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A mesh-free method for the numerical solution of the KdV–Burgers equation

Sirajul Haq [a], Siraj-Ul-Islam [b,*], Marjan Uddin [a]

[a] Faculty of Engineering Sciences, GIK Institute, Topi 23640, NWFP, Pakistan
[b] University of Engineering and Technology, Basic Sciences, Peshawar, NWFP 25120, Pakistan

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ABSTRACT

This paper formulates a simple classical radial basis functions (RBFs) collocation (Kansa) method for the numerical solution of the nonlinear dispersive and dissipative KdV–Burgers' (KdVB) equation. The computed results show implementation of the method to nonlinear partial differential equations. This method has an edge over traditional methods such as finite-difference and finite element methods because it does not require a mesh to discretize the problem domain, and a set of scattered nodes in the domain of influence provided by initial data is required for the realization of the method. Accuracy of the method is assessed in terms of error norms L_2, L_∞, number of nodes in the domain of influence, parameter dependent RBFs and time step length. Numerical experiments demonstrate accuracy and robustness of the method for solving nonlinear dispersive and dissipative problems.

1. Introduction

Kortewege–de Vries–Burgers' equation is a nonlinear partial differential equation, which is given by

$$u_t + \varepsilon u u_x - \nu u_{xx} + \mu u_{xxx} = 0, \tag{1}$$

where ε, ν and μ are positive parameters. This equation was initially formulated by Gardner [1]. This model arises in many physical applications such as propagation of undular bores in shallow water waves [2], propagation of waves in elastic tube filled with a viscous fluid [3] and weakly nonlinear plasma waves with certain dissipative effects [4]. It represents long wavelength approximations where effects of the nonlinear advection term uu_x is counterbalanced by the dispersion u_{xxx}. A number of theoretical issues related the KdVB equation have received considerable attention. The traveling wave solution in particular has been studied extensively. Demiray [5], Antar and Demiray [6] derived KdVB equation as the governing evolution equation for wave propagation in fluid-filled elastic or viscoelastic tubes in which the effects of dispersion, dissipation and nonlinearity were present. Eq. (1) is combination of the Burgers' equation ($\mu = 0$) and the KdV equation ($\nu = 0$). Burgers' equation [7] was first used by Burger for the study of turbulence in 1939, whereas KdV equation [8] was first suggested by Kortewege and de Vries who used this as a nonlinear model to study the change in shape of long waves moving in a rectangular channel. KdVB equation has been solved by many authors exactly and numerically. Zaki [9] has used quintic B-spline finite elements method whereas Talaat and El-Danaf [10] has used septic B-spline to obtain numerical methods for solution of KdVB equation. In [11,12], ADM and variational iteration methods were used to obtain exact solution of KdVB equation.

* Corresponding author. Tel.: +92 091 5568209.
E-mail addresses: siraj_jcs@yahoo.co.in (S. Haq), siraj.islam@gmail.com (Siraj-Ul-Islam), marjankhan1@hotmail.com (M. Uddin).

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In the last decade, mesh-free methods using radial basis functions have been extensively used, because in classical numerical methods, such as finite-difference method, finite element method, finite volume method, generating the mesh for the application of these methods, especially in two or three dimensional problems is nontrivial task. In 1990, Kansa [13] introduced a technique for solving partial differential equations by collocation method using radial basis functions. In this approach, the solution is approximated by radial basis functions, and the collocation method is used to compute the unknown coefficients. Hardy's multiquadric (MQ), Duchon's thin plate splines (TPS), Gaussians (GA), inverse multiquadric (IMQ) and inverse quadric (IQ) are the globally supported RBFs which are commonly used in the literature for solving partial differential equations. The existence, uniqueness and convergence of the method were discussed by Micchelli [14], Madych and Nelson [15], Franke and Schaback [16]. Since the original idea for applying the RBFs method to solve PDEs was proposed by Kansa, latter on, this approach was solidified by Golberg et al. [17]. Hon and Mao [18] extended the use of RBFs for the numerical solutions of various ordinary and partial differential equations including nonlinear Burgers' equation with shock waves. These RBFs contain shape parameter and by varying the value of the shape parameter one can produce most accurate results.

In this paper, we have used three RBFs, multiquadric, Gaussian and inverse quadric, to solve the KdV–Burgers' equation. The results so obtained are compared with those already exist in the literature. It is shown that the results obtained are better and advance than the previous ones.

Rest of the paper is organized as follow. In Section 2, we discuss the mesh-free method. In Section 3, numerical results of KdVB equation with its special cases are presented. Lastly in Section 4, the results are summarized.

