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Locally o-minimal structures and structures with locally o-minimal open core

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We study first-order expansions of ordered fields that are definably complete, and moreover either are locally o-minimal, or have a locally o-minimal open core. We give a characterisation of structures with locally o-minimal open core, and we show that dense elementary pairs of locally o-minimal structures have locally o-minimal open core.

1. Introduction

We study locally o-minimal structures and structures with locally o-minimal open core. All structures considered will be definably complete expansions of ordered fields: structures which are locally o-minimal but not definably complete (e.g., weakly o-minimal structures), while important and interesting, are outside the scope of this article; we refer the reader interested in locally o-minimal structures which are not definably complete to [20].

One of the natural generalisations of o-minimality (for definably complete structures) is requiring that every unary set definable (with parameters) is *locally* a finite union of points and intervals.

Definition 1.1. $\mathbb{K}$ is **locally o-minimal** if, for every $X \subset \mathbb{K}$ definable, and for every $x \in \mathbb{K}$, there exists $y > x$ such that, either $(x, y) \subseteq X$, or $(x, y) \subseteq \mathbb{K} \setminus X$.

Much of the theory of o-minimal structures can be generalised to locally o-minimal ones: for instance, a version of the Monotonicity Theorem (Section 5.1; see also [19]) and Miller's dichotomy (Theorem 5.18) hold also for locally o-minimal structures; moreover, a weak version of cell decomposition holds (Theorem 5.6; see also [19] for a different kind of cell decomposition); besides, A. Pillay's theorem that a group definable in an o-minimal structure can be equipped with a definable

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topology making it a topological group can be extended to locally o-minimal structures (Theorem 5.25). In Section 3.1 and Section 5 we will study locally o-minimal structures more in details. One advantage of locally o-minimal structures over o-minimal ones is that they form an elementary class: that is, an ultraproduct of locally o-minimal structures is locally o-minimal.

A further generalisation of local o-minimality is given by structures with locally o-minimal open core.

Definition 1.2. The **open core** of $\mathbb{K}$ is the reduct of $\mathbb{K}$ generated by all definable open subsets of $\mathbb{K}^n$, for every $n \in \mathbb{N}$.

Definition 1.3. $\mathbb{K}$ has locally o-minimal open core if the structure generated by all open definable sets is locally o-minimal.

The main results of this article can be summed up in the following two theorems.

Theorem A. *(See Theorem 3.3.) The open core of $\mathbb{K}$ is locally o-minimal iff every definable discrete subset of $\mathbb{K}$ is bounded.*

In [6, §9] we defined d-minimal topological structures, and we proved some results about the theory of dense elementary pairs of such structures (see Definition 6.2). We will show that locally o-minimal structures are d-minimal topological structures; thus, we will be able to apply the results in [6] to the present situation, and we will spell out some consequences.

Given $d \leqslant n \in \mathbb{N}$, we denote by $\prod_d^n : \mathbb{K}^n \to \mathbb{K}^d$ the projection onto the first d coordinates. Given $X \subseteq \mathbb{K}^{n+m}$ and $\bar{c} \in \mathbb{K}^n$, we define $X_{\bar{c}} := \{\bar{y} \in \mathbb{K}^m\colon\ \langle \bar{c}, \bar{y}\rangle \in X\}$.

Definition 1.4. (See [6, 9.1].) Let $\mathbb{K}^* \succcurlyeq \mathbb{K}$ be ω-saturated. $\mathbb{K}$ is a **d-minimal topological structure** if:

(DM1) For every $X \subseteq \mathbb{K}^*$ definable (with parameters in $\mathbb{K}^*$), if X has empty interior, then X is a finite union of discrete sets;

(DM2) For every $X \subseteq \mathbb{K}^n$ definable and discrete, $\prod_1^n(X)$ has empty interior;

(DM3) Given $X \subseteq \mathbb{K}^2$ and $U \subseteq \prod_1^2(X)$ definable sets, if U is open and nonempty, and X_a has nonempty interior for every $a \in U$, then X has nonempty interior.

Notice that in the original definition [6, 9.1] there are some additional conditions; but those conditions are trivially satisfied when $\mathbb{K}$ is an ordered field.

