JOURNAL OF ALGEBRA 138, 313–326 (1991)

Etale Galois Coverings of Degree p of the Affine Plane

YOSHIFUMI TAKEDA

Department of Mathematics, Nara Women's University, Kita-Uoya Nishimachi, Nara 630, Japan

Communicated by Joseph Harris

Received March 10, 1989

INTRODUCTION

Let k be an algebraically closed field of characteristic $p>0$ and let $\varphi: \mathfrak{X} \to \mathbb{A}^2$ be an étale Galois covering of degree p, i.e., an étale finite morphism from a normal surface $\mathfrak{X}$ onto the affine plane $\mathbb{A}^2$ such that the function field extension $k(\mathfrak{X})$ over $k(\mathbb{A}^2)$ is a Galois extension of degree p. Then $\mathfrak{X}$ is a $\mathbb{Z}/p\mathbb{Z}$-torsor over $\mathbb{A}^2$ in the sense of étale topology. Moreover, it is known that this covering is of Artin–Schreier type (see Kambayashi and Srinivas [1], Miyanishi [2]). In other words, we can write $\mathfrak{X} = \operatorname{Spec} k[s, t, \xi]/(\xi^p - \xi - f(s, t))$, where (s, t) is a coordinate system on $\mathbb{A}^2$ and $f(s, t) \in k[s, t]$. In the present article, we consider a smooth completion of $\mathfrak{X}$ to study this covering and compute some birational invariants of the completion of $\mathfrak{X}$.

1. PRELIMINARIES

Let V be a nonsingular projective surface over k and let W be a normal surface. We call a finite morphism $\sigma: W \to V$ an Artin–Schreier covering if the function field extension $k(W)$ over $k(V)$ is a Galois extension of degree p. We have the following lemmas on Artin–Schreier coverings. For the proofs, we refer to Takeda [4].

LEMMA 1.1. *$\sigma_* \mathcal{O}_W$ has a canonical filtration of $\mathcal{O}_V$-submodules,*

$$\mathcal{O}_V = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_{p-1} = \sigma_* \mathcal{O}_W$$

such that

(1) *$\mathcal{F}_i$ is a locally free sheaf of rank $i+1$;*

0021-8693/91 $3.00

(2) $\mathscr{F}_1/\mathscr{F}_0$ *is an invertible sheaf and* $\mathscr{F}_i/\mathscr{F}_{i-1}$ *is a torsion-free* $\mathscr{O}_V$*-module of rank* 1 *for* $1 \leqslant i \leqslant p-1$.

Let g be a generator of the Galois group $\mathrm{Gal}(k(W)/k(V))$. Then g acts biregularly on W and V is the quotient variety $W/\langle g \rangle$. The locally free sheaf $\mathscr{F}_i$ is defined over an affine open set U of V as

$$\Gamma(U, \mathscr{F}_i) = \{h \in \Gamma(U, \mathscr{O}_W);\ g(h) - h \in \Gamma(U, \mathscr{F}_{i-1})\},$$

where $0 \leqslant i < p$ and $\mathscr{F}_{-1} = (0)$. We say that an Artin-Schreier covering $\sigma\colon W \to V$ is of simple type if $\mathscr{F}_i/\mathscr{F}_{i-1} \cong (\mathscr{F}_1/\mathscr{F}_0)^{\otimes i}$ for $1 \leqslant i \leqslant p-1$.

LEMMA 1.2. *Suppose that* $\sigma\colon W \to V$ *is an Artin–Schreier covering of simple type. Then there exist an affine open covering* $\mathfrak{U} = \{U_\lambda\}$ *of* V *and* $s_\lambda, t_\lambda \in \mathscr{O}_V(U_\lambda)$ *such that*

$$\sigma^{-1}(U_\lambda) = \mathrm{Spec}\, \mathscr{O}_V(U_\lambda)[\xi_\lambda]/(\xi_\lambda^p - s_\lambda^{p-1}\xi_\lambda - t_\lambda).$$

Moreover, $\{s_\lambda\} \in H^0(V, \mathscr{L})$, *where* $\mathscr{L} \cong (\mathscr{F}_1/\mathscr{F}_0)^{-1}$.

Conversely, if an Artin–Schreier covering $\sigma\colon W \to V$ is given by the equations as in Lemma 1.2 with respect to an affine open covering $\mathfrak{U} = \{U_\lambda\}$, then the locally free sheaf $\mathscr{F}_i$ in Lemma 1.1 is generated over U_λ by $1, \xi_\lambda, ..., \xi_\lambda^i$ as an $\mathscr{O}_{U_\lambda}$-module. Hence $\mathscr{F}_i/\mathscr{F}_{i-1}$ is generated by the class ξ_λ^i (modulo $\mathscr{F}_{i-1}$) as an $\mathscr{O}_{U_\lambda}$-module. This implies that $\sigma\colon W \to V$ is an Artin–Schreier covering of simple type.

By Lemma 1.2, every Artin–Schreier covering of simple type is locally a hypersurface. So, the dualizing sheaf ω_W is an invertible sheaf. We can compute some invariants of the covering as follows:

LEMMA 1.3. *With the same notations and assumptions as in the previous lemma, we have*

(1) $\omega_W = \sigma^*(\omega_V \otimes \mathscr{L}^{p-1})$.

(2) $\chi(\mathscr{O}_W) = p(\chi(\mathscr{O}_V) + (1/4)(p-1)(\mathscr{L}, \omega_V) + (1/12)(p-1)(2p-1)(\mathscr{L}^2))$.

LEMMA 1.4. *In addition to the assumptions in Lemma* 1.3, *suppose, furthermore, that* $\mathscr{L}$ *is ample. Then* $H^1(W, \mathscr{O}_W) = H^1(V, \mathscr{O}_V)$.

