ELSEVIER

Linear Algebra and its Applications 302–303 (1999) 17–32

LINEAR ALGEBRA AND ITS APPLICATIONS

www.elsevier.com/locate/laa

On the power method in max algebra

Ludwig Elsner [a,*,1], P. van den Driessche [b,2]

[a] *Fakultät für Mathematik, Universität Bielefeld, Postfach 100 131, D-33501 Bielefeld, Germany*

[b] *Department of Mathematics and Statistics, University of Victoria, Victoria, BC, Canada, V8W 3P4*

Received 27 October 1997; accepted 14 July 1998

Submitted by U. Rothblum

Dedicated to Hans Schneider on the occasion of his 70th birthday.

Abstract

The eigenvalue problem for an irreducible nonnegative matrix $A = [a_{ij}]$ in the max algebra system is $A \otimes x = \lambda x$, where $(A \otimes x)_i = \max_j(a_{ij}x_j)$ and λ turns out to be the maximum circuit geometric mean, $\mu(A)$. A power method algorithm is given to compute $\mu(A)$ and eigenvector x. The algorithm is developed by using results on the convergence of max powers of A, which are proved using nonnegative matrix theory. In contrast to an algorithm developed in [4], this new method works for any irreducible nonnegative A, and calculates eigenvectors in a simpler and more efficient way. Some asymptotic formulas relating $\mu(A)$, the spectral radius and norms are also given.

1. Introduction

For an $n \times n$ matrix A, the conventional eigenequation for eigenvalue λ and corresponding eigenvector x is $Ax = \lambda x$. In the max algebra system, for $A = [a_{ij}]$ nonnegative, the eigenequation is $\max_j(a_{ij}x_j) = \lambda x_i$, for $i = 1, 2, \ldots, n$. Our aim

* Corresponding author. Tel.: +49 521 106 4800; fax: +49 521 106 4743; e-mail: elsner@mathematik.uni-bielefeld.de

1 Research partially supported by SFB 343, Diskrete Strukturen in der Mathematik.

2 This research was partially supported by an NSERC Research grant and the University of Victoria Committee on Faculty Research and Travel. E-mail: pvdd@smart.math.uvic.ca.

PII: S 0 0 2 4 - 3 7 9 5 (9 8) 1 0 1 7 1 - 4

is to give a power method algorithm to compute the eigenvalue and corresponding eigenvectors in the max algebra.

To motivate this aim and to place it in context, we need some definitions and notation. As in [2], [14, IV3], the *max algebra* system consists of the set of nonnegative numbers with sum $a \oplus b = \max\{a, b\}$ and the standard product ab for $a, b \geqslant 0$. Thus, for compatible nonnegative matrices A, B, their product is denoted by $A \otimes B$, where $(A \otimes B)_{ik} = \max_j(a_{ij}b_{jk})$. In particular, for a vector $x \geqslant 0$, the product $(A \otimes x)_i = \max_j(a_{ij}x_j)$. The usual associative and distributive laws hold in this algebra.

The *max-plus algebra* system consists of the set $\mathbf{R} \cup \{-\infty\}$ with operations of maximization and addition; see e.g. [1,3,6,9], and, in particular, [4] for a power algorithm. (Note that the term "max-algebra" is sometimes used in the literature (e.g. [3,4]) for what is here called the max-plus algebra.) The max-plus algebra is isomorphic by the exponential map to the max algebra. As illustrated in [2], the max algebra enables us to use powerful tools of nonnegative matrices, monotonicity and analysis. The isomorphism can then be used to give results in the max-plus algebra where the eigenequation arises in the study of discrete event dynamical systems [1].

Throughout Sections 1–3 we assume that $A = [a_{ij}]$ is an $n \times n$ nonnegative matrix. The weighted directed graph $\mathscr{G}(A)$ associated with A has vertex set $\{1, 2, \ldots, n\}$ and an edge (i, j) from i to j with weight a_{ij} if and only if $a_{ij} > 0$. A *circuit* of length k is a sequence of k edges $(i_1, i_2), (i_2, i_3), \ldots, (i_{k-1}, i_k), (i_k, i_1)$, where $i_1, i_2, \ldots, i_k$ are distinct. This has *circuit product* $a_{i_1,i_2}a_{i_2,i_3} \ldots a_{i_k,i_1}$ with the positive kth root as the *circuit geometric mean*. The maximum circuit geometric mean in $\mathscr{G}(A)$ is denoted by $\mu(A)$ and plays an important role. A circuit with circuit geometric mean equal to $\mu(A)$ is called a *critical circuit*, and vertices on critical circuits are called *critical vertices*. Assuming that simultaneous row and column permutations have been performed on A so that the critical vertices are in the leading rows and columns, the *critical matrix* of A, denoted by $A^{\mathrm{C}} = [a^{\mathrm{C}}_{ij}]$, is formed from the principal submatrix of A on the rows and columns corresponding to critical vertices by setting $a^{\mathrm{C}}_{ij} = a_{ij}$ if (i, j) is in a critical circuit, and $a^{\mathrm{C}}_{ij} = 0$ otherwise. Thus the critical graph $\mathscr{G}(A^{\mathrm{C}})$ has vertex set $V^{\mathrm{C}} = \{$critical vertices$\}$.

The following basic spectral results are well known in the max-plus algebra (see e.g. [1,3,6,9,14]), and are proved in [2] for the max algebra.

Theorem 1.1. *Let A be an $n \times n$ nonnegative irreducible matrix. Then there exists a unique eigenvalue $\mu(A)$ that is positive, and a positive eigenvector x, such that $A \otimes x = \mu(A)x$. This eigenvector is unique (up to scalar multiples) if and only if $\mathscr{G}(A^{\mathrm{C}})$ is strongly connected.*

The notation $A^2_\otimes$ means $A \otimes A$, $A^k_\otimes$ means the $\otimes$ kth power of A, and $\sum_\oplus$ is used for $\oplus$ summation. We denote the identity matrix by I, and the matrix with every entry 1 by J, the order is usually clear from the context. Vector e has every entry 1, thus $J = ee^{\mathrm{T}}$. Inequalities are in the component-wise sense, for example, $A \leqslant J$ means that $a_{ij} \leqslant 1$ for all $1 \leqslant i, j \leqslant n$. For x, y nonnegative vectors, $x \otimes y^{\mathrm{T}} = xy^{\mathrm{T}}$, and $x \leqslant y$ implies that $A \otimes x \leqslant A \otimes y$. The following properties are useful in the sequel. For any permutation matrix P, $P \otimes A \otimes P^{\mathrm{T}} = PAP^{\mathrm{T}}$, and for any positive diagonal matrix D, $D \otimes A \otimes D^{-1} = DAD^{-1}$.