Theorem B. *(See Section 6.) Let $\mathbb{K}$ be locally o-minimal. Then, $\mathbb{K}$ is a d-minimal topological structure. Moreover, let A be a proper, dense, elementary substructure of $\mathbb{K}$, and let $\langle \mathbb{K}, A\rangle$ be the expansion of $\mathbb{K}$ with a new predicate for A. Then, $\mathbb{K}$ is the open core of $\langle \mathbb{K}, A\rangle$. Moreover, the theory of $\langle \mathbb{K}, A\rangle$ is uniquely determined by the theory of $\mathbb{K}$.*

As it is clear from Theorem A, the class of structures with locally o-minimal open core is also elementary; a related question about structures with o-minimal open core is still open (see the discussion after Corollary 3.6).

Definably complete structures were explicitly defined and studied in [13]. The open core of $\mathbb{K}$ was defined already in [15], where they study the case when $\mathbb{K}$ is an expansion of $\mathbb{R}$. Structures with o-minimal open core are the main topic of [2]; here, instead, they are only a side remark, because we show that many of the results of [2] can be generalised to structures with locally o-minimal open core; moreover, we answer some questions left open there. Besides, some of the techniques used in [2] will be also employed here (see Section 2.4 and Section 4). Locally o-minimal structures were introduced in [14], and [2] proved some results on definably complete locally o-minimal structures (see Section 3.1). In his recent preprints [19,18], H. Schoutens studies locally o-minimal structures without the assumption that they expand a field (he calls them “DCTC structures”), as a step in the project of studying “o-minimalistic” structures (i.e., structures that are elementarily equivalent to ultraproducts of o-minimal structures: see also [17] for a development in that direction); notice that our motivation is different: we study structures with locally o-minimal open core as a necessary step in the study of definably complete structures. Schoutens also independently obtains some of the results exposed here; in particular, he achieves a “quasi-cell decomposition” for sets definable in locally o-minimal structures, and a kind of cell decomposition for a certain class of locally o-minimal structures, that he calls “tame”.

One of the natural examples of structures with o-minimal open core is given by dense elementary pairs of o-minimal structures $A \prec B$, studied in [21]. The main results of [21] can be generalised to elementary pairs of locally o-minimal structures, with a similar proof (Theorem B; see Section 6). Similar results hold in the more general setting of d-minimal structures, which will be studied in another article [4].

While o-minimal structures are geometric (that is, they eliminate the quantifier $\exists^\infty$ and the algebraic closure acl satisfies the Exchange Principle), no such result is true for locally o-minimal structures: more precisely, if $\mathbb{K}$ is a sufficiently saturated locally o-minimal non-o-minimal structure, then $\mathbb{K}$ does *not* eliminate the quantifier $\exists^\infty$ and acl does *not* satisfy the Exchange Principle (this is true not only for locally o-minimal structures, but for d-minimal structures in general: the proof will be given in [4]). However, we still have a notion of dimension for locally o-minimal structures, given by the topology (and which for o-minimal structures coincides with the usual o-minimal dimension), which can be usefully employed in the study of locally o-minimal structures, and in particular in establishing Theorem B, in the same way as the o-minimal dimension is used in o-minimal structures: see Section 2.4 and Section 5.4.

All the structures considered will moreover be definably Baire (see [9]): this will not be explicit in the definitions, but will follow quite easily from them. Also, many of the proofs will rely on this Baire property. Therefore, a preliminary study of definably complete and of definably Baire structures is essential in order to understand locally o-minimal structures, and it will be carried out in Section 2. P. Hieronymi proved in [10] that every definably complete structure is also definably Baire; however, his proof relies on some of the results in Section 2 of this article. Thus, in this article we will not use Hieronymi's theorem, and we will maintain the notion of "definably Baire structures".

Further results on definably complete structures can be found in [5]. In the study of definably complete structures, the following trichotomy plays a fundamental role: if $\mathbb{K}$ is a definably complete structure, then

(1) either $\mathbb{K}$ does not define an unbounded discrete set (i.e., $\mathbb{K}$ is locally o-minimal);
(2) or $\mathbb{K}$ is "restrained" (i.e., for every definable function $f : \mathbb{K} \to \mathbb{K}$, for every definable *discrete* set $D \subset \mathbb{K}$, $f(D)$ is nowhere dense in $\mathbb{K}$) but not locally o-minimal;
(3) or $\mathbb{K}$ is not restrained.

In this paper, we examine the first case; in [7], we study restrained structures: they form the largest class of definably complete structures which partake of several "tameness" properties of structures with (locally) o-minimal open core. The study of non-restrained structures outside $\mathbb{R}$ is still in the conjectural state.