We now consider an étale Galois covering $\varphi\colon \mathfrak{X} \to \mathbb{A}^2$ of degree p. Let (s, t) be a coordinate system on $\mathbb{A}^2$. Then there exists a polynomial $f(s, t)$ such that $\mathfrak{X} = \mathrm{Spec}\, k[s, t, \xi]/(\xi^p - \xi - f(s, t))$. If $g(s, t) = f(s, t) + (h(s, t))^p - h(s, t)$ for some $h(s, t) \in k[s, t]$, then $\mathrm{Spec}\, k[s, t, \xi]/(\xi^p - \xi - f)$ and $\mathrm{Spec}\, k[s, t, \xi]/(\xi^p - \xi - g)$ are isomorphic to each other as finite coverings over $\mathbb{A}^2$. So, we choose $f(s, t)$ in such a way that the degree of $f(s, t)$ is the smallest under the above operation and we say that $\mathfrak{X}$ is defined by a polynomial $f(s, t)$. We consider $\mathbb{A}^2$ as an open set of $\mathbb{P}^2$ and let B be the line at infinity. Take the normalization $\pi: \tilde{\mathfrak{X}} \to \mathbb{P}^2$ in the function field $k(\mathfrak{X})$. This is an Artin–Schreier covering of $\mathbb{P}^2$ which is not necessarily of simple type. Let (X, Y, Z) be a homogeneous coordinate system on $\mathbb{P}^2$ and let $U_X = \{X \neq 0\}$, $U_Y = \{Y \neq 0\}$, and $U_Z = \{Z \neq 0\}$ constitute the standard open covering of $\mathbb{P}^2$. We define a Gorenstein scheme $\bar{\mathfrak{X}}$ and a finite morphism $\psi: \bar{\mathfrak{X}} \to \mathbb{P}^2$ by giving local data as

$$\psi^{-1}(U_Z) = \operatorname{Spec} k[s, t, \xi]/(\xi^p - \xi - f(s, t)),$$
$$\psi^{-1}(U_Y) = \operatorname{Spec} k[x, z, \eta]/(\eta^p - z^{m(p-1)}\eta - z^e\bar{f}(x, 1, z)),$$
$$\psi^{-1}(U_X) = \operatorname{Spec} k[u, v, \zeta]/(\zeta^p - v^{m(p-1)}\zeta - v^e\bar{f}(1, u, v)),$$

where $\eta = (Z^m/Y^m)\xi$, $\zeta = (Z^m/X^m)\xi$, $s = X/Z$, $t = Y/Z$, $x = X/Y$, $z = Z/Y$, $u = Y/X$, $v = Z/X$, $d = \deg f(s, t)$, $\bar{f}(X, Y, Z) = Z^d f(X/Z, Y/Z)$, and $d + e = mp$ $(e, m \in \mathbb{Z},\ 0 \leqslant e < p)$.

Remark 1.5. In general, $\bar{\mathfrak{X}}$ is not normal. We have, however, the following assertions by the same arguments as in the case of an Artin–Schreier covering of simple type (see [4, Sect. 1]).

1.5.1. There exists a filtration of $\mathcal{O}_{\mathbb{P}^2}$-modules

$$\mathcal{O}_{\mathbb{P}^2} = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_{p-1} = \psi_* \mathcal{O}_{\bar{\mathfrak{X}}}$$

such that $\mathcal{F}_i/\mathcal{F}_{i-1} = \mathcal{O}_{\mathbb{P}^2}(-imB) = \mathcal{O}_{\mathbb{P}^2}(-im)$ for $1 \leqslant i \leqslant p-1$.

Let g be a generator of $\operatorname{Gal}(k(\bar{\mathfrak{X}})/k(\mathbb{P}^2))$. Then we can choose g so that the action of g on $\bar{\mathfrak{X}}$ is given by $g(\xi) = \xi + 1$, $g(\eta) = \eta + z^m$, and $g(\zeta) = \zeta + v^m$. If we know that $\bar{\mathfrak{X}}$ is normal, the remark after Lemma 1.2 tells us that the covering $\psi: \bar{\mathfrak{X}} \to \mathbb{P}^2$ is an Artin-Schreier covering of simple type.

1.5.2. $\omega_{\bar{\mathfrak{X}}} = \psi^* \mathcal{O}_{\mathbb{P}^2}(-3 + (p-1)m)$.

Moreover, since B is ample, we have

1.5.3. $H^1(\bar{\mathfrak{X}}, \mathcal{O}_{\bar{\mathfrak{X}}}) = 0$.

Clearly $\tilde{\mathfrak{X}}$ is the normalization of $\bar{\mathfrak{X}}$. Let $\rho: \mathfrak{Y} \to \tilde{\mathfrak{X}}$ be the minimal resolution of the singularities of $\tilde{\mathfrak{X}}$. Then we have the diagram

$$\begin{array}{ccccc} & & \tilde{\mathfrak{X}} & \xleftarrow{\ \rho\ } & \mathfrak{Y} \\ & & \downarrow & & \\ \mathfrak{X} & \longrightarrow & \bar{\mathfrak{X}} & & \\ \varphi \downarrow & & \downarrow \psi & & \\ \mathbb{A}^2 & \longrightarrow & \mathbb{P}^2 & & \end{array}$$

Throughout this article, we keep the notations in this section.

2. Case of $\deg f \equiv 0 \pmod{p}$

We consider the following Hypothesis I on $f(s, t)$:

$$\deg f(s, t) \equiv 0 \quad \text{modulo } p \quad \text{and}$$

$$\left(\frac{\partial}{\partial X}\bar{f}, \frac{\partial}{\partial Y}\bar{f}, \frac{\partial}{\partial Z}\bar{f}\right)(P) \neq (0, 0, 0) \quad \text{for every} \quad P \in B.$$

We say that $\varphi: \mathfrak{X} \to \mathbb{A}^2$ satisfies Hypothesis I if $\mathfrak{X}$ can be written as $\mathfrak{X} = \operatorname{Spec} k[s, t, \xi]/(\xi^p - \xi - f)$ such that f and its homogenization $\bar{f}$ satisfy Hypothesis I. Then we have $d = \deg f = mp$ (see the definition of $\bar{\mathfrak{X}}$ in Section 1). Applying the Jacobian criterion, we have:

Proposition 2.1. *Suppose that $\varphi: \mathfrak{X} \to \mathbb{A}^2$ satisfies Hypothesis* I *and $(p, m) \neq (2, 1)$. Then $\bar{\mathfrak{X}}$ is nonsingular.*

In this case, we have $\bar{\mathfrak{X}} = \tilde{\mathfrak{X}} = \mathfrak{Y}$, whence $\bar{\mathfrak{X}}$ is normal, and $\pi \circ \rho : \mathfrak{Y} \to \mathbb{P}^2$ is, by Remark 1.5, an Artin–Schreier covering of simple type. Hence we can use the formulas in Section 1 to describe the covering $\pi \circ \rho : \mathfrak{Y} \to \mathbb{P}^2$.

Corollary 2.2. *With the same assumptions as in Proposition* 2.1, *we have*

(1) $K_{\mathfrak{Y}} = (\pi \circ \rho)^* ((-3 + m(p-1))B)$, $(K_{\mathfrak{Y}}^2) = 9p - 6p(p-1)m + p(p-1)^2 m^2$.