2. Description of the method

Consider the Kortewege–de Vries–Burgers' equation (1) with boundary conditions

$$u(a,t) = g_1(t), \quad u(b,t) = g_2(t), \quad t > 0 \tag{2}$$

and initial condition

$$u(x,0) = f(x), \quad a \leqslant x \leqslant b, \tag{3}$$

where ε, ν, μ are positive parameters, g_1, g_2 are given functions of t and $f(x)$ is bounded. We discritize the above equation in time t using Crank–Nicolson rule given by

$$\frac{u^{n+1} - u^n}{\delta t} + \frac{\varepsilon}{2}\left\{(uu_x)^{n+1} + (uu_x)^n\right\} - \frac{\nu}{2}\left\{(u_{xx})^{n+1} + (u_{xx})^n\right\} + \frac{\mu}{2}\left\{(u_{xxx})^{n+1} + (u_{xxx})^n\right\} = 0, \tag{4}$$

where $t^{n+1} = t^n + \delta t, u^{n+1} = u(x, t^{n+1})$ and δt is time step size.

The nonlinear term in the above equation is linearized as follow:

$$(uu_x)^{n+1} = u^n u_x^{n+1} + u^{n+1} u_x^n - u^n u_x^n. \tag{5}$$

Using Eq. (5) in Eq. (4), we obtain

$$u^{n+1} + \frac{\delta t}{2}\left[\mu(u_{xxx})^{n+1} - \nu(u_{xx})^{n+1} + \varepsilon\left(u^n u_x^{n+1} + u^{n+1} u_x^n\right)\right] = u^n + \frac{\delta t}{2}\left[\nu\left(u_{xx}\right)^n - \mu\left(u_{xxx}\right)^n\right]. \tag{6}$$

We approximate solution of Eq. (1) by

$$u^n(x_i) = \sum_{j=0}^{N} \lambda_j^n \phi(r_{ij}), \tag{7}$$

where $x_i = i\delta x, i = 0, 1, 2, \ldots, N, \delta x$ is space step, are collocation points in interval $[a, b], r_{ij} = \|x_i - x_j\|$ is distance between the collocation points x_i and x_j, λ_j^n are unknown coefficients to be determined and $\phi(r_j)$ is a radial basis function. In this paper, we are using the following three types of radial basis functions:

Multiquadric (MQ) $\phi(r) = (r^2 + c^2)^{\frac{1}{2}}$.
Inverse quadric (IQ) $\phi(r) = (r^2 + c^2)^{-1}$.
Gaussian (GA) $\phi(r) = \exp(-c^2 r^2)$.

Substituting approximation (7) in Eq. (6) for all interior points $x_i, i = 1, 2, 3, \ldots, N-1$, we get the following approximations

$$\sum_{j=0}^{N} \lambda_j^{n+1} \phi(r_{ij}) - \frac{\delta t}{2}\left[\nu \sum_{j=0}^{N} \lambda_j^{n+1} \phi''(r_{ij}) - \mu \sum_{j=0}^{N} \lambda_j^{n+1} \phi'''(r_{ij}) + \varepsilon\left\{\sum_{j=0}^{N} \lambda_j^{n} \phi(r_{ij}) \sum_{j=0}^{N} \lambda_j^{n+1} \phi'(r_{ij}) + \sum_{j=0}^{N} \lambda_j^{n} \phi'(r_{ij}) \sum_{j=0}^{N} \lambda_j^{n+1} \phi(r_{ij})\right\}\right]$$
$$= \sum_{j=0}^{N} \lambda_j^{n} \phi(r_{ij}) + \frac{\delta t}{2}\left[\nu \sum_{j=0}^{N} \lambda_j^{n} \phi''(r_{ij}) - \mu \sum_{j=0}^{N} \lambda_j^{n} \phi'''(r_{ij})\right]. \tag{8}$$

The boundary conditions (2) become as

$$\sum_{j=0}^{N} \lambda_j^{n+1} \phi(r_{0j}) = g_1(t), \tag{9}$$

$$\sum_{j=0}^{N} \lambda_j^{n+1} \phi(r_{Nj}) = g_2(t) \tag{10}$$

for boundary points x_0 and x_N, respectively. The system of Eqs. (8)–(10) can be solved for unknown coefficients λ_j^n by Gaussian elimination method. By substituting the values of λ_j^n in Eq. (7), solution of the problem (1) at time level n can be obtained.