The index of imprimitivity h of a nonnegative irreducible matrix A is the number of eigenvalues of modulus $\rho(A)$, where $\rho(A)$ is the spectral radius of A, and if $h = 1$ then A is primitive; see e.g. [5, Section 3.4]. The value h is also equal to the g.c.d. of all circuit lengths of the graph $\mathscr{G}(A)$. In particular, it is known that A^h is the direct sum of h primitive matrices (e.g. [13, p. 82, Corollary 2] or [5, p. 74, Theorem 3.4.5]).

Section 2 is devoted to results on the convergence of the max powers, $A_{\otimes}^{k}$. We find conditions under which $A_{\otimes}^{t+d} = A_{\otimes}^{t}$ for sufficiently large t and fixed d, and study in detail the case $d = 1$, i.e., when the max powers of A converge in finitely many steps. Here d is the cyclicity of A, that is the l.c.m of the indices of imprimitivity of the irreducible blocks of A^{C}; see Eq. (2.14). These results are needed for our power method algorithm. We note that equivalent results in the max-plus algebra are known; nevertheless we choose to give proofs, as our approach using basic tools of nonnegative matrices is quite different. Our results are also more general as we do not need to assume that $A_{\otimes}^{d}$ is irreducible (cf. [1, Theorem 3.112]). Our power method, given in Section 3 is more efficient than the algorithm given for the max-plus algebra in [4, Algorithm 3.1], and mentioned for the first time in [1, pp. 457–458]. Except for irreducibility we need no further assumptions on A, and our method for the calculation of the eigenvectors, being essentially different from [4], is simpler and works in all cases. Finally in Section 4, we give some asymptotic formulas that stress analogies between the standard product, the Hadamard product and the max product, and also between $\rho(A), \mu(A)$, and norms.

2. Results on $A_{\otimes}^{k}$

The following result is needed for the proofs of the main theorems of this section.

Lemma 2.1. *Let*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

be a nonnegative $n \times n$ matrix where $A \leqslant J$, A_{11} is $p \times p$ with $p \geqslant 1$ and each diagonal entry equal to 1, *and $A_{22} < J$. Then $\lim_{k\to\infty} A_{\otimes}^{k}$, denoted by A_{∞}, exists.*

If, in addition, either A is irreducible or $p = n$, then there exists k_0 such that $A_\otimes^k = A_\infty$ for all $k \geqslant k_0$.

Proof. Let

$$A_\otimes^k = \begin{bmatrix} A_{11}^{(k)} & A_{12}^{(k)} \\ A_{21}^{(k)} & A_{22}^{(k)} \end{bmatrix}.$$

The assumption that $A_{11} \geqslant I$, and the identities $A_\otimes^{k+1} = A \otimes A_\otimes^k = A_\otimes^k \otimes A$, give the following inequalities:

$$A_{11}^{(k+1)} = A_{11} \otimes A_{11}^{(k)} \oplus A_{12} \otimes A_{21}^{(k)} \geqslant A_{11}^{(k)}, \tag{2.1}$$

$$A_{12}^{(k+1)} = A_{11} \otimes A_{12}^{(k)} \oplus A_{12} \otimes A_{22}^{(k)} \geqslant A_{12}^{(k)}, \tag{2.2}$$

$$A_{21}^{(k+1)} = A_{21}^{(k)} \otimes A_{11} \oplus A_{22}^{(k)} \otimes A_{21} \geqslant A_{21}^{(k)}. \tag{2.3}$$

Hence the sequences $\{A_{11}^{(k)}\}$, $\{A_{12}^{(k)}\}$, and $\{A_{21}^{(k)}\}$ are each monotonic increasing and bounded above by J. Thus they converge to matrices $\tilde{A}_{11}$, $\tilde{A}_{12}$, and $\tilde{A}_{21}$, respectively.

Define

$$\tilde{A}_\otimes^k = \begin{bmatrix} A_{11}^{(k)} & A_{12}^{(k)} \\ A_{21}^{(k)} & A_{21}^{(k)} \otimes A_{12}^{(k)} \end{bmatrix} \quad \text{and} \quad A_\infty = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{21} \otimes \tilde{A}_{12} \end{bmatrix}.$$

We show that there is an $\epsilon > 0$ such that

$$\tilde{A}_\otimes^k \neq A_\infty \Rightarrow ||\tilde{A}_\otimes^k - A_\infty||_{\mathrm{F}} \geqslant \epsilon, \tag{2.4}$$

where $||.||_{\mathrm{F}}$ is the Frobenius norm. To show this, consider an entry of A_∞ that differs from the corresponding entry of $\tilde{A}_\otimes^k$. As $0 \leqslant \tilde{A}_\otimes^k \leqslant A_\infty$, this entry is positive. Since all entries of $\tilde{A}_\otimes^k$ are finite products of entries of A, they have no accumulation point except zero, so for each positive entry of A_∞ there is a punctured neighbourhood of it containing no other product of entries of A. Take ϵ to be the smallest radius of these neighbourhoods, and Eq. (2.4) follows. The contrapositive of Eq. (2.4) implies that, as $\tilde{A}_\otimes^k \to A_\infty$, there exists q such that

$$A_{11}^{(k)} = \tilde{A}_{11}, \quad A_{12}^{(k)} = \tilde{A}_{12}, \quad A_{21}^{(k)} = \tilde{A}_{21} \quad \text{for all } k \geqslant q, \tag{2.5}$$