(2) $\chi(\mathcal{O}_{\mathfrak{Y}}) = p - (3/4)\, p(p-1)m + (1/12)\, p(p-1)(2p-1)m^2$.

(3) $H^1(\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}}) = 0$.

(4) $e(\mathfrak{Y}) = 3p - 3p(p-1)m + p^2(p-1)m^2$, *where $e(\mathfrak{Y})$ is the Euler number of $\mathfrak{Y}$.*

Remark 2.3. Suppose $\operatorname{char} k = p = 2$ and $\deg f(s, t) = 2$, i.e., $m = 1$. Then Hypothesis I is not sufficient for $\bar{\mathfrak{X}}$ to be nonsingular. We know, however, that $\bar{\mathfrak{X}}$ has at most only one rational double point of type A_1 [4, Sect. 2]. Hence the same formulas as in the previous corollary hold in this case, as well.

We shall give some examples.

Example 2.4. Assume $\operatorname{char} k = 3$. Suppose that $\mathfrak{X}$ satisfies Hypothesis I and $m = 1$. Then, since $K_{\mathfrak{Y}} = (\pi \circ \rho)^* (-B)$, it follows that $-K_{\mathfrak{Y}}$ is ample and $(K_{\mathfrak{Y}}^2) = 3$. Hence $\mathfrak{Y}$ is a del Pezzo surface of degree 3.

Example 2.5. Suppose $\operatorname{char} k = 2$ and $\deg f = 2$. If $\bar{\mathfrak{X}}$ is nonsingular, then $\mathfrak{Y}$, which is equal to $\bar{\mathfrak{X}}$, is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ [4, Theorem 3.2]. If

$\bar{\mathfrak{X}}$ has singular points, then $\mathfrak{Y}$ is the Hirzebruch surface of degree 2 [4, Example 2.13].

EXAMPLE 2.6. Assume char $k=2$. Suppose that $\mathfrak{X}$ satisfies Hypothesis I and $m=3$. Then we have $K_{\mathfrak{Y}}=0$ and $H^1(\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}})=0$. Hence $\mathfrak{Y}$ is a $K3$-surface.

3. CASE OF $\deg f \equiv p-1 \pmod p$

In this section, we consider $\varphi: \mathfrak{X} \to \mathbb{A}^2$ defined by $f(s, t)$ with $\deg f \equiv p-1$ modulo p, i.e., $e=1$. Then we have $d=\deg f = mp-1$ (see the definition of $\bar{\mathfrak{X}}$ in Section 1). By the Jacobian criterion, we have the following lemma.

LEMMA 3.1. *Suppose that $\varphi: \mathfrak{X} \to \mathbb{A}^2$ is defined by $f(s, t)$ with $e=1$. Then $\bar{\mathfrak{X}}$ is normal. Moreover, $\psi^{-1}(P)$ is a singular point if and only if $P \in \{\bar{f}(X, Y, Z)=0\} \cap B$.*

By Remark 1.5, we have $\bar{\mathfrak{X}} = \tilde{\mathfrak{X}}$, whence $\bar{\mathfrak{X}}$ is normal, and $\psi: \bar{\mathfrak{X}} \to \mathbb{P}^2$ is an Artin–Schreier covering of simple type. We consider the following Hypothesis II on $f(s, t)$ which imposes a restriction on the singularity of $\bar{\mathfrak{X}}$:

$$\deg f(s, t) \not\equiv 0 \quad \text{modulo } p \quad \text{and}$$

$$\{\bar{f}(X, Y, Z)=0\} \text{ is a reduced curve meeting } B \text{ transversally.}$$

If $f(s, t)$ with $e \neq 0$ is sufficiently general, then $f(s, t)$ satisfies Hypothesis II. We say that $\varphi: \mathfrak{X} \to \mathbb{A}^2$ satisfies Hypothesis II if $\mathfrak{X}$ can be defined by $f(s, t)$ satisfying Hypothesis II.

PROPOSITION 3.2. *Suppose that $\varphi: \mathfrak{X} \to \mathbb{A}^2$ satisfies Hypothesis* II *and $e=1$. Then $\bar{\mathfrak{X}}$ has exactly d rational double points of type A_{p-1}.*

Proof. By Lemma 3.1, we know that $\operatorname{Sing} \bar{\mathfrak{X}}$ are lying over $\{\bar{f}(X, Y, Z)=0\} \cap B$. Let P be a singular point of $\bar{\mathfrak{X}}$. Changing the coordinates if necessary, we may assume $P=(x=0, z=0) \in U_Y$. Since f satisfies Hypothesis II, $\bar{\mathfrak{X}}$ is locally defined by

$$\eta^p - z^{m(p-1)}\eta = zx + (\text{terms of higher degree}).$$

It is now clear that $\psi^{-1}(P)$ is a rational double point of type A_{p-1}. Q.E.D.

COROLLARY 3.3. *Under the same assumptions as in Proposition* 3.2, *the same four formulas as in Corollary* 2.2 *hold.*

Proof. Since $\bar{\mathfrak{X}}$ has only rational double points, we have $K_{\mathfrak{Y}} = \rho^* K_{\bar{\mathfrak{X}}}$ and $H^i(\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}}) = H^i(\bar{\mathfrak{X}}, \mathcal{O}_{\bar{\mathfrak{X}}})$. Hence the assertions follow from Lemmas 1.3 and 1.4. Q.E.D.

We shall give an example.

EXAMPLE 3.4. Assume char $k = 2$. Suppose that $\varphi: \mathfrak{X} \to \mathbb{A}^2$ is defined by $f(s, t)$ such that $\deg f = 5$ and $f(s, t)$ satisfies Hypothesis II. Then we have $K_{\mathfrak{Y}} = 0$ and $H^1(\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}}) = 0$. Hence $\mathfrak{Y}$ is a $K3$-surface.