2. Preliminaries

"Definable" will always mean "definable with parameters".

A linearly ordered structure $\langle \mathbb{K}, < \rangle$ is **definably complete** if every definable subset of $\mathbb{K}$ has a supremum in $\mathbb{K} \sqcup \{\pm\infty\}$ [13].

Proviso. $\mathbb{K}$ will always be a definably complete structure expanding an ordered field.

2.1. Conventions, basic definitions, and notation

$\bar{\mathbb{R}} = \langle \mathbb{R}, 0, 1, +, \cdot, < \rangle$ is the ordered field of real numbers. $\mathbb{R}^{\mathrm{alg}}$ is the subset of $\mathbb{R}$ given by the real algebraic numbers. $\bar{\mathbb{R}}(\mathbb{N}) := \langle \mathbb{R}, +, \cdot, <, \mathbb{N} \rangle$ is the expansion of $\bar{\mathbb{R}}$ with a predicate for the natural numbers.

$d : \mathbb{K}^n \times \mathbb{K}^n \to \mathbb{K}$ is the **distance** function $d(\bar{x}, \bar{y}) := \|\bar{x} - \bar{y}\|$. For every $\bar{x} \in \mathbb{K}^n$ and $0 < r \in \mathbb{K}$, $B(\bar{x}, r)$ is the **open ball** of centre $\bar{x}$ and radius r, while $\overline{B}(\bar{x}, r)$ is the **closed ball**.

Definition 2.1 *(Topological definitions).* Let $X \subseteq \mathbb{K}^n$. $\overline{X}$, also denoted by $\mathrm{cl}(X)$, is the topological **closure** of X inside $\mathbb{K}^n$, while $\mathring{X}$, also denoted by $\mathrm{int}(X)$, is the **interior** of X; $\partial X := \overline{X} \setminus X$ is the **frontier** of X; $\mathrm{bd}(X) := \overline{X} \setminus \mathring{X}$ is the **boundary** of X. X is **constructible** if it is a finite Boolean combination of open sets. X is **locally closed** if for every $x \in X$ there exists a neighbourhood $U \ni x$ such that $X \cap U$ is closed in U. Given a function f, we denote by $\Gamma(f)$ the graph of f.

Fact 2.2. X is locally closed iff it is of the form $C \cap U$, for some closed set C and some open set U. X is constructible iff it is a finite union of locally closed sets. If X is definable, then X is constructible iff it is a finite Boolean combination of *definable* open sets [1,3]. If X is definable, then X is locally closed iff it is of the form $C \cap U$, for some definable closed set C and some definable open set U.

Definition 2.3. (See [9].) $X \subseteq \mathbb{K}^n$ is **nowhere dense** if $\mathring{\overline{X}}$ is empty. $\mathbb{K}$ is **definably Baire** (or simply "Baire" for short) if $\mathbb{K}$ is not the union of a definable increasing family of nowhere dense subsets. X is an $\mathcal{F}_\sigma$**-set** if it is the union of a definable increasing family of closed subsets of $\mathbb{K}^n$, and is a $\mathcal{G}_\delta$-set if its complement is an $\mathcal{F}_\sigma$-set. X is **meager** if it is the union of a definable increasing family of nowhere dense sets; X is **residual** if its complement is meager. X is **almost open** (or a.o. for short) if there exists a definable open set U such that $X \mathbin{\Delta} U$ is meager, where Δ is the **symmetric difference** of sets.

Fact 2.4. (See [9].) If $X \subseteq \mathbb{K}^n$ is an $\mathcal{F}_\sigma$-set and $f : X \to \mathbb{K}^m$ is definable and continuous, then $f(X)$ is an $\mathcal{F}_\sigma$-set.

Fact 2.5 *(Kuratowski–Ulam).* (See [9].) Let $X \subseteq \mathbb{K}^{n+m}$ be definable. If X is meager, then the set $T(X) := \{a \in \mathbb{K}^n\colon\ X_a \text{ is meager}\}$ is residual. If X is a.o., then X is meager iff $T(X)$ is residual. In particular, $\mathbb{K}$ is Baire iff $\mathbb{K}^n$ is Baire, for every $n \geqslant 1$.