In fact it can be proved that $q \leqslant n$, but this is not needed in the sequel. If $p = n$, then $A_\infty = \tilde{A}_{11}$, and the proof is complete with $q = k_0$. We now claim also in the case $p < n$ that A_∞ is the limit of the matrices $A_\otimes^k$. By the above, this is obviously true for the entries in the first p rows and those in the first p columns, so it remains to consider the block in the last $n - p$ rows and columns. Considering the $(1,2)$ block of $A_\otimes^q \otimes A = A_\otimes^{q+1}$, and using Eq. (2.5) gives $(A_\otimes^{q+1})_{12} = \tilde{A}_{12} = \tilde{A}_{11} \otimes A_{12} \oplus \tilde{A}_{12} \otimes A_{22}$. Hence

$$\tilde{A}_{12} \otimes A_{22} \leqslant \tilde{A}_{12}. \tag{2.6}$$

The first step with $k = 1$ of an induction proof of the inequality

$$A_{22}^{(k)} \leqslant \tilde{A}_{21} \otimes \tilde{A}_{12} \oplus (A_{22})_{\otimes}^{k}, \quad \text{for } k \geqslant 1, \tag{2.7}$$

is obvious. To see that Eq. (2.7) holds in general, assume that it is true for k, and consider the case for $k + 1$. Thus, by multiplication and using the induction hypothesis,

$$\begin{aligned} A_{22}^{(k+1)} &= A_{21}^{(k)} \otimes A_{12} \oplus A_{22}^{(k)} \otimes A_{22} \\ &\leqslant \tilde{A}_{21} \otimes \tilde{A}_{12} \oplus (\tilde{A}_{21} \otimes \tilde{A}_{12} \oplus (A_{22})_{\otimes}^{k}) \otimes A_{22} \\ &= \tilde{A}_{21} \otimes \tilde{A}_{12} \oplus \tilde{A}_{21} \otimes \tilde{A}_{12} \otimes A_{22} \oplus (A_{22})_{\otimes}^{k+1} \\ &\leqslant \tilde{A}_{21} \otimes \tilde{A}_{12} \oplus \tilde{A}_{21} \otimes \tilde{A}_{12} \oplus (A_{22})_{\otimes}^{k+1} \quad \text{by (2.6)} \\ &= \tilde{A}_{21} \otimes \tilde{A}_{12} \oplus (A_{22})_{\otimes}^{(k+1)} \end{aligned}$$

as required to show Eq. (2.7) is true for all $k \geqslant 1$. On the other hand, for $k \geqslant 2q$, writing $k = t + s$ with $s, t \geqslant q$,

$$\begin{aligned} A_{22}^{(k)} &= A_{21}^{(s)} \otimes A_{12}^{(t)} \oplus A_{22}^{(s)} \otimes A_{22}^{(t)} \geqslant A_{21}^{(s)} \otimes A_{12}^{(t)} \\ &= \tilde{A}_{21} \otimes \tilde{A}_{12}, \quad \text{from (2.5)}. \end{aligned}$$

Thus for $k \geqslant 2q$, from the above and Eq. (2.7),

$$\tilde{A}_{21} \otimes \tilde{A}_{12} \leqslant A_{22}^{(k)} \leqslant \tilde{A}_{21} \otimes \tilde{A}_{12} \oplus (A_{22})_{\otimes}^{k}.$$

Since $A_{22} < J$, it follows that $\lim_{k\to\infty}(A_{22})_{\otimes}^{k} = 0$, and thus $\lim_{k\to\infty} A_{22}^{(k)} = \tilde{A}_{21} \otimes \tilde{A}_{12}$; completing the claim that $\lim_{k\to\infty} A_{\otimes}^{k} = A_{\infty}$.

If A is irreducible, then due to the fact that the diagonal entries of A_{11} are nonzero, the matrix A is primitive, i.e., in the standard product $A^m > 0$ for some positive integer m, see e.g. [5, Theorem 3.5.2]. From the obvious relation for $n \times n$ nonnegative matrices

$$\frac{AB}{n} \leqslant A \otimes B \leqslant AB$$

and by induction

$$\frac{A^m}{n^{m-1}} \leqslant A_{\otimes}^{m} \leqslant A^m$$

for $A \geqslant 0$ and natural numbers m. So for A primitive it is also true that $A_{\otimes}^{m} > 0$. Thus $\tilde{A}_{21} \otimes \tilde{A}_{12} > 0$, and the same argument as for Eq. (2.4) shows that $A_{22}^{(k)} = \tilde{A}_{21} \otimes \tilde{A}_{12}$ for sufficiently large k, i.e., $k > k_0$. Thus the limit A_{∞} is reached after finitely many steps. $\square$

Note that the last statement of Lemma 2.1 is not in general true for reducible A if $p < n$. This can be seen from the reducible example $A = \mathrm{diag}(1, 2^{-1})$, which satisfies all the assumptions of the first part of Lemma 2.1 with $p = 1$, has $A_\infty = \mathrm{diag}(1, 0)$, but $A_{22}^{(k)} = 2^{-k} \neq 0$ for finite k.

Theorem 2.2 is for irreducible A an important special case (namely $d = 1$) of our Theorem 2.3. It is also the main tool in its proof.

Theorem 2.2. *Assume that A is an $n \times n$ nonnegative matrix with $\mu(A) = 1$, and that its critical matrix A^{C} is the direct sum of primitive matrices. If either A is irreducible or $V^{\mathrm{C}} = \{1, 2, \ldots, n\}$, then $\lim_{k\to\infty} A_\otimes^k$, denoted by A_∞, exists and $A_\otimes^k = A_\infty$ for sufficiently large k.*

Proof. First note that this statement is invariant under diagonal similarities, and simultaneous permutations on rows and columns. Permute the critical vertices so that A^{C} is a direct sum, namely $A^{\mathrm{C}} = A_{11}^{\mathrm{C}} \dot{+} A_{22}^{\mathrm{C}} \dot{+} \cdots \dot{+} A_{rr}^{\mathrm{C}}$, defining a block structure on A. Now by a result of [11], see also [10], A can be scaled by a diagonal similarity so that $a_{ij} \leqslant 1$ for all $i, j = 1, 2, \ldots, n$, with $a_{ij} = 1$ if edge (i, j) is on a critical circuit. As all critical circuits are contained in blocks $A_{11}, A_{22}, \ldots, A_{rr}$, an additional diagonal scaling achieves $a_{ij} < 1$ outside these blocks. So A can be assumed to be in block form