4. CASE OF $\deg f \not\equiv 0, p-1 \pmod p$

In this section, we assume that $p > 2$ and that $\varphi: \mathfrak{X} \to \mathbb{A}^2$ satisfies Hypothesis II and $\deg f \not\equiv 0, p-1$ modulo p, i.e., $2 \leqslant e \leqslant p-1$. By the Jacobian criterion and the Serre criterion, we know that $\bar{\mathfrak{X}}$ is not normal. Hence $\pi: \tilde{\mathfrak{X}} \to \mathbb{P}^2$ is not of simple type. Under these circumstances, we consider the following exact sequence on $\mathbb{P}^2$:

$$0 \to \psi_* \mathcal{O}_{\bar{\mathfrak{X}}} \to \pi_* \mathcal{O}_{\tilde{\mathfrak{X}}} \to \mathcal{H} \to 0.$$

It is clear that $\operatorname{Supp} \mathcal{H} = B$. By Remark 1.5.3 we know $H^1(\mathbb{P}^2, \psi_* \mathcal{O}_{\bar{\mathfrak{X}}}) = 0$. Therefore, $H^0(B, \mathcal{H}) = 0$. This asserts that $\mathcal{H}$ is torsion-free. Hence $\mathcal{H}$ is a locally free sheaf on B. We shall determine $\mathcal{H}$. To begin with, let $P \in B$ be a general point and consider the sequence of stalks at P. We may assume that $P \in U_Y$ and that $\bar{\mathfrak{X}}$ is locally defined by $\eta^p - z^{m(p-1)}\eta - z^e \tilde{f}(x, 1, z) = 0$. Let $F = \tilde{f}(x, 1, z) + z^{m(p-1)-e}\eta$. Then $\eta^p = z^e F$ and $F(\tilde{P}) \neq 0$, where $\tilde{P}$ is a unique point of $\tilde{\mathfrak{X}}$ lying over P. Since p and e are relatively prime, there exist $a, b \in \mathbb{Z}$ such that $ap + be = 1$. We may assume $-e < a < 0$. Set $\tau = \eta^b z^a$. Then we have $\tau \in k(\mathfrak{X})$, $\tau^e = \eta F^{-a}$, and $\tau^p = z F^b$. Hence $x - x(P)$ and τ generate the maximal ideal of the local ring $(\pi_* \mathcal{O}_{\tilde{\mathfrak{X}}})_P$. This implies that $(\pi_* \mathcal{O}_{\tilde{\mathfrak{X}}})_P = (\psi_* \mathcal{O}_{\bar{\mathfrak{X}}})_P [\tau]$ and $\tilde{\mathfrak{X}}$ is nonsingular at $\tilde{P}$. Note that $\tau^n = z^{-\alpha} \eta^\beta F^{-\alpha b - \beta a}$ if $n = -\alpha p + \beta e$ with $\alpha, \beta \in \mathbb{Z}$. It is then easy to verify that $\mathcal{H}_P = \bigoplus_{\alpha, \beta} \mathcal{O}_{P, B} z^{-\alpha} \eta^\beta$, where α and β range over all integers such that $-\alpha p + \beta e > 0$, $0 < \alpha < e$, and $0 < \beta < p$.

We now consider $\mathcal{H}$ at the image point of a singular point of $\tilde{\mathfrak{X}}$. We may assume that every singular point lies over U_X. Let $Q \in B$ such that $\tilde{Q} = \pi^{-1}(Q)$ is a singular point of $\tilde{\mathfrak{X}}$. We know that $\bar{\mathfrak{X}}$ is defined by $\zeta^p - v^{m(p-1)}\zeta - v^e \tilde{f}(1, u, v) = 0$ over U_X. From the above observations, it follows that $(\tilde{f}(1, u, v) + v^{m(p-1)-e}\zeta)(\tilde{Q}) = 0$. Since $f(s, t)$ satisfies Hypothesis II, we know that Q is a zero of order 1 of $\tilde{f} = 0$. Changing the coordinate u if necessary, we may assume $Q = (u = 0, v = 0)$. So, $\bar{\mathfrak{X}}$ is locally defined by $\zeta^p = v^e(u + (\text{terms of higher degree}) + v^{m(p-1)-e}\zeta)$. In order to look into the singularity of $\tilde{\mathfrak{X}}$ at $\tilde{Q}$, it suffices to consider the normalization of a surface defined by $\zeta^p = v^e u$.

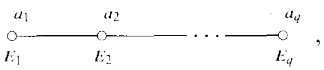
Let $T^p = v$ and $S = \zeta/T^e$. Then $S^p = u$. Consider a derivation $\mathfrak{D} = T(\partial/\partial T) + hS(\partial/\partial S)$ on $k[[S, T]]$, where $h = p - e$ and the invariant subring $k[[S, T]]^{\mathfrak{D}} = \{g \in k[[S, T]] \mid \mathfrak{D}(g) = 0\}$. Note that $k[[S, T]]^{\mathfrak{D}}$ is a normal ring. In the case of a polynomial ring, we have the following useful lemma.

LEMMA 4.1. *Let* $A = k[T, S] \cap k(T^p, S^p, S/T^h)$ *and* $\mathfrak{D} = T(\partial/\partial T) + hS(\partial/\partial S)$. *Then*

(1) $A = k[T, S]^{\mathfrak{D}}$ *and* Spec A *has an isolated singular point which is defined by* $S = T = 0$.

(2) $\{S^\alpha T^\beta \mid \beta + h\alpha \equiv 0 \pmod{p}\}$ *is a k-basis for* A.

(3) *The minimal resolution of singularity of* Spec A *has the following dual graph of exceptional curves*:



where

$$\frac{p}{h} = a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ddots - \cfrac{1}{a_q}}}, \qquad a_i \in \mathbb{Z},\ a_i \geqslant 2.$$

Here a vertex shown by $\bigcirc$ *stands for a nonsingular rational curve. Hence the singularity is rational. The proper transform of the curve* $T^p = 0$ *meets transversally the curve* E_q *and does not meet any other curves* E_i.