Remark 2.6. The open core of $\mathbb{K}$ includes all definable constructible sets and, more generally, all $\mathcal{F}_\sigma$-sets. In fact, if X is an $\mathcal{F}_\sigma$-set, then X is the projection of a closed definable set: if $X = \bigcup_{t \in \mathbb{K}} X_t$, where $(X_t\colon\ t \in \mathbb{K})$ is a definable increasing family of closed subsets of $\mathbb{K}^n$, then $X = \prod_n^{n+1} \mathrm{cl}(\bigcup_{t \in \mathbb{K}} X_t \times \{t\})$.

Definition 2.7. Let $X \subseteq \mathbb{K}^n$ be definable. X is **d-compact** if it is closed and bounded. X is **pseudo-finite** if it is d-compact and discrete.

The following fundamental fact will be used without mention.

Fact 2.8. (See [13].)

(1) Let $(X_t\colon t \in \mathbb{K})$ be a definable decreasing family of d-compact subsets of $\mathbb{K}^n$. If each X_t is nonempty, then $\bigcap_{t\in\mathbb{K}} X_t$ is also nonempty.
(2) Let $X \subseteq \mathbb{K}^n$ be a d-compact set, and $f : X \to \mathbb{K}^m$ be a definable continuous function. Then, $f(X)$ is also d-compact.

See Section 2.5 for more on pseudo-finite sets.

A definable set X is **definably connected** if every clopen definable subset of X is either empty or all of X.

Given a function $f : X \to Y$, where X and Y are topological spaces, we define $\mathcal{D}(f)$ to be the set of points in X where f is discontinuous.

I will now discuss briefly the Proviso that $\mathbb{K}$ expands a field. This assumption is often convenient for notational purposes and to simplify the statements of the theorems (compare e.g. our definition of $\mathcal{F}_\sigma$-sets with the corresponding definition of D_Σ sets in [2]); in those cases, a reader that is interested in definably complete structures that may not expand a field can easily modify definitions, proofs, and statements to his situation. However, sometimes the field assumption is used in an essential way, and the reader assumed above should be more careful.

2.2. Topological preliminaries

Let A be a Hausdorff topological space without isolated points and X be a subset of A (in our applications, A will be $\mathbb{K}^n$).

Remark 2.9.

- ∂X has empty interior.
- $\mathrm{bd}(X) = \partial X \sqcup \partial(A \setminus X)$.
- $\mathrm{bd}(X)$ is closed.
- $\mathrm{bd}(X \cup X') \subseteq \mathrm{bd}(X) \cup \mathrm{bd}(X')$.
- $\mathrm{bd}(X \,\Delta\, X') \subseteq \mathrm{bd}(X) \cup \mathrm{bd}(X')$.
- If X and X' are nowhere dense, then $X \cup X'$ is also nowhere dense.
- If X is locally closed, then $\mathrm{bd}(X)$ is nowhere dense. Therefore, if X is constructible, then $\mathrm{bd}(X)$ is nowhere dense.

Corollary 2.10. *If X is constructible and $\mathring{X} = \emptyset$, then X is nowhere dense. In particular, if X is discrete, then it is nowhere dense.*

Lemma 2.11. *X is locally closed iff ∂X is closed. If $\mathrm{bd}(X)$ is discrete, then X is locally closed.*

Proof. For the first part, if ∂X is closed, then X is open in $\overline{X}$, and hence locally closed. Conversely, if $X = \overline{X} \cap U$ for some open set U, then $\partial X = \overline{X} \setminus U$.

For the second part, if $\mathrm{bd}(X)$ is discrete, then, since $\mathrm{bd}(X)$ is also closed, $\mathrm{bd}(X)$ has no accumulation points in A. Since $\partial X \subseteq \mathrm{bd}(X)$, we have that ∂X is also closed (in A). □

2.3. Preliminaries on definably complete structures

We will prove the analogues for definably complete structures of some easy facts from classical analysis.

Remark 2.12. Let $X \subseteq \mathbb{K}^n$ be definable. If $Y \subseteq \mathbb{K}^n$ is definable and definably connected (e.g., $n = 1$ and Y is an interval), then the following are equivalent:

(1) $Y \cap \mathrm{bd}(X) = \emptyset$;
(2) either $Y \subseteq \mathring{X}$, or $Y \cap \overline{X} = \emptyset$.