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1r} & A_{1,r+1} \\ \vdots & \ddots & \vdots & \vdots \\ A_{r1} & \cdots & A_{rr} & A_{r,r+1} \\ A_{r+1,1} & \cdots & A_{r+1,r} & A_{r+1,r+1} \end{bmatrix}, \tag{2.8}$$

where $A_{ij} < e_i e_j^{\mathrm{T}}$ with the exception of $i = j = 1, 2, \ldots, r$, and A_{ii}^{C} primitive for $i = 1, 2, \ldots, r$. Here e_i is the vector with every entry 1 and dimension equal to the cardinality of the block A_{ii}. Thus there exists a positive integer p such that

$$A_\otimes^p = \begin{bmatrix} A_{11}^{(p)} & \cdots & A_{1r}^{(p)} & A_{1,r+1}^{(p)} \\ \vdots & \ddots & \vdots & \vdots \\ A_{r1}^{(p)} & \cdots & A_{rr}^{(p)} & A_{r,r+1}^{(p)} \\ A_{r+1,1}^{(p)} & \cdots & A_{r+1,r}^{(p)} & A_{r+1,r+1}^{(p)} \end{bmatrix}$$

has $A_{ii}^{(p)} = e_i e_i^{\mathrm{T}}$ for $i = 1, 2, \ldots, r$, and $A_{ij}^{(p)} < e_i e_j^{\mathrm{T}}$ otherwise. This equality follows from primitivity and monotonicity, namely $0 < (A_{ii}^{\mathrm{C}})_\otimes^p \leqslant (A_{ii})_\otimes^p \leqslant e_i e_i^{\mathrm{T}}$, and the fact that $(A_{ii}^{\mathrm{C}})_\otimes^p$ is a (0,1) matrix. From Lemma 2.1, since A is irreducible or $A_{r+1,r+1}^{(p)} = \emptyset$, there exists k_0 such that $A_\otimes^{kp}$ is constant for all $k \geqslant k_0$. This constant matrix is denoted by A_∞, and can be written as

$$A_\infty = \begin{bmatrix} e_1e_1^{\mathrm{T}} & \tilde{A}_{12} & \cdots & \tilde{A}_{1r} & \tilde{A}_{1,r+1} \\ \tilde{A}_{21} & \ddots & & & \tilde{A}_{2,r+1} \\ \vdots & & \ddots & & \vdots \\ \tilde{A}_{r1} & & & e_re_r^{\mathrm{T}} & \tilde{A}_{r,r+1} \\ \tilde{A}_{r+1,1} & \tilde{A}_{r+1,2} & \cdots & \tilde{A}_{r+1,r} & \tilde{A}_{r+1,r+1} \end{bmatrix}.$$

The equality $A_\infty \otimes A_\infty = A_\infty$ leads to

$$\begin{aligned} &\tilde{A}_{ij} = \alpha_{ij}e_ie_j^{\mathrm{T}} \text{ for } i,j \leqslant r \quad \text{with } \alpha_{ii} = 1,\ \alpha_{ij} < 1 \text{ for } i \neq j, \\ &\tilde{A}_{r+1,j} = \eta_j e_j^{\mathrm{T}}, \quad \tilde{A}_{j,r+1} = e_j\xi_j^{\mathrm{T}} \qquad \text{for } j \leqslant r, \\ &\tilde{A}_{r+1,r+1} = \sum_{j=1}^{r}{}^{\oplus} \eta_j\xi_j^{\mathrm{T}}, \quad \text{where } \eta_j,\ \xi_j \text{ are nonnegative vectors.} \end{aligned} \tag{2.9}$$

For example, when $i,j \leqslant r,\ i \neq j$,

$$\tilde{A}_{ij} = \sum_{k=1}^{r+1}{}^{\oplus} \tilde{A}_{ik} \otimes \tilde{A}_{kj} \geqslant \tilde{A}_{ii} \otimes \tilde{A}_{ij} = e_ie_i^{\mathrm{T}} \otimes \tilde{A}_{ij} \geqslant \tilde{A}_{ij}.$$

This implies $\tilde{A}_{ij} = e_ie_i^{\mathrm{T}} \otimes \tilde{A}_{ij}$, so all rows of $\tilde{A}_{ij}$ are equal. Similarly $\tilde{A}_{ij} \geqslant \tilde{A}_{ij} \otimes \tilde{A}_{jj} = \tilde{A}_{ij} \otimes e_je_j^{\mathrm{T}} \geqslant \tilde{A}_{ij}$, and as above it follows that all columns are equal. So $\tilde{A}_{ij} = \alpha_{ij}e_ie_i^{\mathrm{T}}$ for a number α_{ij}, which is < 1. This gives the first relation of Eq. (2.9). Also,

$$\tilde{A}_{j,r+1} = \sum_{k=1}^{r+1}{}^{\oplus} \tilde{A}_{jk} \otimes \tilde{A}_{k,r+1} \geqslant \tilde{A}_{jj} \otimes \tilde{A}_{j,r+1} = e_je_j^{\mathrm{T}} \otimes \tilde{A}_{j,r+1}. \tag{2.10}$$

As above, equality holds in Eq. (2.10) and thus $\tilde{A}_{j,r+1} = e_j\xi_j^{\mathrm{T}}$, where $\xi_j^{\mathrm{T}} = e_j^{\mathrm{T}} \otimes \tilde{A}_{j,r+1}$. Substituting in the first equation of Eq. (2.10) gives:

$$\begin{aligned} e_j\xi_j^{\mathrm{T}} &= \left\{ \sum_{k=1}^{r}{}^{\oplus} \alpha_{jk}e_je_k^{\mathrm{T}} \otimes e_k\xi_k^{\mathrm{T}} \} \oplus \{ \tilde{A}_{j,r+1} \otimes \tilde{A}_{r+1,r+1} \right\} \\ &\geqslant \sum_{k=1}^{r}{}^{\oplus} \alpha_{jk}e_j\xi_k^{\mathrm{T}} \geqslant e_j\xi_j^{\mathrm{T}}. \end{aligned}$$

So there is equality and

$$e_j\xi_j^{\mathrm{T}} = \sum_{k=1}^{r}{}^{\oplus} \alpha_{jk}e_j\xi_k^{\mathrm{T}}. \tag{2.11}$$

In a similar way, $\tilde{A}_{r+1,j} = \eta_j e_j^{\mathrm{T}}$ for $j \leqslant r$, where $\eta_j = \tilde{A}_{r+1,j} \otimes e_j$. Now the last equation in Eq. (2.9) follows from Lemma 2.1, since $\tilde{A}_{r+1,r+1} = \sum_{j=1}^{r}{}_{\oplus} \tilde{A}_{r+1,j} \otimes \tilde{A}_{j,r+1} = \sum_{j=1}^{r}{}_{\oplus} \eta_j \xi_j^{\mathrm{T}}$.