3. Numerical examples

In this section, we solve three examples to check performance of the method. Accuracy of the results are computed using the L_2 and L_∞ error norms given by

$$L_2 = \left\| u^{\text{exact}} - u^{\text{app}} \right\|_2 = \left[\Delta x \sum_{j=1}^{N} \left(u^{\text{exact}} - u^{\text{app}} \right)^2 \right]^{1/2}, \tag{11}$$

$$L_\infty = \left\| u^{\text{exact}} - u^{\text{app}} \right\|_\infty = \max_j \left| u^{\text{exact}} - u^{\text{app}} \right|. \tag{12}$$

We examine our results by calculating the following three conservative laws corresponding to mass, momentum and energy [19]:

$$C_1 = \int_a^b u\,dx,$$

$$C_2 = \int_a^b u^2\,dx,$$

$$C_3 = \int_a^b \left[u^3 - \frac{3\mu}{\varepsilon} (u')^2 \right] dx.$$

Example 3.1. Consider KdV–Burgers' equation (1) with $\varepsilon = 1$ and the initial condition

$$u(x,0) = -\frac{6\nu^2}{25\mu} \left[1 + \tanh\left(\frac{\nu x}{10\mu} \right) + \frac{1}{2} \operatorname{sech}^2\left(\frac{\nu x}{10\mu} \right) \right]. \tag{13}$$

The exact solution of Eq. (1) having initial condition (13) is given by [20]

$$u(x,t) = -\frac{6\nu^2}{25\mu} \left[1 + \tanh(\xi) + \frac{1}{2} \operatorname{sech}^2(\xi) \right], \tag{14}$$

where $\xi = \frac{\nu}{10\mu}\left(x + \frac{6\nu^2}{25\mu} t\right)$. We take the boundary conditions

$$u(a,t) = -\frac{6\nu^2}{25\mu} \left[1 + \tanh\left(\frac{\nu}{10\mu} \left(a + \frac{6\nu^2}{25\mu} t \right) \right) + \frac{1}{2} \operatorname{sech}^2\left(\frac{\nu}{10\mu} \left(a + \frac{6\nu^2}{25\mu} t \right) \right) \right], \tag{15}$$

$$u(b,t) = -\frac{6\nu^2}{25\mu} \left[1 + \tanh\left(\frac{\nu}{10\mu} \left(b + \frac{6\nu^2}{25\mu} t \right) \right) + \frac{1}{2} \operatorname{sech}^2\left(\frac{\nu}{10\mu} \left(b + \frac{6\nu^2}{25\mu} t \right) \right) \right]. \tag{16}$$

We take $\varepsilon = 1, \mu = 0.1, \delta x = 0.5, \delta t = 0.001$ and $\nu = 0.004, 0.04, 0.1$, respectively, to study the effects of viscosity in Eq. (1). In Fig. 1, we present the solution profiles for different values of ν whereas Fig. 2 represents the error graphs of the solutions with MQ, GA and IQ, respectively, at time $t = 10$. It is clear from these graphs that as viscosity ν increases the solution of KdVB equation tends to behave like the solution of Burgers' equation. The L_∞ and L_2 error norms, and the conservative laws are shown in Tables 1–3. From these tables, it is clear that all the three invariants preserves very accurately by the mesh-free method. It is to be noted that the problem is solved by three types of radial basis functions, i.e., multiquadric (MQ), Gaussian (GA) and Inverse quadric (IQ). The tabulated results obtained corresponding to the values of shape parameter $c = 0.02, 0.09$ and 4.5 for MQ, GA, IQ, respectively. It is clear from Tables 1–3, that the results of MQ are marginally better than GA and IQ. Moreover the accuracy of the scheme at advance time levels is relatively lower because of first order finite-difference approximation for time derivative.

Special cases: as discussed in introduction, when $\mu = 0$ and $\nu = 0$ then Eq. (1) will reduce to Burgers' and KdV equations, respectively. To get more insight of the model (1), we discuss each case separately.

Example 3.2. By putting $\mu = 0, \nu = 0.01$ and $\varepsilon = 1$ in Eq. (1), we get Burgers' equation

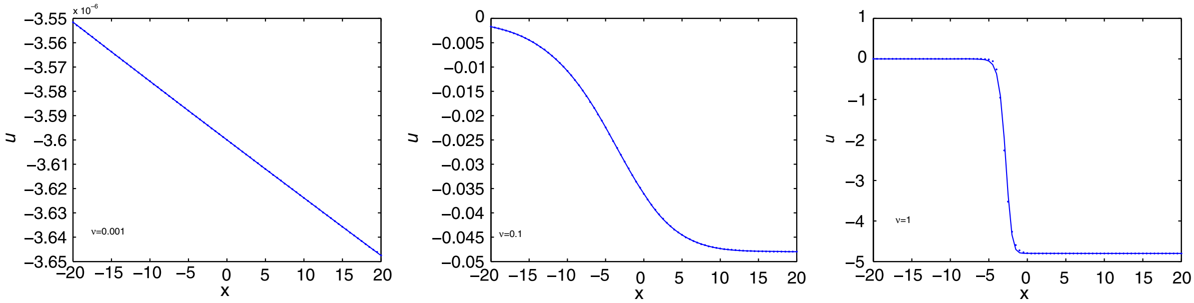

Fig. 1. KdVB type solutions for different values of the viscosity $v = 0.001, 0.1, 1$ from left to right, showing that solution vector for KdVB equation tends to behave like a solution of Burger equation.