Lemma 2.13. *Let $f : [0,1] \to \mathbb{K}$ be definable and continuous. Then, f is uniformly continuous.*

Proof. Assume not; then, there exists $\varepsilon > 0$ such that, for every $\delta > 0$, the set

$$X(\delta) := \left\{\langle x, y\rangle \in [0,1]^2\colon \|x - y\| \leqslant \delta \;\&\; \left|f(x) - f(y)\right| \geqslant \varepsilon\right\}$$

is d-compact and nonempty. Thus, $X := \bigcap_{\delta>0} X(\delta) \neq \emptyset$. If $\langle x, y\rangle \in X$, then $x = y$ and $f(x) \neq f(y)$, absurd. □

Definition 2.14. Let P be a property of definable sets. We say that P is **monotone** if $X \subseteq Y$ and $P(Y)$ imply $P(X)$. We say that P is **additive** if $P(X)$ and $P(Y)$ imply $P(X \cup Y)$. We say that P is **definable** (for $\mathbb{K}$), if for every definable family $(X_y)_{y \in A}$, the set $\{y \in A\colon\ P(X_y)\}$ is definable.

If T is a theory, we say that P is definable for T if P is definable for every model of T.

For instance, "being closed" and "being pseudo-finite" are definable properties. Notice also that a property might be definable for $\mathbb{K}$ without being definable for the theory of $\mathbb{K}$: for instance, if $\mathbb{K}$ is *not* ω-saturated, then "being finite" might be definable for $\mathbb{K}$, without being definable for some $\mathbb{K}' \succ \mathbb{K}$. A type p over $\mathbb{K}$ is definable iff the corresponding property "$X \in p$" is definable [16, §11.b].

Lemma 2.15. *Let $C \subset \mathbb{K}^n$ be d-compact, $f: C \to \mathbb{K}^m$ be definable, and P be a property of definable sets. Assume that P is definable, monotone, and additive, and that, for every $c \in C$, there exists U_c definable neighbourhood of c, such that $P(f(U_c \cap C))$. Then, $P(f(C))$.*

Proof. Let $Q(Y)$ be the property $P(f(Y))$. By replacing P with Q, w.l.o.g. we can assume that $n = m$ and $f: C \to \mathbb{K}^n$ is the inclusion function. Proceed as in the proof of [9, Corollary 3.8], using the property P instead of the property "being meager". □

For instance, we can apply the above lemma to the property "being nowhere dense". We don't know if in the above lemma the assumption that P is monotone is necessary.

2.4. Dimension

The proof of Theorem 3.3 will be by induction on the full dimension: see the next definition.

Definition 2.16. Let $X \subseteq \mathbb{K}^n$ be definable and nonempty. The **dimension** of X is

$$\dim(X) := \max\left\{ d \leqslant n\text{: there exists a coordinate space } L \text{ of dimension } d,\ \text{s.t.} \prod_L^n(X) \text{ has nonempty interior}\right\},$$

where $\prod_L^n$ is the projection from $\mathbb{K}^n$ onto L. By convention, we say that $\dim(\emptyset) = -1$. The **full dimension** of X is the pair $\operatorname{fdim}(X) := \langle d, k\rangle$, where $d = \dim(X)$ and $1 \leqslant k$ is the number of coordinate spaces L of dimension d, s.t. $\prod_L^n(X)$ has nonempty interior.

The set of full dimensions is ordered lexicographically, with the dimension component more important. Therefore, by induction on the full dimension we mean induction first on d and then on k. Dimension and full dimension were already defined in [2].

Lemma 2.17. *Let $X \subseteq \mathbb{K}^{n+m}$ be a definable set of dimension n. Let $A := \{a \in \mathbb{K}^n\colon\ \dim(X_a) > 0\}$. If X is an $\mathcal{F}_\sigma$, then A is an $\mathcal{F}_\sigma$. If moreover $\mathbb{K}$ is Baire, then $\mathring{A} = \emptyset$ (i.e., A has dimension less than n).*

Proof. The proof is essentially the same as in [2, 2.8(3) and 3.4], using Fact 2.5. □

In the above lemma we cannot drop the assumption that X is an $\mathcal{F}_\sigma$; for instance, let $\mathbb{K} := \bar{\mathbb{R}}(\mathbb{N})$, $n = 1$, and

$$X := \bigl\{\langle x, y\rangle \in \mathbb{R}^2\colon\ (x \in \mathbb{Q}\ \&\ 0 < y < 1)\ \vee\ (x \notin \mathbb{Q}\ \&\ 1 < y < 2)\bigr\}.$$

Lemma 2.18. *Assume that $\mathbb{K}$ is Baire. Let $X_1, X_2 \subseteq \mathbb{K}^n$ be definable. If X_1 and X_2 are both $\mathcal{F}_\sigma$, then* $\operatorname{fdim}(X_1 \cup X_2) = \max(\operatorname{fdim}(X_1), \operatorname{fdim}(X_2))$ *and in particular* $\dim(X_1 \cup X_2) = \max(\dim(X_1), \dim(X_2))$.