For $i = 1, 2, \ldots, r$, define $x_i = (\alpha_{1i} e_1^{\mathrm{T}}, \alpha_{2i} e_2^{\mathrm{T}}, \ldots, \alpha_{ri} e_r^{\mathrm{T}}, \eta_i^{\mathrm{T}})^{\mathrm{T}}$, and $y_i = (0, \ldots, 0, e_i^{\mathrm{T}}, 0, \ldots, 0, \xi_i^{\mathrm{T}})^{\mathrm{T}}$. Recall that $\alpha_{ii} = 1$. Consider the $(j, r+1)$ block of $\sum_{k=1}^{r}{}_{\oplus} x_k \otimes y_k^{\mathrm{T}}$. It is given by $\sum_{k=1}^{r}{}_{\oplus} \alpha_{jk} e_j \xi_k^{\mathrm{T}} = e_j \xi_j^{\mathrm{T}}$ from Eq. (2.11). This is also equal to $\tilde{A}_{j,r+1}$, which is the $(j, r+1)$ block of A_∞. By also comparing blocks in columns $1, 2, \ldots, r$ and the $(r+1, r+1)$ block, it can be easily seen from Eq. (2.9) that

$$A_\infty = \sum_{i=1}^{r}{}_{\oplus} x_i \otimes y_i^{\mathrm{T}}. \tag{2.12}$$

Now we claim that $A \otimes A_\infty \geqslant A_\infty$. From $A \otimes A_\infty = A_\infty \otimes A$, it follows that

$$\sum_{i=1}^{r}{}_{\oplus} (A \otimes x_i) \otimes y_i^{\mathrm{T}} = \sum_{i=1}^{r}{}_{\oplus} x_i \otimes (y_i^{\mathrm{T}} \otimes A). \tag{2.13}$$

Define $z_j = (0, \ldots, 0, e_j^{\mathrm{T}}, 0, \ldots, 0)^{\mathrm{T}}$ for $j = 1, 2, \ldots, r$, then $y_i^{\mathrm{T}} \otimes z_j = \delta_{ij}$ and $y_i^{\mathrm{T}} \otimes A \otimes z_j \leqslant 1$ with equality when $i = j$. From Eq. (2.13)

$$\sum_{i=1}^{r}{}_{\oplus} (A \otimes x_i) \otimes y_i^{\mathrm{T}} \otimes z_j = A \otimes x_j = \sum_{i=1}^{r}{}_{\oplus} x_i \otimes (y_i^{\mathrm{T}} \otimes A \otimes z_j) \geqslant x_j,$$

and by Eq. (2.12),

$$A \otimes A_\infty = \sum_{i=1}^{r}{}_{\oplus} A \otimes x_i \otimes y_i^{\mathrm{T}} \geqslant \sum_{i=1}^{r}{}_{\oplus} x_i \otimes y_i^{\mathrm{T}} = A_\infty.$$

Thus our claim is proved, and can be used recursively to give

$$A_\infty \leqslant A \otimes A_\infty \leqslant A_\otimes^2 \otimes A_\infty \leqslant \cdots \leqslant A_\otimes^p \otimes A_\infty = A_\infty.$$

So $A \otimes A_\infty = A_\infty$ and $\lim_{k\to\infty} A_\otimes^k = A_\infty$, and in fact $A_\otimes^k = A_\infty$ for $k \geqslant pk_0$. $\square$

Note that the assumption that A_{ii}^{C} are primitive cannot be dropped in general from Theorem 2.2. For example, consider the matrix

$$A = \begin{bmatrix} 2^{-2} & 1 & 2^{-1} \\ 1 & 2^{-2} & 2^{-1} \\ 2^{-1} & 2^{-1} & 2^{-2} \end{bmatrix},$$

where A_{11} is 2×2 irreducible, but A_{11}^{C} is not primitive. Then

$$A_{\otimes}^{2k} = \begin{bmatrix} 1 & 2^{-2} & 2^{-1} \\ 2^{-2} & 1 & 2^{-1} \\ 2^{-1} & 2^{-1} & 2^{-2} \end{bmatrix}$$

and $A_{\otimes}^{2k+1} = A$. Thus $\lim_{k\to\infty} A_{\otimes}^{k}$ does not exist.

In general A^{C} is the direct sum of irreducible matrices, namely $A^{\mathrm{C}} = A_{11}^{\mathrm{C}} \dot{+} A_{22}^{\mathrm{C}} \dot{+} \cdots \dot{+} A_{rr}^{\mathrm{C}}$ with indices of imprimitivity s_i for $i = 1, 2, \ldots, r$. So defining the *cyclicity* of A by

$$d = \mathrm{l.c.m.}(s_1, s_2, \ldots, s_r), \tag{2.14}$$

then by the results in the Introduction on powers of irreducible matrices $(A^{\mathrm{C}})^d$ is the direct sum of $\sum_{i=1}^{r} s_i$ primitive matrices. We can now formulate a general result concerning the behaviour of the powers of A.

Theorem 2.3. *Assume that $A \geqslant 0$ is irreducible, $\mu(A) = 1$, and let d be defined as in Eq.* (2.14). *Then there exists t_0 such that for all $t \geqslant t_0$*

$$A_{\otimes}^{t+d} = A_{\otimes}^{t}. \tag{2.15}$$

Proof. Firstly, note that, letting $B = A_{\otimes}^{d}$,

$$(A_{\otimes}^{d})^{\mathrm{C}} = B^{\mathrm{C}} = (A^{\mathrm{C}})_{\otimes}^{d},$$

since a product of d factors, each of which is less than or equal to 1, is equal to 1 if and only if each factor is 1. As $(A^{\mathrm{C}})^d / n^{d-1} \leqslant (A^{\mathrm{C}})_{\otimes}^{d} \leqslant (A^{\mathrm{C}})^d$ and $(A^{\mathrm{C}})^d$ is the direct sum of primitive matrices, then so is $(A^{\mathrm{C}})_{\otimes}^{d} = B^{\mathrm{C}}$. Also $\mu(B) = 1$. If B is irreducible, then Theorem 2.2 can be applied directly to B. If B is reducible, then by [13, p. 81,Theorem 9], B is the direct sum of irreducible matrices, and Theorem 2.2 can be applied to each diagonal block. Hence there is τ such that $B_{\otimes}^{\tau} = B_{\otimes}^{\tau+1}$ and thus for $t_0 = \tau d$, $A_{\otimes}^{t_0+d} = A_{\otimes}^{t_0}$. But then Eq. (2.15) holds for all $t \geqslant t_0$. $\square$

We mention again that Theorem 2.2 for A irreducible, is the special case of Theorem 2.3 with $d = 1$. Versions of these theorems under the additional assumption that $A_{\otimes}^{d}$ is irreducible are well-known in the max-plus algebra, see e.g. [1, Theorem 3.108, 3.112], [6, Theorem 4], [7, p. 45]. However, our proofs are different and use standard tools from nonnegative matrix theory. We now use our results for a power method algorithm.