For the proof, we refer to Miyanishi and Russell [3, Lemma 3.2]. In our case, it is clear that $u, v, \zeta \in k[[S, T]]^{\mathfrak{D}}$ and $k((u, v, \zeta)) = k((S^p, T^p, S/T^h))$. From Lemma 4.1(2), it follows that $\{T^{-\alpha p + \beta e} S^\beta \mid \alpha, \beta \in \mathbb{Z},\ \beta > 0,\ -\alpha p + \beta e > 0\}$ is a k-basis of $k[[S, T]]^{\mathfrak{D}}$. It is then easy to verify that $\{T^{-\alpha p + \beta e} S^{\beta + np} \mid \alpha, \beta, n \in \mathbb{Z},\ n > 0,\ 0 < \alpha < e,\ 0 < \beta < p,\ -\alpha p + \beta e > 0\}$ is a k-basis of $k[[S, T]]^{\mathfrak{D}}/k[[S^p, T^p, T^e S]]$. Hence $\{T^{-\alpha p + \beta e} S^\beta \mid \alpha, \beta \in \mathbb{Z},\ 0 < \alpha < e,\ 0 < \beta < p,\ -\alpha p + \beta e > 0\}$ is a $k[[S^p]]$-basis. Since $T^{-\alpha p + \beta e} S^\beta = v^{-\alpha} \zeta^\beta$, we know $\mathscr{H}_Q = \bigoplus_{\alpha, \beta} \mathcal{O}_{Q, B} v^{-\alpha} \zeta^\beta$, where α and β range over all integers such that $0 < \alpha < e$, $0 < \beta < p$, and $-\alpha p + \beta e > 0$. On the other hand, we have already shown that $\mathscr{H}|_{B^0} = \bigoplus_{\alpha, \beta} \mathcal{O}_{B^0} z^{-\alpha} \eta^\beta$, where α and β are the same as above and $B^0 = B - \{Q \in B \mid \pi^{-1}(Q) \text{ is a singular point}\}$. Note that $z^{-\alpha} \eta^\beta = v^{-\alpha} \zeta^\beta u^{\alpha - \beta m}$. The transition functions of $\mathscr{H}$ over $U_Y \cap U_X$ are $\{u^{\alpha - \beta m}\}$. By these observations we have the following:

THEOREM 4.2. *Suppose that* $\varphi: \mathfrak{X} \to \mathbb{A}^2$ *satisfies Hypothesis* II *and* $e \neq 1$. *Set* $\mathscr{H} = \bigoplus_{\alpha,\beta} \mathcal{O}_B(\alpha - \beta m)$, *where* α *and* β *range over all integers such that* $0 < \alpha < e$, $0 < \beta < p$, *and* $-\alpha p + \beta e > 0$. *Then we have the following exact sequence on* $\mathbb{P}^2$:

$$0 \to \psi_* \mathcal{O}_{\tilde{\mathfrak{X}}} \to \pi_* \mathcal{O}_{\tilde{\mathfrak{X}}} \to \mathscr{H} \to 0.$$

COROLLARY 4.3. *Suppose* $e = p - 1$. *Then we can write* $\mathscr{H}$ *in the following way*:

$$\begin{aligned}\mathscr{H} = \mathcal{O}_B(1 - 2m) \oplus \mathcal{O}_B(1 - 3m) \oplus \cdots &\oplus \mathcal{O}_B(1 - (p-1)m) \\ \oplus \mathcal{O}_B(2 - 3m) \oplus \cdots &\oplus \mathcal{O}_B(2 - (p-1)m) \\ &\ddots \vdots \\ &\oplus \mathcal{O}_B(p - 2 - (p-1)m).\end{aligned}$$

Proof. Straightforward.

COROLLARY 4.4. *With the same assumptions as in Corollary* 4.3, *we have*

$$\chi(\mathcal{O}_{\mathfrak{Y}}) = (1/6)(p+2)(p^2 - 2p + 3) - (1/12)\, p(p-1)(4p+1)m + (1/12)\, p(p-1)(2p-1)m^2.$$

Proof. From the previous corollary, it follows that $\chi(\mathscr{H}) = (1/6)(p+3)(p-1)(p-2) - (1/3)\, p(p-1)(p-2)m$. Therefore, by Remark 1.5.1, we have $\chi(\mathcal{O}_{\tilde{\mathfrak{X}}}) = (1/6)(p+2)(p^2 - 2p + 3) - (1/12)\, p(p-1)(4p+1)m + (1/12)\, p(p-1)(2p-1)m^2$. Meanwhile, we know that $\tilde{\mathfrak{X}}$ has only rational singularities. Hence $\chi(\mathcal{O}_{\mathfrak{Y}}) = \chi(\mathcal{O}_{\tilde{\mathfrak{X}}})$. The assertion follows from these observation. Q.E.D.

Next, we shall determine the canonical divisor of $\mathfrak{Y}$.

LEMMA 4.5. *The canonical divisor* $K_{\tilde{\mathfrak{X}}}$ *of* $\tilde{\mathfrak{X}}$ *is written as* $K_{\tilde{\mathfrak{X}}} = \pi^*((-3 + m(p-1))B) - (p-1)(e-1)\, D$, *where* D *is the set-theoretic inverse image of* B *by* π *and* $\pi^* B = pD$.

In the proof of this lemma, we use the following:

LEMMA 4.6 (Miyanishi [2]). *Let* C *be an étale Galois covering of* $\mathbb{A}^1 = \operatorname{Spec} k[t]$ *defined by*

$$\xi^p - \xi = c_0 t^n + \cdots + c_{n-1} t,$$

where $c_0, \ldots, c_{n-1} \in k$, $c_0 \neq 0$, $n > 0$, *and* $c_i = 0$ *whenever* $i \equiv 0 \pmod p$. *Then the genus of* C *is equal to* $(p-1)(n-1)/2$.

Proof. See [2, Lemma 2.1].

Proof of Lemma 4.5. First, note that we can write the canonical divisor of $\tilde{\mathfrak{X}}$ as

$$K_{\tilde{\mathfrak{X}}} = \pi^*((-3 + m(p-1))B) - rD,$$

where $-rD$ is the contribution coming from the conductor ideal for the extension $\psi_* \mathcal{O}_{\mathfrak{X}} \subset \pi_* \mathcal{O}_{\tilde{\mathfrak{X}}}$. We have only to determine r. Let L be a general line on $\mathbb{P}^2$. We may assume that π^*L is an irreducible curve contained in the nonsingular part of $\tilde{\mathfrak{X}}$. We claim that π^*L is nonsingular. Indeed, let $P \in L \cap B$ and suppose that L is defined by $x = 0$ near P after a change of coordinates. Recall that $(\pi_* \mathcal{O}_{\tilde{\mathfrak{X}}})_P = (\psi_* \mathcal{O}_{\mathfrak{X}})_P [\tau]$, where $\tau = \eta^b z^a$ with $ap + be = 1$. In particular, $(\mathcal{O}_{\tilde{\mathfrak{X}}, \tilde{P}})^\wedge = k[[\tau, x]]$ and $(\mathcal{O}_{\pi^*L, \tilde{P}})^\wedge = k[[\tau]]$, where $\tilde{P} = \pi^{-1}(P)$. This shows that π^*L is a nonsingular curve. Now, by applying Lemma 4.6, we have $g(\pi^*L) = (p-1)(d-1)/2$. Meanwhile, by the adjunction formula, we have

$$\begin{aligned}(p-1)(d-1) - 2 &= (K_{\tilde{\mathfrak{X}}}, \pi^*L) + ((\pi^*L)^2) \\ &= p(-3 + m(p-1)) - r(D, \pi^*L) + p.\end{aligned}$$

Since $d + e = mp$, we have $r(D, \pi^*L) = (p-1)(e-1)$. On the other hand, the arguments given at the beginning of this section show that $\pi^*B = pD$. Hence $\pi_* D = B$ and $(D, \pi^*L) = 1$. This implies that $r = (p-1)(e-1)$.