Proof. Assume, for a contradiction, that, for some $m \leqslant n$, $\prod_m^n(X_1 \cup X_2)$ has nonempty interior, while $\prod_m^n(X_i)$ has empty interior, for $i = 1, 2$. However, since $\prod_m^n(X_i)$ is an $\mathcal{F}_\sigma$, this means that $\prod_m^n(X_i)$ is meager, for $i = 1, 2$, and therefore $\prod_m^n(X_1 \cup X_2) = \prod_m^n(X_1) \cup \prod_m^n(X_2)$ is also meager, absurd. □

In the above lemma we cannot drop the assumptions that the X_i are $\mathcal{F}_\sigma$: for instance, let $X_1 = \mathbb{Q}$ and $X_2 = \mathbb{R} \setminus \mathbb{Q}$ in the structure $\bar{\mathbb{R}}(\mathbb{N})$.

Example 2.19. It is not true that, if $X \subseteq \mathbb{K}^n$ is definable and constructible, then $\dim(\overline{X}) = \dim(X)$. In fact, let $\mathbb{K} := \bar{\mathbb{R}}(\mathbb{N})$, and $X \subset \mathbb{R}^2$ defined by:

$$X := \bigl\{\langle x, y\rangle\colon\ x = p/q \in \mathbb{Q}\ \&\ 0 < p < q \in \mathbb{N}\ \&\ (p, q) = 1\ \&\ y = 1/q\bigr\}.$$

Notice that X is locally closed (and *a fortiori* constructible), $\overline{X} = X \cup ([0, 1] \times \{0\})$, and $\dim(X) = 0$, while $\dim(\overline{X}) = 1$.

2.5. Pseudo-finite sets

We claim that pseudo-finite sets are a first-order analogue of finite subsets of $\mathbb{R}^n$. This subsection will give some evidence for the above claim (but see [17] for an obstruction to the claim). Moreover, we will prove some properties of pseudo-finite sets which will be central to our proof of Theorem 3.3.

Let $X \subseteq \mathbb{K}^n$ be definable.

Remark 2.20. If X is discrete, then it is nowhere dense in $\mathbb{K}^n$.

Lemma 2.21. *If X and X' are pseudo-finite, then $X \times X'$ is also pseudo-finite. Moreover, if X is pseudo-finite, then every definable subset of X is also pseudo-finite.*

Proof. Clear: any subset of closed discrete set is also closed and discrete. □

Lemma 2.22. *Let X be pseudo-finite. If $f : X \to \mathbb{K}^m$ is definable, then $f(X)$ is also pseudo-finite.*

Notice that f in the above lemma is necessarily continuous.

Proof. We want to prove that $f(X)$ is bounded and has no accumulation points in $\mathbb{K}^m$.

Claim 1. *$f(X)$ has no accumulation points in $\mathbb{K}^m$.*

Assume, for a contradiction, that $y \in \mathbb{K}^m$ is an accumulation point for $f(X)$. For every $r > 0$, let $U(r) := f^{-1}(B(y,r) \setminus \{y\})$; note that each $U(r)$ is nonempty. By Lemma 2.21, each $U(r)$ is closed in X. Since X is d-compact, $\bigcap_r U(r) \neq \emptyset$, which is absurd.

If $f(X)$ is bounded, we are done. If $f(X)$ is unbounded, let $\phi : \mathbb{K}^m \to (0,1)^m$ be a definable homeomorphism, and $g := \phi \circ f$. Thus, $g(X)$ is bounded, and, by Claim 1, it is pseudo-finite. Thus, $f(X) = \phi^{-1}(g(X))$ is d-compact, and in particular bounded, and therefore, again by Claim 1, pseudo-finite. □

Corollary 2.23. *Let $X \subseteq \mathbb{K}^n$ and $X' \subseteq \mathbb{K}^{n'}$ be definable. Then, $X \times X'$ is pseudo-finite iff X and X' are pseudo-finite.*

Corollary 2.24. *X is pseudo-finite iff every projection of X on the coordinate axis is pseudo-finite.*

Definition 2.25. Let $\delta(X) := \inf\{d(x,x'){:}\ x,x' \in X\ \&\ x \neq x'\}$.