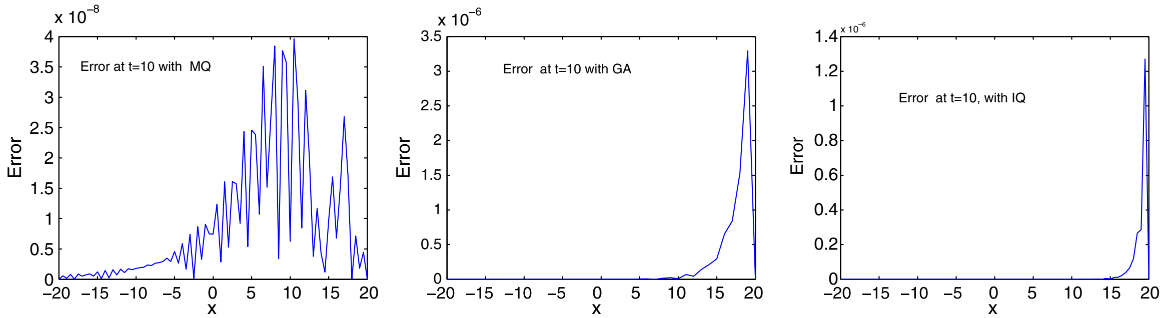

Fig. 2. Errors (exact solution–numerical solution) at $t = 10$.

Table 1
Invariants and error norms for single soliton $v = 0.004$.

Time	1	2	3	10
MQ				
C_1	−2.331E−003	−2.331E−003	−2.331E−003	−2.331E−003−
C_2	1.343E−007	1.343E−007	1.343E−007	1.343E−007
C_3	−8.619E−012	−8.622E−012	−8.624E−012	−8.630E−012
L_∞	6.822E−009	1.150E−008	1.485E−008	2.479E−008
L_2	8.845E−009	1.652E−008	2.338E−008	6.046E−008
GA				
C_1	−2.360E−003	−2.360E−003	−2.360E−003	−2.364E−003
C_2	1.360E−007	1.360E−007	1.360E−007	1.364E−007
C_3	−8.137E−012	−8.195E−012	−8.451E−012	−2.161E−011
L_∞	7.913E−009	5.128E−008	1.677E−007	3.294E−006
L_2	5.378E−009	3.488E−008	1.199E−007	3.706E−006
IQ				
C_1	−2.331E−003	−2.331E−003	−2.332E−003	−2.332E−003
C_2	1.343E−007	1.343E−007	1.344E−007	1.344E−007
C_3	-9.202E−012	−1.148E−011	−1.368E−011	−1.705E−011
L_∞	4.077E−007	7.475E−007	9.830E−007	1.270E−006
L_2	2.574E−007	4.982E−007	6.709E−007	8.858E−007

$$u_t + uu_x - vu_{xx} = 0 \tag{17}$$

subject to the initial condition

$$u(x,0) = \frac{\{\alpha + \beta + (\beta - \alpha)e^{\gamma}\}}{(1 + e^{\gamma})}, \tag{18}$$

where $\gamma = (\frac{\alpha}{v})(x - \eta)$ and α, β, η, v are the parameters. The exact solution [20] of the above problem is given by

$$u(x,t) = \frac{[\alpha + \beta + (\beta - \alpha)exp(\zeta)]}{\{1 + exp(\zeta)\}}, \tag{19}$$

where $\zeta = (\frac{\alpha}{v})(x - \beta t - \eta)$.

Table 2
Invariants and error norms for single soliton $v = 0.04$.

