Linear Algebra and its Applications 437 (2012) 2773–2780

Contents lists available at SciVerse ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

Terwilliger algebras of wreath products by quasi-thin schemes ☆

Kijung Kim

Department of Mathematics, Pusan National University, Busan 609-735, Republic of Korea

ARTICLE INFO

Article history:
Received 14 September 2011
Accepted 1 July 2012
Available online 28 July 2012

Submitted by H. Schneider

AMS classification:
05E15
05E30

Keywords:
Terwilliger algebra
Wreath product
Quasi-thin scheme

ABSTRACT

The structure of Terwilliger algebras of wreath products by thin schemes or one-class schemes was studied in [A. Hanaki, K. Kim, Y. Maekawa, Terwilliger algebras of direct and wreath products of association schemes, J. Algebra 343 (2011) 195–200]. In this paper, we will consider the structure of Terwilliger algebras of wreath products by quasi-thin schemes. This gives a generalization of their result.

1. Introduction

The Terwilliger algebra is a new algebraic tool for the study of association schemes introduced by Terwilliger in [10–12]. In general, this algebra is a non-commutative, finite dimensional, and semisimple $\mathbb{C}$-algebra. In the theory of association schemes, the wreath product is a method to construct new association schemes. Recently, Bhattacharyya, Song and Tanaka began to study Terwilliger algebras of wreath products of one-class association schemes in [1]. In particular, Song and Xu gave a complete structural description of Terwilliger algebras for wreath products of one-class association schemes in [9]. Terwilliger algebras of wreath products by thin schemes or one-class schemes were studied in [4]. In this paper, we give a generalization of their result by replacing thin schemes with quasi-thin schemes.

The remainder of this paper is organized as follows. In Section 2, we review notations and basic results on coherent configurations and Terwilliger algebras as well as important results on quasi-thin

☆ This work was supported by the National Research Foundation of Korea Grant funded by the Korean Government (Ministry of Education, Science and Technology) [NRF-2010-355-C00002].

E-mail address: knukkj@pusan.ac.kr

0024-3795/$ - see front matter © 2012 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.laa.2012.07.005

schemes. In Section 3, based on the fact that one point extensions of quasi-thin schemes coincide with their Terwilliger algebras, we determine all central primitive idempotents of Terwilliger algebras of wreath products by quasi-thin schemes. In Section 4, we state our main theorem.

2. Preliminaries

In this section, to unify notations and terminologies given in [2,4,7,8], we combine them. We assume that the reader is familiar with the basic notions of association schemes in [13].

2.1. Coherent configurations and coherent algebras

Let X be a finite set and S a partition of $X \times X$. Put by $S^{\cup}$ the set of all unions of the elements of S. A pair $\mathcal{C} = (X, S)$ is called a *coherent configuration* on X. if the following conditions hold:

(1) $1_X := \{(x, x) \mid x \in X\} \in S^{\cup}$.
(2) For $s \in S$, $s^* := \{(y, x) \mid (x, y) \in s\} \in S$.
(3) For all $s, t, u \in S$ and all $x, y \in X$,

$$p_{st}^{u} := |\{z \in X \mid (x, z) \in s, \ (z, y) \in t\}|$$

is constant whenever $(x, y) \in u$.

The elements of X, S and $S^{\cup}$ are called the *points*, the *basic relations* and the *relations*, respectively. The numbers $|X|$ and $|S|$ are called the *degree* and *rank*. Any set $\Delta \subseteq X$ for which $1_{\Delta} \in S$ is called the *fiber*. The set of all fibers is denoted by $\mathrm{Fib}(\mathcal{C})$. The coherent configuration $\mathcal{C}$ is called *homogeneous* or a *scheme* if $1_X \in S$. If Y is a union of fibers, then the *restriction* of $\mathcal{C}$ to Y is defined to be a coherent configuration

$$\mathcal{C}_Y = (Y, S_Y),$$

where S_Y is the set of all non-empty relations $s \cap (Y \times Y)$ with $s \in S$. For $s \in S$, let σ_s denote the matrix in $\mathrm{Mat}_X(\mathbb{C})$ that has entries

$$(\sigma_s)_{xy} = \begin{cases} 1 & \text{if } (x, y) \in s; \\ 0 & \text{otherwise.} \end{cases}$$

We call σ_s the *adjacency matrix* of $s \in S$. Then $\bigoplus_{s \in S} \mathbb{C}\sigma_s$ becomes a subalgebra of $\mathrm{Mat}_X(\mathbb{C})$. We call $\bigoplus_{s \in S} \mathbb{C}\sigma_s$ the *adjacency algebra* of S, and denote it by $\mathcal{A}(S)$. A linear subspace $\mathcal{A}$ of $\mathrm{Mat}_X(\mathbb{C})$ is called a *coherent algebra* if the following conditions hold:

(1) $\mathcal{A}$ contains the identity matrix I_X and the all-one matrix J_X.
(2) $\mathcal{A}$ is closed with respect to the ordinary and Hadamard multiplications.
(3) $\mathcal{A}$ is closed with respect to transposition.