3. Power method in the max algebra

In this section we study a power method for calculating eigenvectors of an irreducible matrix A in the max algebra. It also provides a sequence of upper and lower bounds for the eigenvalue $\mu(A)$.

It turns out that in this situation the results are different from those in the conventional case of the ordinary matrix algebra (see e.g. [16, Ch. 2]), which we nevertheless follow in spirit. In the max algebra, our assumptions are more general than those in [4], and we obtain an eigenvector directly.

In the sequel we make use of the following result.

Lemma 3.1. *Let $A \geqslant 0$ be irreducible, and $x \geqslant 0,\ x \neq 0$.*

$$\text{If } \alpha, \beta \in \mathbf{R}_+ \text{ and } \alpha x \leqslant A \otimes x \leqslant \beta x, \text{ then } \alpha \leqslant \mu(A) \leqslant \beta. \tag{3.1}$$

Proof. Let $y \geqslant 0$ be a left eigenvector of A. Then $y^{\mathrm{T}} \otimes A = \mu(A) y^{\mathrm{T}}$ with $y > 0$ by Theorem 1.1 applied to A^{T} and using the obvious relation $\mu(A) = \mu(A^{\mathrm{T}})$. From $A \otimes x \leqslant \beta x$, we have $\mu(A) y^{\mathrm{T}} \otimes x = y^{\mathrm{T}} \otimes A \otimes x \leqslant \beta y^{\mathrm{T}} \otimes x$ and thus $\mu(A) \leqslant \beta$. The proof of the second statement is similar. □

For $x > 0$ we define $q_i(x) = (A \otimes x)_i / x_i$ and

$$\bar{q}(x) = \max_i q_i(x) = \max_i \max_j a_{ij} \frac{x_j}{x_i}$$

$$\underline{q}(x) = \min_i q_i(x) = \min_i \max_j a_{ij} \frac{x_j}{x_i}.$$

Then by Lemma 3.1, as $\underline{q}(x) x \leqslant A \otimes x \leqslant \bar{q}(x) x$,

$$\underline{q}(x) \leqslant \mu(A) \leqslant \bar{q}(x). \tag{3.2}$$

This inequality can be found for $x = e$ in [3, Cor. 4.2], in the max-plus algebra version. It is however obvious that applying the result of [3] to $D^{-1}AD$, where $D = \operatorname{diag}(x_1, x_2, \ldots, x_n)$, and using $\mu(A) = \mu(D^{-1}AD)$, Eq. (3.2) is also obtained. The classical analogue of Eq. (3.2) was first given in [8].

We consider now for given $x(0) > 0$ the sequence $x(t),\ t = 1, 2, \ldots$ generated by

$$x(t+1) = A \otimes x(t), \quad t \geqslant 0 \tag{3.3}$$

and show that the iterates $x(t)$ yield a sequence of increasingly better upper and lower bounds for $\mu(A)$.

Lemma 3.2. *Let $A \geqslant 0$ be irreducible. Define $x(t),\ t = 0, 1, \ldots$ as above, and*

$$\bar{q}_t = \bar{q}(x(t)), \quad \underline{q}_t = \underline{q}(x(t)).$$

Then

$$\underline{q}_t \leqslant \underline{q}_{t+1} \leqslant \mu(A) \leqslant \bar{q}_{t+1} \leqslant \bar{q}_t, \quad t = 0, 1, \ldots \tag{3.4}$$

Proof. By the irreducibility of A, all $x(t)$ are strictly positive and hence $\underline{q}_t,\ \bar{q}_t$ are well defined. To show $\bar{q}_{t+1} \leqslant \bar{q}_t$, multiply the inequality

$$A \otimes x(t) \leqslant \bar{q}_t x(t)$$

by A. This leads to

$$A \otimes x(t+1) = A \otimes A \otimes x(t) \leqslant A \otimes \bar{q}_t x(t) = \bar{q}_t x(t+1)$$

and hence by definition, $\bar{q}_{t+1} \leqslant \bar{q}_t$. Similarly $\underline{q}_t \leqslant \underline{q}_{t+1}$. The other inequalities of Eq. (3.4) follow from Eq. (3.2). $\square$

In the conventional case, for $x > 0$ define $\underline{q}(x)$ and $\bar{q}(x)$ as the minimum and maximum of the numbers $q_i(x) = (Ax)_i/x_i$, $i = 1, \ldots, n$. For $\tilde{x}(t+1) = A\tilde{x}(t)$, with the standard matrix product, defining $\tilde{\underline{q}}_t = \underline{q}(\tilde{x}(t)), \bar{\tilde{q}}_t = \bar{q}(\tilde{x}(t))$ also gives:

$$\tilde{\underline{q}}_t \leqslant \tilde{\underline{q}}_{t+1} \leqslant \rho(A) \leqslant \bar{\tilde{q}}_{t+1} \leqslant \bar{\tilde{q}}_t.$$

Here it is known that for A primitive $\lim_{t\to\infty} \tilde{\underline{q}}_t = \lim_{t\to\infty} \bar{\tilde{q}}_t = \rho(A)$ and that the $x(t)$, suitably normalized, converge to the (suitably normalized) eigenvector of A corresponding to $\rho(A)$.

In the max algebra case we have

$$\mu_1(A) := \lim_{t\to\infty} \underline{q}_t \leqslant \mu(A) \leqslant \mu_2(A) := \lim_{t\to\infty} \bar{q}_t \tag{3.5}$$

and the existence of a vector $z > 0$, an accumulation point of the suitably normalized sequence $\{x(t)\}_{t=0,1,\ldots}$ satisfying

$$\mu_1(A) = \underline{q}(A^s_\otimes \otimes z), \quad \mu_2(A) = \bar{q}(A^s_\otimes \otimes z), \qquad s = 0, 1, \ldots.$$

The example

$$A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \text{ with } x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{gives } x(1) = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

$$x(2) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad x(3) = 2\begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ and } z = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \mu_1(A) = 1 < \sqrt{2}$$
$$= \mu(A) < \mu_2(A) = 2,$$

and shows that primitivity of A alone does not guarantee convergence of either q_t to $\mu(A)$ or $x(t)$ to an eigenvector. We prove the following result.