Time	1	2	3	10
MQ				
C_1	−2.198E−001	−2.198E−001	−2.199E−001	−2.200E−001
C_2	1.300E−003	1.300E−003	1.300E−003	1.301E−003
C_3	−8.368E−006	−8.369E−006	−8.371E−006	−8.381E−006
L_∞	2.936E−006	4.204E−006	4.126E−006	5.800E−006
L_2	3.727E−007	2.207E−008	1.928E−006	1.297E−005
GA				
C_1	−2.223E−001	−2.223E−001	−2.223E−001	−2.229E−001
C_2	−1.315E−003	1.315E−003	1.316E−003	1.323E−003
C_3	−8.477E−006	−8.481E−006	−8.491E−006	−8.654E−006
L_∞	−1.482E−006	−8.668E−006	2.665E−005	2.987E−004
L_2	2.865E−006	9.908E−006	2.575E−005	4.084E−004
IQ				
C_1	−2.198E−001	−2.200E−001	−2.202E−001	−2.186E−001
C_2	1.300E−003	1.302E−003	1.304E−003	1.284E−003
C_3	−8.381E−006	−8.592E−006	−8.545E−006	−1.349E−005
L_∞	3.925E−005	2.465E−004	3.567E−004	1.669E−003
L_2	2.842E−005	2.251E−004	4.205E−004	1.878E−003

Table 3
Invariants and error norms for single soliton $v = 0.1$.

Time	1	2	3	10
MQ				
C_1	−1.205E−000	−1.206E−000	−1.207E−000	−1.215E−000
C_2	4.863E−002	4.868E−002	4.874E−002	4.912E−002
C_3	−2.150E−003	−2.152E−003	−2.155E−003	−2.172E−003
L_∞	1.540E−005	3.076E−005	4.604E−005	1.498E−004
L_2	1.004E−005	1.732E−005	2.874E−005	1.342E−004
GA				
C_1	−1.217E−000	−1.218E−000	−1.220E−000	−1.228E−000
C_2	4.921E−002	4.926E−002	4.933E−002	4.979E−002
C_3	−2.178E−003	−2.181E−003	−2.184E−003	−2.207E−003
L_∞	1.540E−005	6.794E−005	1.622E−004	4.886E−004
L_2	2.564E−005	8.394E−005	1.864E−004	1.058E−003
IQ				
C_1	−1.205E−000	−1.206E−000	−1.207E−000	−1.216E−000
C_2	4.863E−002	4.870E−002	4.876E−002	4.919E−002
C_3	−2.151E−003	−2.154E−003	−2.156E−003	−2.178E−003
L_∞	1.314E−004	2.330E−004	1.741E−004	4.436E−004
L_2	1.169E−004	3.476E−004	3.315E−004	1.218E−003

Table 4
Error norms for single soliton.

Time	0.1	0.3	0.5	0.8	1
MQ					
L_∞	1.064E−005	1.292E−005	1.449E−005	2.082E−004	2.497E−005
L_2	1.926E−005	3.979E−005	5.523E−005	7.480E−005	8.654E−005
GA					
L_∞	1.22E−003	3.686E−003	6.166E−003	9.956E−003	1.251E−002
L_2	1.338E−005	7.503E−005	2.028E−004	5.464E−004	8.601E−004
IQ					
L_∞	1.22E−003	3.686E−003	6.166E−003	9.956E−003	1.251E−002
L_2	2.597E−006	1.254E−005	2.865E−005	6.361E−005	9.321E−005

For numerical computations we choose $\alpha = 0.4, \beta = 0.6$ and $\eta = 0.125, \delta x = 0.2, \delta t = 0.001$ in order to compare our results with those given in [9]. We solve the problem (17) with initial condition (18) using radial basis functions MQ, GA and IQ. The L_∞ and L_2 error norms are computed and are given in Table 4 which shows clearly that RBF method preserved all the three invariants. In Fig. 3, we display the exact and numerical solution, whereas Fig. 4 represents the error graphs of the