Let B be the set of primitive idempotents of $\mathcal{A}$ with respect to the Hadamard multiplication. Then B is a linear basis of $\mathcal{A}$ consisting of $\{0, 1\}$-matrices such that

$$\sum_{s \in B} \sigma_s = J_X \quad \text{and} \quad \sigma_s \in B \Leftrightarrow \sigma_s^t \in B.$$

Remark 2.1. There are bijections between the sets of coherent configurations and coherent algebras as follows:

$$S \mapsto \mathcal{A}(S) \quad \text{and} \quad \mathcal{A} \mapsto \mathcal{C}(\mathcal{A}),$$

where $\mathcal{C}(\mathcal{A}) = (X, S')$ with $S' = \{s \in X \times X \mid \sigma_s \in B\}$.

Let $\mathcal{C} = (X, S)$ be a coherent configuration. For each $x \in X$, we define $xs := \{y \in X \mid (x, y) \in s\}$. A point $x \in X$ is called *regular* if

$$|xs| \leqslant 1, \quad s \in S.$$

In particular, if the set of all regular points is non-empty, then $\mathcal{C}$ is called *1-regular*.

Let $\mathcal{C} = (X, S)$ be a scheme and T a closed subset containing the thin residue of S. Put by $S_{(T)}$ the set of all basic relations $s_{\Delta,\Gamma} := s \cap (\Delta \times \Gamma)$, where $s \in S$ and $\Delta, \Gamma \in X/T := \{xT \mid x \in X\}$. Then by [3] or [7], the pair $\mathcal{C}_{(T)} = (X, S_{(T)})$ is a coherent configuration called the *thin residue extension* of $\mathcal{C}$.

2.2. Terwilliger algebras and one point extensions

Let (X, S) be a scheme. For $U \subseteq X$, we denote by ε_U the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ with entries $(\varepsilon_U)_{xx} = 1$ if $x \in U$ and $(\varepsilon_U)_{xx} = 0$ otherwise. Note that $J_{U,V} := \varepsilon_U J_X \varepsilon_V$ and $J_U := J_{U,U}$ for $U, V \subseteq X$.

The *Terwilliger algebra* of (X, S) with respect to $x_0 \in X$ is defined as a subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by $\{\sigma_s \mid s \in S\} \cup \{\varepsilon_{x_0 s} \mid s \in S\}$ (see [10]). The Terwilliger algebra will be denoted by $\mathcal{T}(X, S, x_0)$ or $\mathcal{T}(S)$. Since $\mathcal{A}(S)$ and $\mathcal{T}(S)$ are closed under transposed conjugate, they are semisimple $\mathbb{C}$-algebras. The set of irreducible characters of $\mathcal{T}(S)$ and $\mathcal{A}(S)$ will be denoted by $\mathrm{Irr}(\mathcal{T}(S))$ and $\mathrm{Irr}(\mathcal{A}(S))$, respectively. The *trivial character* $1_{\mathcal{A}(S)}$ of $\mathcal{A}(S)$ is a map $\sigma_s \mapsto n_s$, where $n_s := p_{ss^*}^{1_X}$ is called the *valency* of s, and the corresponding central primitive idempotent is $|X|^{-1} J_X$. The *trivial character* $1_{\mathcal{T}(S)}$ of $\mathcal{T}(S)$ corresponds to the central primitive idempotent $\sum_{s \in S} n_s^{-1} \varepsilon_{x_0 s} J_X \varepsilon_{x_0 s}$ of $\mathcal{T}(S)$. For $\chi \in \mathrm{Irr}(\mathcal{A}(S))$ or $\mathrm{Irr}(\mathcal{T}(S))$, e_χ will be the corresponding central primitive idempotent of $\mathcal{A}(S)$ or $\mathcal{T}(S)$. For convenience, we denote $\mathrm{Irr}(\mathcal{A}(S)) \setminus \{1_{\mathcal{A}(S)}\}$ and $\mathrm{Irr}(\mathcal{T}(S)) \setminus \{1_{\mathcal{T}(S)}\}$ by $\mathrm{Irr}(\mathcal{A}(S))^\times$ and $\mathrm{Irr}(\mathcal{T}(S))^\times$, respectively.

Let $\mathcal{C} = (X, S)$ be a coherent configuration and $x \in X$. Denote by S_x the set of basic relations of the smallest coherent configuration on X such that

$$1_x \in S_x \quad \text{and} \quad S \subset S_x^{\cup}.$$

Then the coherent configuration $\mathcal{C}_x = (X, S_x)$ is called a *one point extension* of $\mathcal{C}$. It is easy to see that given $s, t, u \in S$ the set xs and the relation $u_{xs,st}$ are unions of some fibers and some basic relations of $\mathcal{C}_x$, respectively.

Remark 2.2. A one point extension $\mathcal{C}_x$ of a scheme $\mathcal{C}$ is related to $\mathcal{T}(X, S, x)$. In fact, $\mathcal{C}_x \supseteq \mathcal{T}(X, S, x)$.

2.3. Direct sums, direct products and wreath products

Let $\mathcal{C} = (X, S)$ and $\mathcal{C}' = (X', S')$ be coherent configurations. Put by $X \sqcup X'$ the disjoint union of X and X', and by $S \boxplus S'$ the union of the set $S \cup S'$ and the set of all relations $\Delta \times \Delta'$, $\Delta' \times \Delta$, where Δ, Δ' are fibers of $\mathcal{C}$ and $\mathcal{C}'$, respectively. Then the pair

$$\mathcal{C} \boxplus \mathcal{C}' = (X \sqcup X', S \boxplus S')$$

is a coherent configuration called the *direct sum* of $\mathcal{C}$ and $\mathcal{C}'$. Set $S \times S' = \{s \times s' \mid s \in S, s' \in S'\}$, where $s \times s'$ is the relation on $X \times X'$ consisting of all pairs $((\alpha, \alpha'), (\beta, \beta'))$ with $(\alpha, \beta) \in s$ and $(\alpha', \beta') \in s'$. Then the pair

$$\mathcal{C} \times \mathcal{C}' = (X \times X', S \times S')$$

is a coherent configuration called the *direct product* of $\mathcal{C}$ and $\mathcal{C}'$. The adjacency matrix of $s \times s' \in S \times S'$ is given by the Kronecker product $\sigma_s \otimes \sigma_{s'}$.