Theorem 3.3. *Let $A \geqslant 0$ be irreducible, and consider the sequence $x(t+1) = A \otimes x(t)$, $t = 0, 1, \ldots$ where $x(0) > 0$ is given. If the critical matrix A^{C} is the direct sum of primitive matrices, then $x(t)$ is an eigenvector of A for some finite t_0 (and hence for all $t \geqslant t_0$), and $\underline{q}_t = \mu(A) = \bar{q}_t$. Otherwise there exist integers p, t and a positive number s such that*

$$A^p_\otimes \otimes x(t) = sx(t), \tag{3.6}$$

in which case

$$\mu(A) = s^{1/p}, \tag{3.7}$$

$$z = x(t) \oplus \frac{1}{\mu(A)} A \otimes x(t) \oplus \cdots \oplus \frac{1}{\mu(A)^{p-1}} A_{\otimes}^{p-1} \otimes x(t) \tag{3.8}$$

is an eigenvector of A.

Proof. By Theorem 2.2, if A^{C} is the direct sum of primitive matrices, then

$$\frac{1}{\mu(A)^t} A_{\otimes}^t = A_{\infty}$$

for all $t \geqslant t_0$, where $A \otimes A_\infty = \mu(A)A_\infty$. For such a t it follows that $x(t) = A_\otimes^t \otimes x(0) = \mu(A)^t A_\infty \otimes x(0)$, showing that $x(t)$ is in the range of A_∞ and hence an eigenvector. So $\underline{q}_t = \mu(A) = \bar{q}_t$ and $\mu_1(A) = \mu(A) = \mu_2(A)$. In general, if A is only irreducible, we have by Theorem 2.3 that there exist t_0 and a minimal integer $p \leqslant d$, where d is given by Eq. (2.14), such that

$$\frac{1}{\mu(A)^{t+p}} A_{\otimes}^{t+p} = \frac{1}{\mu(A)^t} A_{\otimes}^t \quad \text{for all } t \geqslant t_0.$$

Hence

$$A_\otimes^p \otimes x(t) = x(t+p) = \mu(A)^p x(t) \quad \text{for all } t \geqslant t_0$$

and we see that Eq. (3.6) holds with $s = \mu(A)^p$. It is verified in a straight forward manner that z given by Eq. (3.8) satisfies $A \otimes z = \mu(A)z$. □

Observe that $p = 1$ gives the case of convergence. So we have shown that the following algorithm will stop after finitely many steps, giving a vector z and number $\mu(A)$ satisfying $A \otimes z = \mu(A)z$.

Algorithm 3.4. Let $A \geqslant 0$, irreducible, $x(0) > 0$.
For $t = 0, 1, \ldots$
1. $x(t+1) = A \otimes x(t)$,
2. *For* $r = 1, 2, \ldots, t+1$
   If $x(t+1) = sx(t+1-r)$ *for some* $s > 0$, *then*
   $\mu(A) = s^{1/r}$
   $z = x(t+1-r) \oplus \frac{1}{\mu(A)} x(t+2-r) \oplus \cdots \oplus \frac{1}{\mu(A)^{r-1}} x(t)$
   $p = r$
   stop.

We remark that the number p calculated by this algorithm and in Theorem 3.3 is in general only a lower bound for d, the cyclicity of A, see Eq. (2.14). If by chance $x(0)$ is an eigenvector, p will be 1, regardless of the cyclicity of A. If $\mathcal{G}(A^{\mathrm{C}})$ has r strongly connected components, then r linearly independent (in the max sense) eigenvectors exist [2, Theorem 5]. These can be found by starting with different $x(0)$. If $x(0) \geqslant 0$, $x(0) \neq 0$, but after some iterations $x(k) > 0$, then the previous results apply. Note also that Eq. (3.8) gives an eigenvector without any further calculation, in contrast to the power algorithm for the max-plus algebra given in [4, Algorithm 3.1].

Example 3.5. Let

$$A = \begin{bmatrix} 0 & e^3 & 0 \\ e^2 & 0 & e \\ e & e^2 & e^2 \end{bmatrix},$$

cf. [4, Example 5.2]. Then A is irreducible, A^{C} is the leading 2×2 principal submatrix of A and is irreducible but not primitive. Take $x(0) = (1, 0, 0)^{\mathrm{T}}$, then $x(1) = (0, e^2, e)^{\mathrm{T}},\ x(2) = (e^5, e^2, e^4)^{\mathrm{T}},\ x(3) = (e^5, e^7, e^6)^{\mathrm{T}},\ x(4) = e^5 x(2)$, giving $\mu_1(A) = 1,\ \mu_2(A) = e^5$ and $A^2_{\otimes} \otimes x(2) = e^5 x(2)$. Thus $\mu(A) = e^{\frac{5}{2}}$, with eigenvector $z = x(2) \oplus e^{-\frac{5}{2}} A \otimes x(2) = (e^5, e^{\frac{9}{2}}, e^4)^{\mathrm{T}}$.

4. Asymptotic formulas

In this section we give some asymptotic formulas for $\mu(A)$ that involve spectral radii and matrix norms. They complement some well known relations for other matrix products, see the table at the end of this section. These results are certainly of more theoretical interest than the power algorithm developed in the previous section.

For $A \in M_n = \{n \times n$ complex matrices$\}$, it is well known that for any matrix norm

$$\lim_{k \to \infty} ||A^k||^{1/k} = \rho(A), \tag{4.1}$$

see e.g. [15, Corollary 5.6.14]. The *Hadamard power* of A is defined via the Hadamard (componentwise) product as $A^k_{\odot} = [a^k_{ij}]$, and it is clear from the definition of μ for A nonnegative that

$$\mu(A^k_{\odot}) = (\mu(A))^k, \tag{4.2}$$

see e.g. [14, p. 140]. For A nonnegative and irreducible, then $A \otimes x = \mu(A)x$ with $x > 0$ by Theorem 1.1, thus

$$\mu(A^k_{\otimes}) = (\mu(A))^k. \tag{4.3}$$

By continuity this holds also for reducible $A \geqslant 0$. It is also well known (see [10, (6.11)], [14, p. 138]) that, for an $n \times n$ nonnegative matrix A,

$$\mu(A) \leqslant \rho(A) \leqslant n\mu(A), \tag{4.4}$$

which together with Eq. (4.2) implies that

$$\lim_{k\to\infty}[\rho(A^k_\odot)]^{1/k} = \mu(A) \tag{4.5}$$

as given in [10, (6.1)], [12, Theorem 1], [14, p. 140]. Also from Eq. (4.4) and the fact that $\rho(A^k) = (\rho(A))^k$, we have the result of [2, Theorem 14], namely

$$\lim_{k\to\infty}[\mu(A^k)]^{1/k} = \rho(A). \tag{4.6}$$

We now give corresponding asymptotic formulas when the standard and Hadamard products are replaced by the max product.