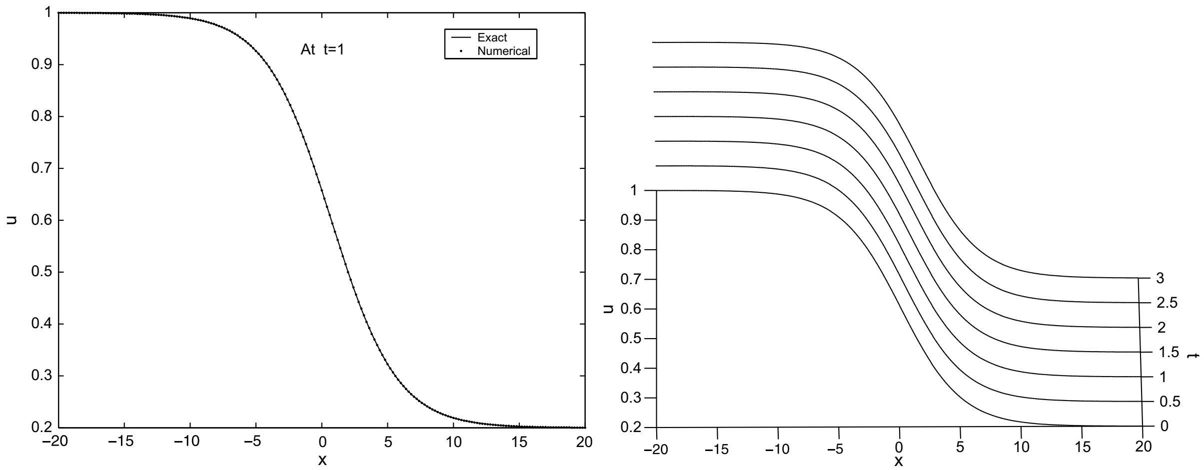

Fig. 3. Solution graph of Burgers' equation.

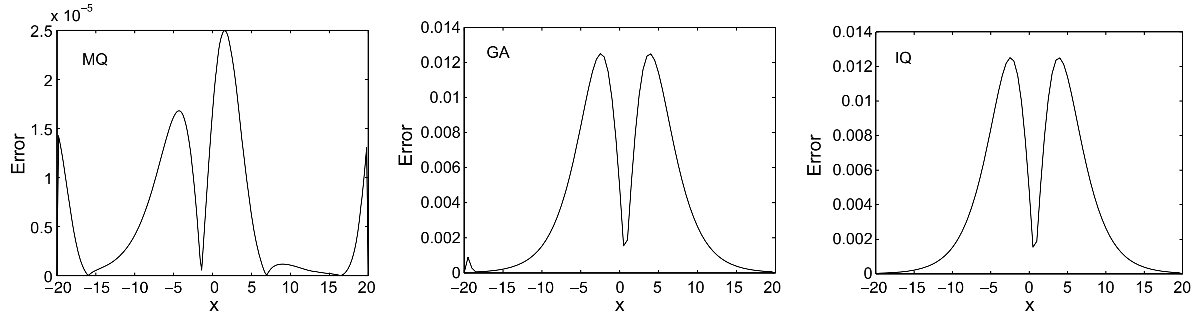

Fig. 4. Error (exact–numerical) at $t = 1$ of Burgers' equation.

Table 5
Invariants and error norms for single soliton.

Time	C_1	C_2	C_3	L_∞	L_2
MQ					
0.1	−4.000	5.333	−.400	2.024E−003	9.966E−007
1	−4.000	5.333	−6.400	9.855E−004	1.088E−004
2	−4.000	5.333	−6.400	1.096E−003	3.405E−004
3	−4.000	5.333	−6.400	9.016E−004	3.445E−005
GA					
0.1	−4.000	5.333	−6.400	2.023E−003	1.152E−005
1	−4.000	5.333	−6.400	9.512E−004	1.688E−004
2	−4.000	5.333	−6.400	1.194E−003	2.842E−004
3	−4.000	5.333	−6.400	8.000E−004	4.844E−004
IQ					
0.1	−4.000	5.333	−6.400	2.024E−003	6.275E−005
1	−4.000	5.333	−6.400	9.730E−004	1.077E−003
2	−3.998	5.333	−6.400	1.270E−003	3.607E−003
3	−3.995	5.333	−6.400	2.688E−003	1.026E−002

solutions with MQ, GA and IQ, respectively, at time $t = 1$. By the present method the error norms at $t = 1$ are $L_\infty = 2.497 \times 10^{-5}$ (MQ) and $L_\infty = 1.251 \times 10^{-2}$ (GA and IQ) whereas $L_\infty = 0.0026$ using quintic B-spline finite elements scheme [9], $L_\infty = 0.005$ by collocation with B-spline [9], $L_\infty = 0.096$ by standard Galerkin approach [9], $L_\infty = 0.082$ by product approximation Galerkin method [9] and $L_\infty = 0.151$ by compact finite-difference method [9]. From the above analysis it is clear that results obtained by MQ are superior than all the methods reported in [9]. We have used the shape parameter $c = 0.001, 0.3$ and 6 for MQ, GA and IQ, respectively, in our calculations.