Let (X, S) and (Y, T) be schemes. For $s \in S$, set $\tilde{s} = \{((x, y), (x', y)) \mid (x, x') \in s,\ y \in Y\}$. For $t \in T$, set $\bar{t} = \{((x, y), (x', y')) \mid x, x' \in X,\ (y, y') \in t\}$. Also set $S \wr T = \{\tilde{s} \mid s \in S\} \cup \{\bar{t} \mid t \in T \setminus \{1_Y\}\}$. Then $(X \times Y, S \wr T)$ is a scheme called the *wreath product* of (X, S) by (Y, T). For the adjacency matrices, we have $\sigma_{\tilde{s}} = \sigma_s \otimes I_Y$, $\sigma_{\bar{t}} = J_X \otimes \sigma_t$. Note that $(x_0, y_0)\tilde{s} = (x_0 s, y_0) = \{(x, y_0) \mid x \in x_0 s\}$,

$(x_0, y_0)\bar{t} = (X, y_0t) = \{(x, y) \mid x \in X,\ y \in y_0t\}$ and $\varepsilon_{(x_0,y_0)\tilde{s}} = \varepsilon_{x_0s} \otimes \varepsilon_{y_01_Y}, \varepsilon_{(x_0,y_0)\bar{t}} = \sum_{s\in S} \varepsilon_{x_0s} \otimes \varepsilon_{y_0t} = I_X \otimes \varepsilon_{y_0t}$.

2.4. Quasi-thin schemes

A scheme $\mathcal{C} = (Y, T)$ is called *quasi-thin* if $T = T_1 \cup T_2$, where T_i is the set of basic relations with valency i ($i \in \{1, 2\}$).

Lemma 2.1 ([6], Lemma 4.1). *For $t \in T_2$, there exists a unique $t^\perp$ such that $tt^* = \{1_Y, t^\perp\}$.*

In [8], any element from the set $T^\perp = \{t^\perp \mid t \in T_2\}$ is called an *orthogonal* of $\mathcal{C}$. If $|T^\perp| = 1$ and $T^\perp \subseteq T_1$, then $H := \{1_Y\} \cup T^\perp$ is the thin residue. Considering $\mathcal{C}_{(H)} = (Y, T_{(H)})$, it follows that given $\Delta, \Gamma \in \mathrm{Fib}(\mathcal{C}_{(H)})$ either the set $\Delta \times \Gamma \in T_{(H)}$ or the set $\Delta \times \Gamma \notin T_{(H)}$. Denote the latter case by $\Delta \sim \Gamma$. Then $\sim$ is an equivalence relation on the set $\mathrm{Fib}(\mathcal{C}_{(H)})$.

Next, we state two results on quasi-thin schemes.

Theorem 2.2 ([8], Theorem 6.1). *Let $\mathcal{C} = (Y, T)$ be a quasi-thin scheme and $y_0 \in Y$. Then $\mathcal{A}(T_{y_0}) = \mathcal{T}(Y, T, y_0)$.*

Theorem 2.3 ([8], Theorem 5.2 and Corollary 6.4). *Let $\mathcal{C} = (Y, T)$ be a quasi-thin scheme with $T^\perp \neq \emptyset$.*

(1) If $|T^\perp| = 1$ and $T^\perp \subseteq T_2$, then

$$T = T_1\{1_Y, u\},$$

where $y_0 \in Y$ and $T^\perp = \{u\}$.

(2) If $|T^\perp| = 1$ and $T^\perp \subseteq T_1$, then

$$\mathcal{C}_{(H)} = \boxplus_{i\in I}\mathcal{C}_i,$$

where I is the set of classes given in the above $\sim$, Y_i is the union of fibers belonging a class $i \in I$, and $\mathcal{C}_i = (\mathcal{C}_{(H)})_{Y_i}$.

(3) If $|T^\perp| \geqslant 2$, then $\mathcal{C}_{y_0} = (Y, T_{y_0})$ is 1-regular. In particular, any point of y_0T_2 is regular.

Example 2.3. Some examples for each case of Theorem 2.3 can be found in [5].

(1) as 12 No. 51
(2) as 12 No. 48
(3) as 28 No. 175, No. 176

3. Wreath products by quasi-thin schemes

Let (X, S) and (Y, T) be schemes. Fix $x_0 \in X$ and $y_0 \in Y$ and consider $(X\times Y, S\wr T)$ and $\mathcal{T}(X\times Y, S\wr T, (x_0, y_0))$. In the rest of this section, we assume that (Y, T) is a quasi-thin scheme with $T^\perp \neq \emptyset$.

3.1. The restriction of $\mathcal{T}(S \wr T)$ to $X \times (Y \setminus \{y_0\})$

$\mathcal{T}(S \wr T)$ is generated by $\{J_X \otimes \sigma_t, I_X \otimes \varepsilon_{y_0t} \mid t \in T \setminus \{1\}\} \cup \{\sigma_s \otimes I_Y, \varepsilon_{x_0s} \otimes \varepsilon_{\{y_0\}} \mid s \in S\}$.