Q.E.D.

We need some more notations to write down the canonical divisor of $\mathfrak{Y}$. Suppose that $\varphi: \mathfrak{X} \to \mathbb{A}^2$ satisfies Hypothesis II and $e \neq 1$. Let $\tilde{D}$ be the proper transform of D by ρ. Then $\tilde{D}$ is a nonsingular rational curve. Let $\{Q_1, ..., Q_d\}$ exhaust all points of $\mathbb{P}^2$ whose inverse images by π are the singular points of $\tilde{\mathfrak{X}}$. By Lemma 4.1, $(\pi \circ \rho)^{-1}(Q_i)$ has the configuration shown in Fig. 1, where $E_{i,j}$ is a nonsingular rational curve whose self-intersection number is $-a_j$, a_j being the same as given in Lemma 4.1. Set

$$\delta_j = \det \begin{pmatrix} a_{j+1} & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & a_q \end{pmatrix} \quad \text{for} \quad 0 \leqslant j < q \text{ and } \delta_q = 1$$

and set

$$\delta^j = \det \begin{pmatrix} a_1 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & a_{j-1} \end{pmatrix} \quad \text{for} \quad 1 < j \leqslant q+1 \text{ and } \delta^1 = 1.$$

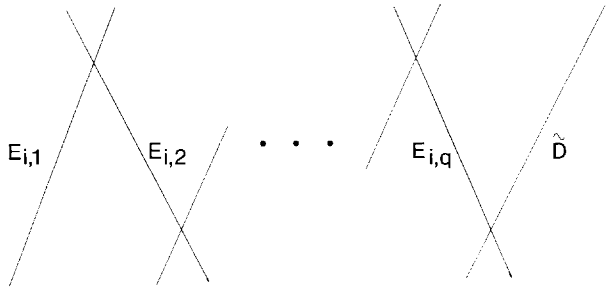


FIGURE 1

Note that $\delta_0 = \delta^{q+1} = p$, $\delta_{j-1} = a_j \delta_j - \delta_{j+1}$ for $1 \leqslant j \leqslant q-1$ and $\delta^{j+1} = a_j \delta^j - \delta^{j-1}$ for $2 \leqslant j \leqslant q$.

We are now ready to write down the canonical divisor of $\mathfrak{Y}$.

THEOREM 4.7. *We retain the same notations and assumptions as above. Then we have*

$$K_{\mathfrak{Y}} = N\tilde{D} + \sum_{j=1}^{q} (\Delta_j + N\delta^j/p)(E_{1,j} + \cdots + E_{d,j}),$$

where $N = (-3 + m(p-1))\,p - (p-1)(e-1)$ *and* $\Delta_j = (1/p)(\delta_j + \delta^j) - 1$ *for* $1 \leqslant j \leqslant q$.

Proof. Note that $(\pi \circ \rho)^{-1}(Q_i)$ and $(\pi \circ \rho)^{-1}(Q_j)$ are disjoint from each other if $i \neq j$. Hence we have only to describe $K_{\mathfrak{Y}}$ near $(\pi \circ \rho)^{-1}(Q_i)$ for every $1 \leqslant i \leqslant d$. Near $(\pi \circ \rho)^{-1} Q_i$, the canonical divisor is written as $K_{\mathfrak{Y}} = N\tilde{D} + n_1 E_{i,1} + \cdots + n_q E_{i,q}$ with $n_j \in \mathbb{Z}$. By applying the adjunction formula for the curves $E_{i,j}$, we have

$$\begin{aligned} a_1(n_1+1) - (n_2+1) &= 1 \\ -(n_1+1) + a_2(n_2+1) - (n_3+1) &= 0 \\ &\;\vdots \\ -(n_{q-1}+1) + a_q(n_q+1) \qquad &= N+1. \end{aligned}$$

We can solve these equations in terms of δ_j and δ^j to obtain $n_j + 1 = (1/p)(\delta_j + \delta^j(N+1))$, $1 \leqslant j \leqslant q$. We have thus the stated formula for $K_{\mathfrak{Y}}$.

Q.E.D.

It is easy to verify that $(\tilde{D}^2) = -\delta^q m + (1 + \delta^q e)/p$. Furthermore, we have the following corollaries which are immediate consequences of the above theorem.

COROLLARY 4.8. *We have*

$$(K_{\mathfrak{Y}}^2)=(1/p)[N^2-(mp-e)\{\delta_1+\delta^q+2-2p+(a_1+\cdots+a_q)\,p-2pq\}],$$

where $N^2=\{(e+2)p-(e-1)\}^2-2(p-1)\,p\{(e+2)p-(e-1)\}m+p^2(p-1)^2m^2$.

COROLLARY 4.9. *If* $e=p-1$, *then we have*

$$K_{\mathfrak{Y}}=(mp(p-1)-p^2-2)\tilde{D}+(m(p-1)-p-1)(E_{1,1}+\cdots+E_{d,1}),$$

where $(\tilde{D}^2)=-m+1$ *and* $((E_{i,1})^2)=-p$. *Moreover,*

$$(K_{\mathfrak{Y}}^2)=p^3+p^2-p+8-(2p-1)\,p^2m+p(p-1)^2m^2.$$

We have the following topological datum of $\mathfrak{Y}$:

PROPOSITION 4.10. *Suppose that* $\varphi\colon\mathfrak{X}\to\mathbb{A}^2$ *satisfies Hypothesis* II. *If* $d=mp-e\gg 0$ *with e fixed, i.e.,* $m\gg 0$, *then we have* $b_1(\mathfrak{Y})=0$, *where* $b_1(\mathfrak{Y})$ $(=\text{rank } H^1_{et}(\mathfrak{Y},\mathbb{Q}_l)$ *with a prime* $l\neq p)$ *is the first Betti number of* $\mathfrak{Y}$.