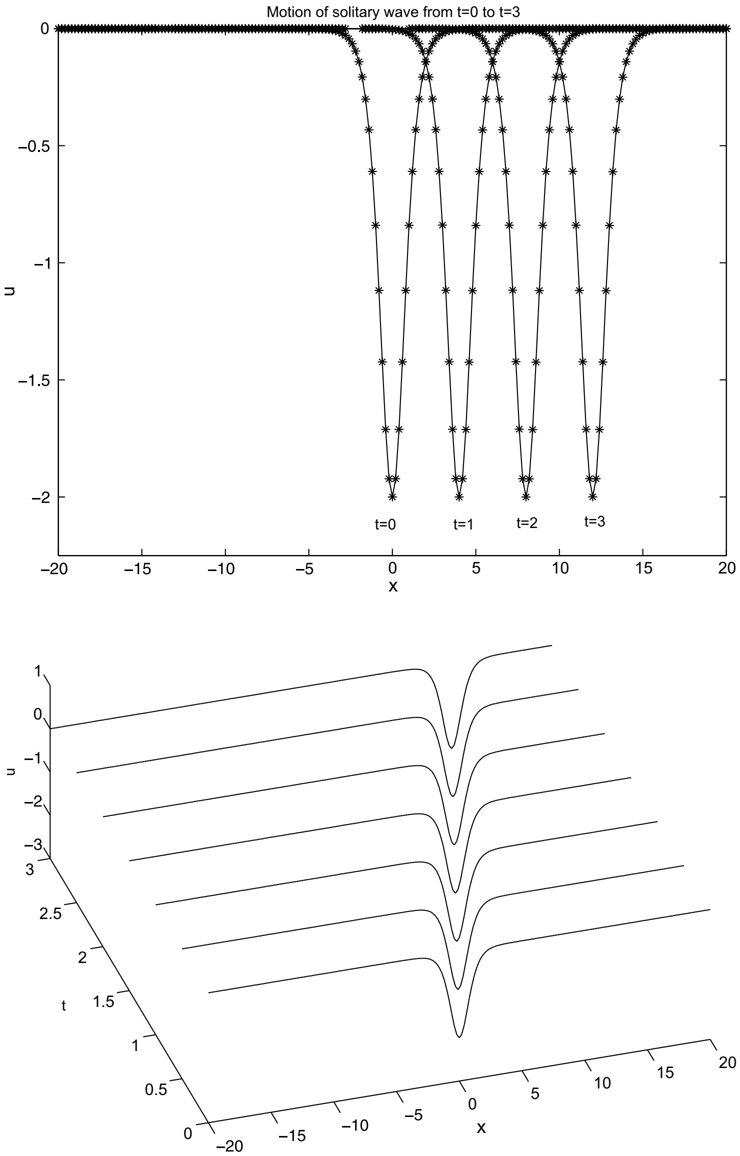

Fig. 5. Motion of solitary wave at $t = 0–3$.

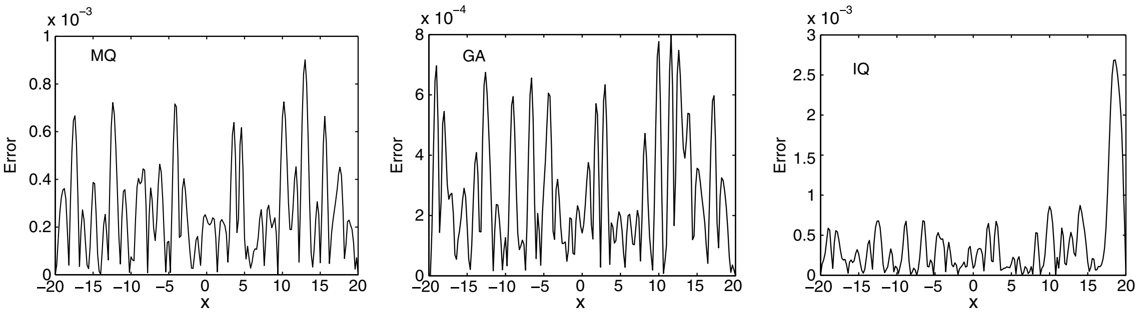

Fig. 6. Error (exact-numerical) at $t = 3$ of KdV equation.

Example 3.3. If we put the parameters $\nu = 0, \varepsilon = -6$, and $\mu = 1$ in Eq. (1), the equation reduces to KdV equation