Since $\sum_{s\in S} \varepsilon_{x_0s}\otimes\varepsilon_{\{y_0\}} = I_X\otimes\varepsilon_{\{y_0\}}$ and $\sum_{s\in S} \sigma_s\otimes I_Y = J_X\otimes I_Y$, we consider a subalgebra $\mathcal{U}$ generated by $\{J_X\otimes\sigma_t, I_X\otimes\varepsilon_{y_0t} \mid t \in T\}$. It is easy to see that $\mathcal{U}$ is generated by $\{|X|^{-1}J_X\otimes\varepsilon_{y_0t_1}\sigma_t\varepsilon_{y_0t_2} \mid t_1, t_2 \in T\}$ and isomorphic to $\mathcal{T}(T)$. So by Theorem 2.2, a basis $B(\mathcal{U})$ of $\mathcal{U}$ can be determined by $\mathcal{C}(\mathcal{T}(T))$, i.e. $B(\mathcal{U}) = \{J_X \otimes \sigma_c \mid c \in \mathcal{R}\}$, where $\mathcal{R}$ is the set of basic relations of $\mathcal{C}(\mathcal{T}(T))$.

We consider $\varepsilon_X \otimes \varepsilon_{Y\setminus\{y_0\}} \mathcal{T}(S \wr T) \varepsilon_X \otimes \varepsilon_{Y\setminus\{y_0\}}$. Since $\mathcal{T}(S \wr T)$ is generated by $B(\mathcal{U}) \cup \{\sigma_s \otimes I_Y, \varepsilon_{x_0 s} \otimes \varepsilon_{\{y_0\}} \mid s \in S\}$, $(\varepsilon_X \otimes \varepsilon_{Y\setminus\{y_0\}}) \mathcal{T}(S \wr T)(\varepsilon_X \otimes \varepsilon_{Y\setminus\{y_0\}})$ is generated by $\{J_X \otimes \sigma_c \mid c \in \mathcal{R}_{Y\setminus\{y_0\}}\} \cup \{\sigma_s \otimes I_{Y\setminus\{y_0\}} \mid s \in S\}$. Thus, we can determine a basis of $\varepsilon_X \otimes \varepsilon_{Y\setminus\{y_0\}} \mathcal{T}(S \wr T) \varepsilon_X \otimes \varepsilon_{Y\setminus\{y_0\}}$ with respect to the set of basic relations of $\mathcal{C}(\mathcal{T}(T))_{Y\setminus\{y_0\}}$.

3.2. A basis of $\mathcal{A}(T_{y_0}) = \mathcal{T}(Y, T, y_0)$

By Theorem 2.2, $\mathcal{A}(\mathcal{C}_{y_0}) = \mathcal{T}(Y, T, y_0)$. In order to find a basis of $\mathcal{T}(Y, T, y_0)$, it is enough to know all basic relations of $\mathcal{C}_{y_0}$. In particular, we focus on $\Delta \times \Gamma \in T_{y_0}$ or $\notin T_{y_0}$ for $\Delta, \Gamma \in \mathrm{Fib}(\mathcal{C}_{y_0})$ of size 2.

Lemma 3.1. *If $\mathcal{C} = (Y, T)$ belongs to case (1) or (3) in Theorem 2.3, then $\Delta \times \Gamma \notin T_{y_0}$ for $\Delta, \Gamma \in \mathrm{Fib}(\mathcal{C}_{y_0})$ of size 2.*

Proof. In the case of Theorem 2.3(1), each $t \in T_2$ is represented by $t_1 u$ for some $t_1 \in T_1$. For $t, t' \in T_2$, $\sigma_t \sigma_{t'^*} = \sigma_{t_1} \sigma_u \sigma_{u^*} \sigma_{t_1'^*} = \sigma_{t_1}(2\sigma_{1_Y} + \sigma_u)\sigma_{t_1'^*} = 2\sigma_{t_1 t_1'} + \sigma_{t_1}\sigma_u \sigma_{t_1'^*}$. So the coefficient of $\sigma_{t_1}\sigma_u\sigma_{t_1'^*}$ implies that $\Delta \times \Gamma \notin T_{y_0}$ for $\Delta, \Gamma \in \mathrm{Fib}(\mathcal{C}_{y_0})$ of size 2.

In the case of Theorem 2.3(3), clearly $\Delta \times \Gamma \notin T_{y_0}$ for $\Delta, \Gamma \in \mathrm{Fib}(\mathcal{C}_{y_0})$ of size 2. $\square$

Lemma 3.2. *Suppose that $\mathcal{C} = (Y, T)$ belongs to case (2) in Theorem 2.3. For distinct $i_1, i_2 \in I$, if fibers $\Delta \subseteq Y_{i_1}$ and $\Gamma \subseteq Y_{i_2}$, then $\Delta \times \Gamma \in T_{y_0}$.*