Proof. It is sufficient to show that the Albanese variety Alb($\mathfrak{Y}$) has dimension zero. Consider the Albanese map $\gamma\colon\mathfrak{Y}\to\text{Alb}(\mathfrak{Y})$ and take the Stein factorization $\gamma=\nu\circ\mu\colon\mathfrak{Y}\xrightarrow{\mu}\mathfrak{A}\xrightarrow{\nu}\text{Alb}(\mathfrak{Y})$, where $\mathfrak{A}$ is normal. Note that $K_{\mathfrak{Y}}$ is connected and effective if $m\gg 0$ (cf. Theorem 4.7). Suppose that $\dim\mathfrak{A}=2$. Then μ is a birational morphism. Since every irreducible component of $K_{\mathfrak{Y}}$ is a rational curve and ν is a finite morphism, $K_{\mathfrak{Y}}$ is contracted to a point by μ. Hence $(K_{\mathfrak{Y}}^2)<0$. On the other hand, we know that $(K_{\mathfrak{Y}}^2)>0$ for $m\gg 0$ by Corollary 4.8. Hence, if $m\gg 0$, this case cannot occur. Next, suppose $\dim\mathfrak{A}=1$. Then μ is a fibration such that $K_{\mathfrak{Y}}$ is contained in a fibre of μ. Hence $(K_{\mathfrak{Y}}^2)\leqslant 0$ and this case cannot occur, either, if $m\gg 0$. So, $\dim\text{Alb}(\mathfrak{Y})=0$ and $b_1(\mathfrak{Y})=0$ if $m\gg 0$. Q.E.D.

We know that $H^1(\mathfrak{Y},\mathcal{O}_{\mathfrak{Y}})=0$ for $e=0$ (resp. $e=1$) if $\mathfrak{X}$ satisfies Hypothesis I (resp. II). Hence, in these cases, we have $b_1(\mathfrak{Y})=0$ as well. From this and Proposition 4.10, it follows that the first Betti number of $\mathfrak{Y}$ is zero provided $f(s,t)$ is general and $d=\deg f(s,t)\gg 0$.

We shall give some examples.

EXAMPLE 4.11. Suppose that $\varphi\colon\mathfrak{X}\to\mathbb{A}^2$ satisfies Hypothesis II and $d=1$ (i.e., $m=1$ and $e=p-1$). By Theorems 4.2 and 4.7, we have $\chi(\mathcal{O}_{\mathfrak{Y}})=1$, $K_{\mathfrak{Y}}=-(p+2)\tilde{D}-2E$, $(E^2)=-p$, and $(K_{\mathfrak{Y}}^2)=8$. We know that $\mathfrak{Y}$ is the Hirzebruch surface of degree p with the minimal section E.

EXAMPLE 4.12. Assume char $k=3$. Suppose that $\mathfrak{X}$ satisfies Hypothesis II and $d=4$ (i.e., $m=2$ and $e=2$). Then we have $\chi(\mathcal{O}_{\mathfrak{Y}})=2$, $K_{\mathfrak{Y}}=\tilde{D}$, and $(K_{\mathfrak{Y}}^2)=-1$. Let $\theta\colon \mathfrak{Y}\to\mathfrak{Z}$ be the contraction of $\tilde{D}$. Then $\mathfrak{Z}$ is a $K3$-surface.

5. IRREGULARITY OF $\mathfrak{Y}$ IN THE CASE OF $e=p-1$

In this section, we assume that $p>2$. We shall prove

THEOREM 5.1. *Suppose that* $\varphi\colon\mathfrak{X}\to\mathbb{A}^2$ *satisfies Hypothesis* II *and* $e=p-1$. *Then we have* $H^1(\mathfrak{Y},\mathcal{O}_{\mathfrak{Y}})=0$ *for* $m\geqslant p-1$.

Our proof of this theorem consists of three steps.

STEP I. CLAIM. *It is sufficient to show that*

$$H^1(\mathbb{P}^2,\pi_*\mathcal{O}_{\mathfrak{X}}\otimes\mathcal{O}_{\mathbb{P}^2}(m(p-1)-p-1))=0\qquad\textit{for}\quad m\geqslant p-1.$$

Proof of Claim. Note that $K_{\mathfrak{Y}}-(p-2)\tilde{D}=(\pi\circ\rho)^*((m(p-1)-p-1)B)$ by Corollary 4.9. Hence we have the exact sequence

$$0\to(\pi\circ\rho)^*\mathcal{O}_{\mathbb{P}^2}(m(p-1)-p-1)\to\omega_{\mathfrak{Y}}\to\mathcal{O}_{(p-2)\tilde{D}}\otimes\omega_{\mathfrak{Y}}\to 0.\qquad(*)$$

On the other hand, it is easy to verify that $(K_{\mathfrak{Y}},\tilde{D})=m-3$. So, we have the exact sequences

$$\begin{array}{ccccccccc} & & & & & & \mathcal{O}_{\tilde{D}}(m-3) & & \\ & & & & & & \| & & \\ 0 & \longrightarrow & \mathcal{O}_{\tilde{D}}(2m-4) & \longrightarrow & \mathcal{O}_{2\tilde{D}}\otimes\omega_{\mathfrak{Y}} & \longrightarrow & \mathcal{O}_{\tilde{D}}\otimes\omega_{\mathfrak{Y}} & \longrightarrow & 0 \\ & & \vdots & & \vdots & & \vdots & & \\ 0 & \to & \mathcal{O}_{\tilde{D}}((p-2)m-p) & \to & \mathcal{O}_{(p-2)\tilde{D}}\otimes\omega_{\mathfrak{Y}} & \to & \mathcal{O}_{(p-3)\tilde{D}}\otimes\omega_{\mathfrak{Y}} & \to & 0. \end{array}$$

By taking the cohomology groups, we have $H^1(\mathfrak{Y},\mathcal{O}_{(p-2)\tilde{D}}\otimes\omega_{\mathfrak{Y}})=0$ whenever $m>1$. Therefore, for $m>1$, the cohomology exact sequence associated with $(*)$ gives exact sequences

$$H^1(\mathfrak{Y},(\pi\circ\rho)^*\mathcal{O}_{\mathbb{P}^2}(m(p-1)-p-1))\to H^1(\mathfrak{Y},\omega_{\mathfrak{Y}})\to 0,$$
$$0\to H^2(\mathfrak{Y},(\pi\circ\rho)^*\mathcal{O}_{\mathbb{P}^2}(m(p-1)-p-1))\to H^2(\mathfrak{Y},\omega_{\mathfrak{Y}})\to 0.$$

So, if we know that $H^1(\mathfrak{Y},(\pi\circ\rho)^*\mathcal{O}_{\mathbb{P}^2}(m(p-1)-p-1))=0$, then our theorem is verified by virtue of the Serre duality. Meanwhile, since π is a finite morphism and ρ is a resolution of rational singularities, we know that $H^i(\mathfrak{Y},(\pi\circ\rho)^*\mathcal{O}_{\mathbb{P}^2}(m(p-1)-p-1))=H^i(\mathbb{P}^2,\pi_*\mathcal{O}_{\mathfrak{X}}\otimes\mathcal{O}_{\mathbb{P}^2}(m(p-1)-p-1))$. Our claim will follow from these observations. Q.E.D.