$$u_t - 6uu_x + u_{xxx} = 0 \tag{20}$$

subject to the initial condition

$$u(x,0) = -2\,\mathrm{sech}^2(x). \tag{21}$$

Exact solution [20] of the above problem is given by

$$u(x,t) = -2\,\mathrm{sech}^2(x - 4t). \tag{22}$$

For our numerical solution, we take the boundary conditions

$$u(a,t) = -2\,\mathrm{sech}^2(a - 4t), \tag{23}$$

$$u(b,t) = -2\,\mathrm{sech}^2(b - 4t). \tag{24}$$

The radial basis functions MQ, GA and IQ are used to find solutions of Eq. (20) subject to the initial condition (21). For numerical computation we choose $\delta x = 0.2$, $\delta t = 0.001$ and values of the shape parameter for the three radial basis functions for MQ, GA, IQ are, respectively, $c = 0.8, 3.3$ and 1.5. In Table 5, L_∞, L_2 error norms as well as the three invariants C_1, C_2 and C_3 for time up to $t = 3$ are given. From this table it is clear that all the three invariants are nearly constant and the L_∞ error norm is less than 2.688×10^{-3}. In Fig. 5, numerical and exact solutions are plotted on the same diagram which shows an excellent agreement. It is clear from the figure that as the time increases the solution moves towards the right with a constant speed. The difference between the exact and numerical solutions (error) are plotted in Fig. 6 for all the three methods when $t = 3$.

4. Conclusions

We have applied the collocation method using three standard RBFs MQ, GA and IQ for the numerical solution of nonlinear KdV–Burgers' equation. The results show that this scheme is an efficient approach for the solution of such type of nonlinear equations. It is noted that time marching process reduces the solution accuracy due to the time truncation errors. As for as its application is concerned we have found that RBFs method is very much simple and straightforward, irrespective of the dimension and geometry of the problem.

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References

[1] C.H. Su, C.S. Gardner, Derivation of the Kortewege–de Vries and Burgers' equation, J. Math. Phys. 10 (1969) 536–539.
[2] R.S. Jonson, Shallow water waves in a viscous fluid – the undular bore, Phys. Fluids 15 (1972) 1993–1999.
[3] R.S. Jonson, A nonlinear equation incorporating damping and dispersion, J. Phys. Mech. 42 (1970) 49–60.
[4] H. Grad, P.N. Hu, Unified shock profile in a plasma, Phys. Mech. 10 (1967) 2596–2601.
[5] H. Demeiray, Nonlinear waves in a thick walled viscoelastic tube filled with an inviscid fluid, Int. J. Eng. Sci. 36 (1998) 359–362.
[6] N. Antar, H. Demiray, Nonlinear waves in an inviscid fluid contained in an pre-stressed thin viscoelastic tube, ZAMP 48 (1997) 325–340.
[7] J.M. Burger, Mathematical examples illustrating relations occurring in the theory of turbulent fluid motion, Trans. Royal. Neth. Acad. Sci. Amsterdam 17 (1939) 1–53.
[8] D.J. Kortewege, G. de Vries, On the change of form of long waves advancing in a rectangular channel and on a new type of long stationary waves, Philos. Mag. 39 (1895) 422–443.
[9] S.I. Zaki, Aquintic B-spline finite elements scheme for the KdVB equation, Comput. Meth. Appl. Mech. Eng. 188 (2000) 121–134.
[10] S. Talaat, Aly El-Danaf, Septic B-spline method of the Kortewege–de Varies–Burgers' equation, Commun. Nonlinear Sci. Numer. Simul. 13 (2008) 554–566.
[11] D. Kaya, M. Aassila, Decomposition method for the solution of the nonlinear Kortewege–de Vries equation, Phys. Lett. A 299 (2006) 201–206.
[12] A.A. Soliman, A numerical simulation and explicit solutions of KdV–Burgers' and Lax's seventh-order KdV equations, Chaos Soliton. Fract. 29 (2006) 294–302.
[13] E.J. Kansa, Multiquadrics scattered data approximation scheme with applications to computational fluid-dynamics. I. Surface approximations and partial derivative estimates, Appl. Math. Comput. 19 (1990) 127–145.
[14] C.A. Micchelli, Interpolation of scattered data: distance matrix and conditionally positive definite functions, Constr. Approx. 2 (1986) 11–22.
[15] W.R. Madych, S.A. Nelson, Multivariate interpolation and conditionally positive definite functions II, Math. Comput. 54 (1990) 211–230.
[16] C. Franke, R. Schaback, Convergence order estimates of meshless collocation methods using radial basis functions, Adv. Comput. Math. 8 (1998) 381–399.
[17] M.A. Golberg, C.S. Chen, S.R. Karur, Improved multiquadric approximation for partial differential equations, Eng. Anal. Bound. Elem. 18 (1996) 9–17.
[18] Y.C. Hon, X.Z. Mao, An efficient numerical scheme for burgers equation, Appl. Math. Comput. 95 (1998) 37–50.
[19] R.M. Miura, G.S. Gardner, M.D. Kruskal, Kortewege–de Vries equation and generalization. II. Existence of conservation laws and constants of motion, J. Math. Phys. 6 (1968) 1204–1209.
[20] Dogan Kaya, An application of the decomposition method for the KdVB equation, Appl. Math. Comput. 152 (2004) 279–288.