Proof. First, we show that $(C_{(H)})_{Y'} = (C_{y_0})_{Y'}$, where $Y' = \cup_{t \in T_2} y_0 t$. Since $\mathcal{A}(\mathcal{C}_{y_0}) = \mathcal{T}(T) = \langle \{\varepsilon_{y_0 t_1} \sigma_t \varepsilon_{y_0 t_2} \mid t_1, t_2, t \in T\} \rangle$, $\varepsilon_{Y'} \mathcal{A}(\mathcal{C}_{y_0}) \varepsilon_{Y'} = \langle \{\varepsilon_{y_0 t_1} \sigma_t \varepsilon_{y_0 t_2} \mid t_1, t_2 \in T_2, t \in T\} \rangle$. By thin residue extension, $\mathcal{A}(\mathcal{C}_{(H)}) = \langle \{\varepsilon_\Delta \sigma_t \varepsilon_{\Delta t} \mid \Delta \in Y/H, t \in T\} \rangle$. So we have $\varepsilon_{Y'} \mathcal{A}(\mathcal{C}_{(H)}) \varepsilon_{Y'} = \langle \{\varepsilon_\Delta \sigma_t \varepsilon_{\Delta t} \mid \Delta \in Y/H, t \in T, \Delta = y_0 t', \Delta t = y_0 t'' \text{ for some } t', t'' \in T_2\} \rangle$. Thus, $(C_{(H)})_{Y'} = (C_{y_0})_{Y'}$.

Now we consider $(C_{y_0})_{Y'}$. Note that $Y' = \cup_{i \in I \setminus \{i_0\}} Y_i$ and $Y_{i_0} = \cup_{t \in T_1} y_0 t$, where i_0 is a class of I such that $y_0 \in Y_{i_0}$. Since $\mathcal{C}_{(H)} = \boxplus_{i \in I} \mathcal{C}_i$ and $(C_{(H)})_{Y'} = (C_{y_0})_{Y'}$, if fibers $\Delta \subseteq Y_{i_1}$ and $\Gamma \subseteq Y_{i_2}$ for distinct $i_1, i_2 \in I \setminus \{i_0\}$, then $\Delta \times \Gamma \in T_{y_0}$. Clearly, for i_0 and $i_1 \in I \setminus \{i_0\}$, if fibers $\Delta \subseteq Y_{i_0}$ and $\Gamma \subseteq Y_{i_1}$, then $\Delta \times \Gamma, \Gamma \times \Delta \in T_{y_0}$. $\square$

3.3. Central primitive idempotents of $\mathcal{T}(X \times Y, S \wr T, (x_0, y_0))$

Set $F^{(t)} = (x_0, y_0)\bar{t} = (X, y_0 t)$ and $U^{(t)} = (S \wr T)_{(x_0, y_0)\bar{t}}$ for $t \in T$. If $t \in T_1$, then $(F^{(t)}, U^{(t)})$ is isomorphic to (X, S). If $t \in T_2$, then $(F^{(t)}, U^{(t)})$ is isomorphic to the wreath product of (X, S) by the trivial scheme of degree 2.

For $\chi \in \mathrm{Irr}(\mathcal{T}(U^{(1_Y)}))^\times$, define

$$\tilde{e}_\chi = e_\chi \otimes \varepsilon_{\{y_0\}} \in \mathcal{T}(S \wr T).$$

For $t \in T_1 \setminus \{1_Y\}$ and $\varphi \in \mathrm{Irr}(\mathcal{A}(U^{(t)}))^\times$, define

$$\bar{e}_\varphi = e_\varphi \otimes \varepsilon_{y_0 t} \in \mathcal{T}(S \wr T).$$

For $t \in T_2$ and $\psi \in \mathrm{Irr}(\mathcal{A}(S))^\times$, define

$$\hat{e}_\psi = e_\psi \otimes \varepsilon_{y_0 t} \in \mathcal{T}(S \wr T).$$

Then they are idempotents of $\mathcal{T}(S \wr T)$.

Lemma 3.3 ([4], Lemma 4.2 and 4.4). *For $\chi \in \mathrm{Irr}(\mathcal{T}(U^{(1_Y)}))^\times$, $\tilde{e}_\chi$ is a central primitive idempotent of $\mathcal{T}(S \wr T)$.*

Lemma 3.4 ([4], Lemma 4.3 and 4.4). *For $t \in T_1 \setminus \{1_Y\}$ and $\varphi \in \mathrm{Irr}(\mathcal{A}(U^{(t)}))^{\times}$, $\bar{e}_{\varphi}$ is a central primitive idempotent of $\mathcal{T}(S \wr T)$.*

By mimicking the proof of Lemma 3.4, we get the following lemma.

Lemma 3.5. *For $t \in T_2$ and $\psi \in \mathrm{Irr}(\mathcal{A}(S))^{\times}$, $\hat{e}_{\psi}$ is a central primitive idempotent of $\mathcal{T}(S \wr T)$.*

Proof. First, we show that $\hat{e}_{\psi}$ commutes with $\sigma_s \otimes I_Y, J_X \otimes \sigma_u$ $(u \in T \setminus \{1_Y\})$, $\varepsilon_{x_0 s} \otimes \varepsilon_{\{y_0\}}$, and $I_X \otimes \varepsilon_{y_0 u}$ $(u \in T \setminus \{1_Y\})$. For $s \in S$, $\hat{e}_{\psi}(\sigma_s \otimes I_Y) = \sum_{u \in T} \hat{e}_{\psi}(\sigma_s \otimes \varepsilon_{y_0 u}) = (e_{\psi} \otimes \varepsilon_{y_0 t})(\sigma_s \otimes \varepsilon_{y_0 t})$. Since e_{ψ} commutes with σ_s, we have $\hat{e}_{\psi}(\sigma_s \otimes I_Y) = (\sigma_s \otimes I_Y)\hat{e}_{\psi}$. Since $t \neq 1_Y$, we have $\hat{e}_{\psi}(\varepsilon_{x_0 s} \otimes \varepsilon_{\{y_0\}}) = (\varepsilon_{x_0 s} \otimes \varepsilon_{\{y_0\}})\hat{e}_{\psi} = 0$. Since $e_{1_{\mathcal{A}(S)}} = |X|^{-1} J_X$ and $e_{1_{\mathcal{A}(S)}} e_{\psi} = e_{\psi} e_{1_{\mathcal{A}(S)}} = 0$, we have $\hat{e}_{\psi}(J_X \otimes \sigma_u) = (J_X \otimes \sigma_u)\hat{e}_{\psi} = 0$. Also, $\hat{e}_{\psi}(I_X \otimes \varepsilon_{y_0 u}) = (I_X \otimes \varepsilon_{y_0 u})\hat{e}_{\psi}$ is trivial.