In the above proof, we have also proved that

$$\dim_k H^2(\mathbb{P}^2, \pi_* \mathcal{O}_{\mathfrak{X}} \otimes \mathcal{O}_{\mathbb{P}^2}(m(p-1)-p-1)) = 1 \qquad \text{for} \quad m > 1.$$

For the sake of simplicity, we shall denote $\mathcal{O}_{\mathbb{P}^2}(m(p-1)-p-1)$ by $\mathcal{M}$.

Step II. Claim. *We have*:

(1) $\dim_k H^1(\mathbb{P}^2, \psi_* \mathcal{O}_{\mathfrak{X}} \otimes \mathcal{M}) = 0$.

(2) $\dim_k H^2(\mathbb{P}^2, \psi_* \mathcal{O}_{\mathfrak{X}} \otimes \mathcal{M}) = p(p-1)/2 \qquad$ *if* $\qquad m \geq p-1$.

Proof of Claim. By Remark 1.5.1, there are the following exact sequences on $\mathbb{P}^2$:

$$\begin{array}{ccccccccc}
 & & \mathcal{M} & & & & & & \\
 & & \| & & & & & & \\
0 & \longrightarrow & \mathcal{F}_0 \otimes \mathcal{M} & \longrightarrow & \mathcal{F}_1 \otimes \mathcal{M} & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(m(p-2)-p-1) & \longrightarrow & 0 \\
 & & \vdots & & \vdots & & \vdots & & \\
0 & \longrightarrow & \mathcal{F}_{p-2} \otimes \mathcal{M} & \longrightarrow & \mathcal{F}_{p-1} \otimes \mathcal{M} & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(-p-1) & \longrightarrow & 0. \\
 & & & & \| & & & & \\
 & & & & \psi_* \mathcal{O}_{\mathfrak{X}} \otimes \mathcal{M} & & & &
\end{array}$$

By taking the cohomology groups, we have $\dim_k H^1(\mathbb{P}^2, \psi_* \mathcal{O}_{\mathfrak{X}} \otimes \mathcal{M}) = 0$. Moreover, $\dim_k H^2(\mathbb{P}^2, \psi_* \mathcal{O}_{\mathfrak{X}} \otimes \mathcal{M}) = \dim_k H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-p-1)) = p(p-1)/2$ for $m \geq p-1$. Q.E.D.

Step III. We consider the exact sequence

$$0 \to \psi_* \mathcal{O}_{\mathfrak{X}} \otimes \mathcal{M} \to \pi_* \mathcal{O}_{\mathfrak{X}} \otimes \mathcal{M} \to \mathscr{H} \otimes \mathcal{M} \to 0. \qquad (**)$$

Since

$$\begin{aligned}
\mathscr{H} \otimes \mathcal{M} = \mathcal{O}_B(m(p-3)-p) \oplus \mathcal{O}_B(m(p-4)-p) \quad & \oplus \cdots \oplus \mathcal{O}_B(-p) \\
\oplus \mathcal{O}_B(m(p-4)-p+1) & \oplus \cdots \oplus \mathcal{O}_B(-p+1) \\
& \ddots \vdots \\
& \oplus \mathcal{O}_B(-3),
\end{aligned}$$

we have $\dim H^1(B, \mathscr{H} \otimes \mathcal{M}) = \dim H^1(B, \mathcal{O}_B(-p)) + \cdots + \dim H^1(B, \mathcal{O}_B(-3))$

$= p(p-1)/2 - 1$ if $m \geqslant p-1$. By taking the cohomology exact sequence associated with $(**)$, we have an exact sequence

$$H^1(\mathbb{P}^2, \psi_* \mathcal{O}_{\tilde{X}} \otimes \mathcal{M}) \to H^1(\mathbb{P}^2, \pi_* \mathcal{O}_{\tilde{X}} \otimes \mathcal{M}) \to H^1(B, \mathcal{H} \otimes \mathcal{M})$$
$$\to H^2(\mathbb{P}^2, \psi_* \mathcal{O}_{\tilde{X}} \otimes \mathcal{M}) \to H^2(\mathbb{P}^2, \pi_* \mathcal{O}_{\tilde{X}} \otimes \mathcal{M}) \to 0.$$

We already know that $\dim H^2(\mathbb{P}^2, \pi_* \mathcal{O}_{\tilde{X}} \cdot \mathcal{M}) = 1$, $H^1(\mathbb{P}^2, \psi_* \mathcal{O}_{\tilde{X}} \otimes \mathcal{M}) = 0$, and $\dim H^2(\mathbb{P}^2, \psi_* \mathcal{O}_{\tilde{X}} \otimes \mathcal{M}) = p(p-1)/2$ if $m \geqslant p-1$. Hence it follows that $\dim H^1(\mathbb{P}^2, \pi_* \mathcal{O}_{\tilde{X}} \otimes \mathcal{M}) = 0$ if $m \geqslant p-1$. The proof of our theorem is now completed.

Acknowledgments

The author expresses his sincere gratitude to Professors M. Miyanishi, S. Tsunoda, and K. Nishiguchi for their useful advice.

References

1. T. Kambayashi and V. Srinivas, On étale coverings of the affine space, *in* "Algebraic Geometry-Proceedings of the Third Midwest Algebraic Geometry Conference," pp. 75–82, Lecture Notes in Mathematics, Vol. 1008, Springer-Verlag, Berlin/Heidelberg/New York/Tokyo, 1983.
2. M. Miyanishi, p-Cyclic coverings of the affine space, *J. Algebra* **63** (1980), 279–284.
3. M. Miyanishi and P. Russell, Purely inseparable coverings of exponent one of the affine plane, *J. Pure Appl. Algebra* **28** (1983), 279–317.
4. Y. Takeda, Artin–Schreier coverings of algebraic surfaces, *J. Math. Soc. Japan* **41** (1989), 415–435.