$G_{ap}(u, v) = v j_{ap}(u/v)$ implies

$$(1-z)\, j'_{ap}(z) + j_{ap}(z) \leqslant 0, \qquad z = u/v \in (A, 1/A). \tag{4.10}$$

A simple calculation shows that this can be written as

$$p \geqslant g_a(z) - g_1(z) = h_{1a}(z), \qquad z \in (A, 1/A). \tag{4.11}$$

By Theorem 6 we get immediately that (4.11) holds if and only if

$$\begin{aligned} p \geqslant \sup_{z\in(A,1/A)} h_{1a}(z) &= h_{1a}(A) \qquad \text{if } |a| \geqslant 1 \\ &= h_{1a}(1/A) \qquad \text{if } |a| < 1, \end{aligned}$$

that is, (4.2) or (4.3) is valid.

Similarly, (4.4) is equivalent to

$$p \leqslant h_{1a}(z), \qquad z \in (A, 1/A) \tag{4.12}$$

which gives (4.5) and (4.6).

If $a = 1$, $p = 0$, or $a = -1$, $p = 1$, obviously equality holds for all $x \in F_{ap}(m, M)$, $t \geqslant 0$ in (4.1) and (4.4). Otherwise the inequality is strict in (4.11), (4.12), and (4.9) and according to Theorem 3 also in (4.1), and (4.4) unless $t = 0$ or $x(z) =$ constant a.e.

In the special case $A = 0$ our conditions concerning a, p can essentially be simplified. Making use of the relations (following from (2.14), (2.15))

$$h_{10}(0) = (| a | - a/2), \qquad h_{10}(+\infty) = 1 - (a + | a |)/2$$

we get

COROLLARY 5 (Compare Losonczi [8, Theorem 6]). *Supposing that condition L holds, inequality* (4.1) *is valid for all* $x \in F_{ap}(R_+)$, $t \in [0, \infty)$ *if and only if*

$$p \geqslant ((1 - a)/2) + ((| 1 - | a ||)/2),$$

while the necessary and sufficient condition for (4.4) *in case* $(m, M) = (0, \infty)$ *is*

$$p \leqslant ((1 - a)/2) - ((| 1 - | a ||)/2).$$

We remark that the discrete version of (4.1) was, in case $A = 0$, $a = 1$, investigated by Beckenbach [3].

5. THE INEQUALITY OF DRESHER

The inequality

$$I_a(x + y)_p \leqslant I_a(x)_p + I_a(y)_p \tag{5.1}$$

was studied first by Beckenbach [3] (case $a = 1$) who proved by the method of quasilinearization that the discrete version of (5.1) holds if $a = 1$, $1 \leqslant p \leqslant 1$. Dresher [7] using the method of moments obtained that in case of continuous x, y: $[0, 1] \to [0, \infty)$ the conditions $a + p \geqslant 1 \geqslant p \geqslant 0$ are sufficient for (5.1) to be valid. The same result was also proved by Danskin [4] and Daróczy [5] (discrete case) using other known inequalities. Necessary and sufficient conditions were found by Losonczi [9] when the range of the functions x, y is $(0, \infty)$. In the sequel we investigate (5.1) for the class $F_{ap}(m, M)$, i.e., the range of x, y is only $(m, M) \subset (0, \infty)$, giving a criterion for (5.1) and for the inverse inequality.

THEOREM 9. *Assume that the condition L is satisfied. Then the inequality* (5.1) *holds for all* $x, y \in F_{ap}(m, M)$ *if and only if*

$$\max\{\alpha, \beta - a - 2g_a(A)\} \leqslant p \leqslant \beta, \tag{5.2}$$

while the inverse inequality

$$I_a(x + y)_p \geqslant I_a(x)_p + I_a(y)_p, \qquad x, y \in F_{ap}(m, M) \tag{5.3}$$

is valid exactly when

$$\alpha - a - 2g_a(A) \leqslant p \leqslant \min\{\alpha, \beta - a - 2g_a(A)\} \tag{5.4}$$

where $\alpha, \beta (\alpha < \beta)$ *are the zeros of the polynomial of second degree*

$$P(u) = u^2 - (2g_a(A) + 1)\, u - (a - 1)\, g_a(A). \tag{5.5}$$

Equality is attained in (5.1) *or in* (5.3) *if and only if* $x(z) = Cy(z)$ *a.e. in* X *where* C *is a constant or if* $a = 1, p = 0$ *or if* $a = -1, p = 1$.

(We remark that the polynomial (5.5) always has two different real zeros.)

Proof. According to Theorem 1 and Lemma 1, (5.1) is equivalent to

$$G_{ap}(u + v, t + s) \leqslant G_{ap}(u, t) + G_{ap}(v, s), \qquad u, v, t, s \in (m, M). \tag{5.6}$$