Now we show that $\hat{e}_{\psi}$ is primitive. The map $\pi : \mathcal{T}(S \wr T) \to \hat{e}_{\psi}\mathcal{T}(S \wr T)$ is a projection. Actually, $\hat{e}_{\psi}\mathcal{T}(S \wr T)$ is naturally isomorphic to $e_{\psi}\mathcal{A}(S)$. Since e_{ψ} is a central primitive idempotent of $\mathcal{A}(S)$, $\hat{e}_{\psi}$ is primitive. $\square$

From now on, we define the other central primitive idempotents of $\mathcal{T}(S \wr T)$. Suppose that (Y, T) belongs to case (1) or (3) in Theorem 2.3. We define the following matrices $G_{y_0 t, y_0 t'}$ for $t, t' \in T_2$. Let $G_{y_0 t, y_0 t'} = \frac{1}{2|X|} J_X \otimes (J_{\{y_{t(1)}\},\{y_{t'(1)}\}} + J_{\{y_{t(2)}\},\{y_{t'(2)}\}} - J_{\{y_{t(1)}\},\{y_{t'(2)}\}} - J_{\{y_{t(2)}\},\{y_{t'(1)}\}})$, where $y_0 t = \{y_{t(1)}, y_{t(2)}\}$ and $y_0 t' = \{y_{t'(1)}, y_{t'(2)}\}$. It is easy to see that $\{G_{y_0 t, y_0 t'} \mid t, t' \in T_2\}$ is a linearly independent subset of $\mathcal{T}(S \wr T)$.

Lemma 3.6. *$\langle \{G_{y_0 t, y_0 t'} \mid t, t' \in T_2\} \rangle$ is an ideal. Moreover, $\langle \{G_{y_0 t, y_0 t'} \mid t, t' \in T_2\} \rangle \cong \mathrm{Mat}_{T_2}(\mathbb{C})$.*

Proof. First, we prove that $\sigma_u G_{y_0 t, y_0 t'}, G_{y_0 t, y_0 t'} \sigma_u \in \langle \{G_{y_0 t, y_0 t'} \mid t, t' \in T_2\} \rangle$ for $u \in S \wr T$. Since $(\sigma_u)_{y_0 h, y_0 t} \neq 0$ for some $h \in T_2$ and $y_0 h \times y_0 t \notin \mathcal{C}(\mathcal{T}(S \wr T))$ by Lemma 3.1, we have $\sigma_u G_{y_0 t, y_0 t'} = \pm G_{y_0 h, y_0 t'}$. Similarly, $G_{y_0 t, y_0 t'} \sigma_u \in \langle \{G_{y_0 t, y_0 t'} \mid t, t' \in T_2\} \rangle$ is proved. For $u \in S \wr T$ and $t, t' \in T_2$, clearly $\varepsilon_{(x_0, y_0)u} G_{y_0 t, y_0 t'} = \delta_{(x_0, y_0)u(X \times y_0 t)} G_{y_0 t, y_0 t'} \in \langle \{G_{y_0 t, y_0 t'} \mid t, t' \in T_2\} \rangle$ and $G_{y_0 t, y_0 t'} \varepsilon_{(x_0, y_0)u} \in \langle \{G_{y_0 t, y_0 t'} \mid t, t' \in T_2\} \rangle$. So $\langle \{G_{y_0 t, y_0 t'} \mid t, t' \in T_2\} \rangle$ is an ideal.

Now we prove that $G_{y_0 t, y_0 t'} G_{y_0 t''', y_0 t''} = \delta_{t' t'''} G_{y_0 t, y_0 t''}$. It is enough to show that $G_{y_0 t, y_0 t'} G_{y_0 t', y_0 t''} = G_{y_0 t, y_0 t''}$. By calculation, we have $G_{y_0 t, y_0 t'} G_{y_0 t', y_0 t''} = \frac{1}{2|X|} J_X \otimes (J_{\{y_{t(1)}\},\{y_{t'(1)}\}} + J_{\{y_{t(2)}\},\{y_{t'(2)}\}} - J_{\{y_{t(1)}\},\{y_{t'(2)}\}} - J_{\{y_{t(2)}\},\{y_{t'(1)}\}}) \frac{1}{2|X|} J_X \otimes (J_{\{y_{t'(1)}\},\{y_{t''(1)}\}} + J_{\{y_{t'(2)}\},\{y_{t''(2)}\}} - J_{\{y_{t'(1)}\},\{y_{t''(2)}\}} - J_{\{y_{t'(2)}\},\{y_{t''(1)}\}}) = G_{y_0 t, y_0 t''}$.