Lemma 4.1. *If $A \geqslant 0$ then*

$$\lim_{k\to\infty}||A^k_\otimes||^{1/k} = \mu(A). \tag{4.7}$$

Proof. Since for an operator norm $||A|| \geqslant \rho(A)$ for any $A \in M_n$ [15, Theorem 5.6.9],

$$||A^k_\otimes|| \geqslant \rho(A^k_\otimes) \geqslant \mu(A^k_\otimes) = (\mu(A))^k,$$

by Eqs. (4.4) and (4.3). So if the required limit exists, it must be at least $\mu(A)$. Assume for the other direction that $\mu(A) = 1$, and A is scaled as in Theorem 2.2 so that $a_{ij} \leqslant 1$ for all i, j. Then $A^k_\otimes \leqslant ee^{\mathrm{T}}$, so the maximum row sum norm of $A^k_\otimes$ is at most n. Since the limit does not depend on the choice of norm, its value is at most $1 = \mu(A)$. Hence the limit exists and has value $\mu(A)$. $\square$

By Eqs. (4.3) and (4.4) it is obvious that

$$\lim_{k\to\infty}[\mu(A^k_\otimes)]^{1/k} = \lim_{k\to\infty}[\rho(A^k_\otimes)]^{1/k} = \mu(A). \tag{4.8}$$

As the final relation, we have that

$$\lim_{k\to\infty}||A^k_\odot||^{1/k} = \max_{i,j}|a_{i,j}| \tag{4.9}$$

for $A \in M_n$. This follows by taking the norm $||A|| = n \max_{i,j}|a_{ij}|$; see [15, p. 292].

We summarize the above results for $A \geqslant 0$ in the following table where $\lim_{k\to\infty}[\alpha(A^k_\beta)]^{1/k}$ is given in row α with $\alpha \in \{\mu, \rho, ||\cdot||\}$, and column β, where β denotes the matrix product used (standard, Hadamard or max product).

$\alpha \setminus \beta$	Standard	Hadamard $\odot$	Max $\otimes$
μ	$\rho(A)$	$\mu(A)$	$\mu(A)$
ρ	$\rho(A)$	$\mu(A)$	$\mu(A)$
$\|\|\cdot\|\|$	$\rho(A)$	$\max_{i,j}\|a_{ij}\|$	$\mu(A)$

Note that the three entries not involving μ hold for all $A \in M_n$.

5. Concluding remarks

In Section 3 we developed a power method algorithm for the eigenvalue problem $A \otimes x = \mu(A)x$ in the max algebra. By formulating it in this way we were able to use the well developed machinery of nonnegative matrix theory. But the algorithm also can be viewed via the exponential map as a method in the max-plus algebra, and it turns out to be much simpler than the previously known method.

There are still open problems in this context, we mention just two of them. We would like to know a bound for t_0 in Theorem 3.3, or equivalently for the number of steps in Algorithm 3.4. In the literature, see e.g. [1, Ch. 3], other representations for the eigenvector are given. What are the exact relations to our formula (3.8)?

Acknowledgements

We thank Prof. R.B. Bapat for sending us a preprint of [2], and three referees for helpful comments. This research was undertaken at the University of Victoria, the first author is grateful for the hospitality there.

References

[1] F.L. Baccelli, G. Cohen, G.J. Olsder, J.-P. Quadrat, Synchronization and Linearity: An Algebra for Discrete Event Systems, Wiley, New York, 1992.
[2] R.B. Bapat, A max version of the Perron–Frobenius Theorem, Linear Algebra Appl. 275–276 (1998) 3–18.
[3] R.B. Bapat, D.P. Stanford, P. van den Driessche, Pattern properties and spectral inequalities in max algebra, SIAM J. of Matrix Analysis and Applications 16 (1995) 964–976.
[4] J.G. Braker, G.J. Olsder, The power method in max algebra, Linear Algebra Appl. 182 (1993) 67–89.
[5] R.A. Brualdi, H.J. Ryser, Combinatorial Matrix Theory, Cambridge University Press, Cambridge, 1991.
[6] W. Chen, Q. Xiangdong, D. Shuhui, The eigen-problem and period analysis of the discrete-event system, systems, science and math, Sciences 3 (1990) 243–260.
[7] G. Cohen, D. Dubois, J.-P. Quadrat, M. Viot, Analyse du Comportement Périodique des Systèmes de Production par la Théorie des Dioïdes, INRIA Rep. 191, Le Chesnay, 1983.
[8] L. Collatz, Einschließungssatz für die charakteristischen Zahlen von Matrizen, Math. Zeitschrift 48 (1942) 221–226.
[9] R.A. Cuninghame-Green, Minimax-Algebra, Lecture Notes in Economics and Math. Systems, vol. 166, Springer, Berlin, 1979.

[10] L. Elsner, C.R. Johnson, J.A. Dias da Silva, The Perron root of a weighted geometric mean of nonnegative matrices, Linear and Multilinear Algebra 24 (1988) 1–13.
[11] G.M. Engel, H. Schneider, Diagonal similarity and equivalence for matrices over groups with O, Czechoslovak Math. J. 25 (1975) 389–403.
[12] S. Friedland, Limit eigenvalues of nonnegative matrices, Linear Algebra Appl. 74 (1986) 173–178.
[13] F.R. Gantmacher, The Theory of Matrices II, Chelsea, 1959.
[14] S. Gaubert, Théorie des Systèmes Linéaires dans des Dioïdes, Thése, L'École des Mines de Paris, 1992.
[15] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, Cambridge, 1985.
[16] R.S. Varga, Matrix Iterative Analysis, Prentice Hall, Englewood Cliffs, NJ, 1962.