Since $G_{ap}(u, t) = tj_{ap}(u/v)$, with the notations $z = u/t$, $w = v/s$, $\lambda = t/(t + s)$ we have

$$j_{ap}(\lambda z + (1 - \lambda)\, w) \leqslant \lambda j_{ap}(z) + (1 - \lambda)\, j_{ap}(w), \tag{5.7}$$

for all $z, w \in (A, 1/A)$, $\lambda \in (1/(A + (1/A)), 1/(1 + A))$, that is j_{ap} is Jensen-convex. Thus the necessary and sufficient condition for (5.1) is

$$j''_{ap}(z) \geqslant 0, \qquad z \in (A, 1/A). \tag{5.8}$$

Calculating the second derivative here we get

$$a^{-1}(a + p)\,(a + p - 1)\, z^a - a^{-1}p(p - 1) \geqslant 0, \qquad z \in (A, 1/A), \quad \text{if} \quad a \neq 0, \tag{5.9}$$

while

$$p(p - 1) \ln z + 2p - 1 \geqslant 0, \qquad z \in (A, 1/A), \quad \text{if} \quad a = 0. \tag{5.10}$$

The functions on the left-hand side of (5.9), (5.10) are either constants or strictly monotonic; therefore these inequalities are satisfied for $z \in (A, 1/A)$ exactly when they are satisfied at $z = A$ and $z = 1/A$. Uniting the systems of inequalities so obtained we get the following ones (equivalent to (5.9) and (5.10), respectively)

$$p^2 - (2g_a(A) + 1)\, p - (a - 1)\, g_a(A) \leqslant 0, \tag{5.11}$$

$$p^2 + (2g_a(A) + 2a - 1)\, p + (a - 1)\,(a + g_a(A)) \geqslant 0, \tag{5.12}$$

where now a is arbitrary. The discriminant of the polynomial on the left side of (5.11) (which is exactly the polynomial (5.5))

$$D = 4g_a(A)\,[a + g_a(A)] + 1 = (4a^2A^{-a}/(1 - A^{-a})^2) + 1 \qquad (a \neq 0)$$
$$= (4/\ln^2 A) + 1 \qquad (a = 0)$$

is obviously positive thus the polynomial (5.5) always has two distinct real zeros α, β $(\alpha < \beta)$. The zeros of the polynomial on the left of (5.12) are

$$\alpha - a - 2g_a(A) \qquad \text{and} \qquad \beta - a - 2g_a(A).$$

Equation (5.11) is true if and only if

$$\alpha \leqslant p \leqslant \beta,$$

while (5.12) is valid exactly when

$$p \leqslant \alpha - a - 2g_a(A) \qquad \text{or} \qquad \beta - a - 2g_a(A) \leqslant p.$$

Since $\alpha < \beta$ and $-(a + 2g_a(A)) > 0$, the necessary and sufficient condition for (5.11) and (5.12) to be satisfied is

$$\min\{\alpha, \beta - a - 2g_a(A)\} \leqslant p \leqslant \beta,$$

i.e., (5.2).

The criterion for (5.3) is similarly $j''_{ap}(z) \leqslant 0$, $z \in (A, A^{-1})$ which yields (5.4).

To find the cases of equality in (5.1) or (5.3) we have to determine the set H of (u, v, t, s)'s for which equality holds in (5.6). This is equivalent to finding those triplets (z, w, λ) for which equality holds in (5.7). In (5.8) (or (5.9), (5.10)) the inequality is strict except for the cases $a \neq 0$, $(a + p)(a + p - 1) = 0$, $p(p - 1) = 0$, and $a = 0$, $p(p - 1) = 0$, $2p - 1 = 0$ (when the left-hand sides of (5.9) and (5.10) are identically zero), that is, in cases where $a = 1$, $p = 0$, and $a = -1$, $p = 1$. Apart from these equality holds in (5.7) only for $z = w$ and thus only for $u/t = v/s$ in (5.6). Hence

$$H = \{(u, v, t, s) \mid u/t = v/s,\ u, v, t, s \in (m, M)\}$$

and by Theorem 1 equality holds in (5.1) if and only if

$$x(z)/I_a(x)_p = y(z)/I_a(y)_p$$

a.e. in X, i.e., if and only if $x(z) = Cy(z)$ a.e. in X with a constant C.

On the other hand for $a = 1$, $p = 0$, and $a = -1$, $p = 1$, equality holds for all $x, y \in F_{ap}(m, M)$. Since (5.3) can be treated similarly, our theorem is proved.

Conditions (5.2), (5.4) can again be simplified if $A = 0$. Namely in this case

$$\alpha = (|a| - a)/2, \qquad \beta = 1 + ((|a| - a)/2),$$

$$\max\{\alpha, \beta - a - 2g_a(0)\} = ((1 - a)/2) + ((|1 - |a||)/2),$$

and

$$\min\{\alpha, \beta - a - 2g_a(0)\} = ((1 - a)/2) - ((|1 - |a||)/2).$$

Hence (5.2), (5.4) can be written as

$$((1 - a)/2) + ((|1 - |a||)/2) \leqslant p \leqslant 1 + ((|a| - a)/2), \tag{5.2*}$$

$$-(a + |a|)/2 \leqslant p \leqslant ((1 - a)/2) - ((|1 - |a||)/2). \tag{5.4*}$$

Thus we proved

COROLLARY 6 (Compare Losonczi [9, Theorem 3]). *Assume that the condition L holds; then the inequality*

$$I_a(x+y)_p \leqslant I_a(x)_p + I_a(y)_p$$

holds for all $x, y \in F_{ap}(I)$ *where* I *is* $(0, 1)$, $(1, \infty)$, *or* $(0, \infty)$ *if and only if* (5.2)* *is valid, while the necessary and sufficient condition for* (5.3) *to be satisfied for all* $x, y \in F_{ap}(I)$ *(with the same* I *as above) is* (5.3)*.

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