Finally, we prove that $\langle \{G_{y_0 t, y_0 t'} \mid t, t' \in T_2\} \rangle \cong \mathrm{Mat}_{T_2}(\mathbb{C})$. For $t, t' \in T_2$, let $e_{tt'}$ be the $|T_2| \times |T_2|$ matrix whose (t, t')-entry is 1 and whose other entries are all zero. Then the linear map $\varphi : \langle \{G_{y_0 t, y_0 t'} \mid t, t' \in T_2\} \rangle \to \mathrm{Mat}_{T_2}(\mathbb{C})$ defined by $\varphi(G_{y_0 t, y_0 t'}) = e_{tt'}$ is an isomorphism. $\square$

Define

$$e_{\eta} = \sum_{t \in T_2} \frac{1}{2|X|} J_X \otimes (\varepsilon_{y_0 t} - \overline{\varepsilon_{y_0 t}}),$$

where $\overline{\varepsilon_{y_0 t}} := J_{\{y_{t(1)}\},\{y_{t(2)}\}} + J_{\{y_{t(2)}\},\{y_{t(1)}\}}$. Then by Lemma 3.6, e_{η} is a central primitive idempotent of $\mathcal{T}(S \wr T)$.

Suppose that (Y, T) belongs to case (2) in Theorem 2.3. Put $y_0 \in Y_{i_0}$. For each $i \in I \setminus \{i_0\}$, we consider the set $U_i := \{t \in T_2 \mid y_0 t \subseteq Y_i\}$. Define the following matrices $G_{y_0 t, y_0 t'}$ for $t, t' \in U_i$. Let $G_{y_0 t, y_0 t'} = \frac{1}{2|X|} J_X \otimes (J_{\{y_{t(1)}\},\{y_{t'(1)}\}} + J_{\{y_{t(2)}\},\{y_{t'(2)}\}} - J_{\{y_{t(1)}\},\{y_{t'(2)}\}} - J_{\{y_{t(2)}\},\{y_{t'(1)}\}})$. According to process in the proof of Lemma 3.6, we can prove that $\langle \{G_{y_0 t, y_0 t'} \mid t, t' \in U_i\} \rangle$ is an ideal. For each $i \in I \setminus \{i_0\}$, define

$$e_{\eta_i} = \sum_{t \in U_i} \frac{1}{2|X|} J_X \otimes (\varepsilon_{y_0 t} - \overline{\varepsilon_{y_0 t}}),$$

where $\overline{\varepsilon_{y_0t}} := J_{\{y_{t(1)}\},\{y_{t(2)}\}} + J_{\{y_{t(2)}\},\{y_{t(1)}\}}$. Then e_{η_i} is a central primitive idempotent of $\mathcal{T}(S \wr T)$. We denote $\sum_{i\in I\setminus\{i_0\}} e_{\eta_i}$ by e_η.

Lemma 3.7. *The sum of $e_{1_{\mathcal{T}(S\wr T)}}$, $\tilde{e}_\chi$'s, $\bar{e}_\varphi$'s, $\hat{e}_\psi$'s and e_η is the identity element.*

Proof. It is easy to see that

$$e_{1_{\mathcal{T}(S\wr T)}} = e_{1_{\mathcal{T}(U^{(1_Y)})}} \otimes \varepsilon_{\{y_0\}} + \sum_{t\in T_1\setminus\{1_Y\}} \frac{1}{|X|}\varepsilon_{F(t)}J_{X\times Y}\varepsilon_{F(t)} + \sum_{t\in T_2} \frac{1}{2|X|}\varepsilon_{F(t)}J_{X\times Y}\varepsilon_{F(t)},$$

$$\sum_{\chi\in \mathrm{Irr}(\mathcal{T}(U^{(1_Y)}))^\times} \tilde{e}_\chi = \varepsilon_{F(1)}I_{X\times Y}\varepsilon_{F(1)} - e_{1_{\mathcal{T}(U^{(1_Y)})}} \otimes \varepsilon_{\{y_0\}},$$

$$\sum_{\varphi\in \mathrm{Irr}(\mathcal{A}(U^{(t)}))^\times} \bar{e}_\varphi = \varepsilon_{F(t)}I_{X\times Y}\varepsilon_{F(t)} - \frac{1}{|X|}\varepsilon_{F(t)}J_{X\times Y}\varepsilon_{F(t)}$$

for each $t \in T_1 \setminus \{1_Y\}$,

$$\sum_{\psi\in \mathrm{Irr}(\mathcal{A}(S))^\times} \hat{e}_\psi = \varepsilon_{F(t)}I_{X\times Y}\varepsilon_{F(t)} - \frac{1}{|X|}(J_{X\times y_{t(1)}} + J_{X\times y_{t(2)}})$$

for each $t \in T_2$,

$$e_\eta = \sum_{t\in T_2} \frac{1}{2|X|} J_X \otimes (\varepsilon_{y_0t} - \overline{\varepsilon_{y_0t}}).$$

Thus, we have

$$e_{1_{\mathcal{T}(S\wr T)}} + \sum_{\chi\in \mathrm{Irr}(\mathcal{T}(U^{(1_Y)}))^\times} \tilde{e}_\chi + \sum_{t\in T_1\setminus\{1_Y\}} \sum_{\varphi\in \mathrm{Irr}(\mathcal{A}(U^{(t)}))^\times} \bar{e}_\varphi + \sum_{t\in T_2} \sum_{\psi\in \mathrm{Irr}(\mathcal{A}(S))^\times} \hat{e}_\psi + e_\eta = I_{X\times Y}. \quad \square$$

4. Main result

In conclusion, we have determined the set of all central primitive idempotents of Terwilliger algebras of wreath products by quasi-thin schemes. Combining Section 3 and Theorem 4.1 of [4] gives the following theorem.

Theorem 4.1. *Let (X, S) and (Y, T) be association schemes. Suppose that (Y, T) is a quasi-thin scheme or a one-class scheme. Fix $x_0 \in X$ and $y_0 \in Y$, and consider the wreath product $(X \times Y, S \wr T)$. Then*

(1) If (Y, T) is a thin scheme or a one-class scheme, then

$$\{e_{1_{\mathcal{T}(S\wr T)}}\} \cup \{\tilde{e}_\chi \mid \chi \in \mathrm{Irr}(\mathcal{T}(U^{(1_Y)}))^\times\} \cup \bigcup_{t\in T\setminus\{1_Y\}} \{\bar{e}_\varphi \mid \varphi \in \mathrm{Irr}(\mathcal{A}(U^{(t)}))^\times\}$$

is the set of all central primitive idempotents of $\mathcal{T}(X \times Y,\ S \wr T,\ (x_0, y_0))$.

(2) If (Y, T) has $T^{\perp} \subseteq T_2$ or $|T^{\perp}| \geqslant 2$, then

$$\begin{aligned}\{e_{1_{\mathcal{T}(S\wr T)}}\} \cup \{\tilde{e}_{\chi} \mid \chi \in \mathrm{Irr}(\mathcal{T}(U^{(1_Y)}))^{\times}\} \\ \cup \bigcup_{t \in T_1 \setminus \{1_Y\}} \{\bar{e}_{\varphi} \mid \varphi \in \mathrm{Irr}(\mathcal{A}(U^{(t)}))^{\times}\} \\ \cup \bigcup_{t \in T_2} \{\hat{e}_{\psi} \mid \psi \in \mathrm{Irr}(\mathcal{A}(S))^{\times}\} \cup \{e_{\eta}\}\end{aligned}$$

is the set of all central primitive idempotents of $\mathcal{T}(X \times Y,\ S \wr T,\ (x_0, y_0))$.

(3) If (Y, T) has $|T^{\perp}| = 1$ and $T^{\perp} \subseteq T_1$, then

$$\begin{aligned}\{e_{1_{\mathcal{T}(S\wr T)}}\} \cup \{\tilde{e}_{\chi} \mid \chi \in \mathrm{Irr}(\mathcal{T}(U^{(1_Y)}))^{\times}\} \\ \cup \bigcup_{t \in T_1 \setminus \{1_Y\}} \{\bar{e}_{\varphi} \mid \varphi \in \mathrm{Irr}(\mathcal{A}(U^{(t)}))^{\times}\} \\ \cup \bigcup_{t \in T_2} \{\hat{e}_{\psi} \mid \psi \in \mathrm{Irr}(\mathcal{A}(S))^{\times}\} \\ \cup \{e_{\eta_i} \mid i \in I \setminus \{i_0\}\}\end{aligned}$$

is the set of all central primitive idempotents of $\mathcal{T}(X \times Y,\ S \wr T,\ (x_0, y_0))$.

Acknowledgements

The author would like to thank an anonymous referee for valuable comments.

References

[1] G. Bhattacharyya, S.Y. Song, R. Tanaka, Terwilliger algebras of wreath products of one-class association schemes, J. Algebraic Combin. 31 (2010) 455–466.
[2] S. Evdokimov, I. Ponomarenko, Permutation group approach to association schemes, European J. Combin. 30 (2009) 1456–1476.
[3] S. Evdokimov, I. Ponomarenko, Schemes of a finite projective plane and their extensions, Algebra and Analysis 21 (2009) 90–132.
[4] A. Hanaki, K. Kim, Y. Maekawa, Terwilliger algebras of direct and wreath products of association schemes, J. Algebra 343 (2011) 195–200.
[5] A. Hanaki, I. Miyamoto, Classification of association schemes of small order, online catalogue. Available from: <http://kissme.shinshu-u.ac.jp/as>.
[6] M. Hirasaka, On quasi-thin association schemes with odd number of points, J. Algebra 240 (2001) 665–679.
[7] M. Klin, M. Muzychuk, C. Pech, A. Woldar, P.-H. Zieschang, Association schemes on 28 points as mergings of a half-homogeneous coherent configuration, European J. Combin. 28 (2007) 1994–2025.
[8] M. Muzychuk, I. Ponomarenko, On quasi-thin association schemes, J. Algebra 351 (2012) 467–489.
[9] S.Y. Song, B. Xu, On wreath products of one-class association schemes, 12 August 2010. Available from: <arXiv:1008.2228v1 [math.CO]>.
[10] P. Terwilliger, The subconstituent algebra of an association scheme I, J. Algebraic Combin. 1 (1992) 363–388.
[11] P. Terwilliger, The subconstituent algebra of an association scheme II, J. Algebraic Combin. 2 (1993) 73–103.
[12] P. Terwilliger, The subconstituent algebra of an association scheme III, J. Algebraic Combin. 2 (1993) 177–210.
[13] P.-H. Zieschang, Theory of Association Schemes, Springer Monographs in Mathematics, Springer, Berlin, 2005.