Lemma 2.26. *The following are equivalent*:

(1) *X is pseudo-finite*;
(2) *X is bounded and has no accumulation points in $\mathbb{K}^n$*;
(3) *X is bounded and $\delta(X) > 0$.*

Proof. (1) $\Leftrightarrow$ (2) follows from the definition of pseudo-finite.

(3) $\Rightarrow$ (2) is clear.

(1) $\Rightarrow$ (3). Assume that X is pseudo-finite. We want to prove that $\delta(X) > 0$. Let $Y := X \times X$, and $\Delta(Y)$ be its diagonal. Consider the map $d : Y \setminus \Delta(Y) \to \mathbb{K}$, mapping (x,x') to $d(x,x')$. Note that $Y \setminus \Delta(Y)$ is pseudo-finite, and that $\delta(X) = \inf_{Y \setminus \Delta(Y)} d(y)$. Thus, by Lemma 2.22, d attains a minimum on $Y \setminus \Delta(Y)$, and therefore $\delta(X) > 0$. □

Lemma 2.27. *$X \subseteq \mathbb{K}^n$ is discrete and closed iff, for every $r > 0$, $X \cap B(0,r)$ is pseudo-finite.*

Proof. ($\Rightarrow$) is clear, because if X is discrete and closed, then $X \cap B(0,r)$ is discrete, closed and bounded. ($\Leftarrow$) follows from the fact that X has no accumulation points in $\mathbb{K}^n$. □

The following lemma answers a question in [14, §5], and is used in an essential way in [5].

Lemma 2.28. *If $\mathbb{K}$ defines a discrete subset of $\mathbb{K}$ that is not pseudo-finite, then $\mathbb{K}$ defines an unbounded discrete* closed *subset of $\mathbb{K}$.*

Proof. Let $D \subset \mathbb{K}$ be definable, discrete and not pseudo-finite. If D is not closed, let $a \in \mathbb{K} \setminus D$ be an accumulation point of D, and $D' := \{1/(x-a){:}\ x \in D\}$. Notice that D' is discrete and unbounded. Thus, w.l.o.g., we can assume that D is discrete and unbounded. For every $r > 0$, let

$$D(r) := \{x \in D\colon\ D \cap B(x,r) = \{x\}\},$$

the set of points in D at distance at least r from the other points of D. Each $D(r)$ is discrete and closed. If $D(r)$ is unbounded for some r, we are done. Otherwise, each $D(r)$ is pseudo-finite; let $z(r) := \max(D(r))$, and $Z := \{z(r)\colon\ r > 0\}$. Since D is unbounded, Z is also unbounded.

Claim 2. *Z is closed and discrete.*

Otherwise, Z would have an accumulation point a. For every $r > 0$, let $Z(r) := \{z(r')\colon\ r \leqslant r'\}$. Notice that $Z(r)$ is bounded and $\delta(Z(r)) \geqslant r$; thus, $Z(r)$ is pseudo-finite. Moreover, since $z(r)$ is a decreasing function of r, there exists $r_0 > 0$ such that $z(r) > a + 1$ for every $r < r_0$. Hence, a cannot be an accumulation point of Z, absurd. □

Notice that the implication (1) ⇒ (4) in Theorem 3.3 gives a stronger version of the above lemma (i.e., for subsets of $\mathbb{K}^n$ instead of subsets of $\mathbb{K}$).

Lemma 2.29. *If X is discrete, then it is the union of a definable increasing family of pseudo-finite sets. In particular, X is an $\mathcal{F}_\sigma$.*

Proof. After a change of coordinates, we can assume that X is bounded. For every $r > 0$, define $X(r) := \{x \in X\colon\ X \cap B(x,r) = \{x\}\}$. Since X is discrete, $X = \bigcup_r X(r)$. Therefore, it suffices to prove that, for each r, $X(r)$ is pseudo-finite. Fix $r > 0$. It is clear that $\delta(X(r)) \geqslant r > 0$, and therefore $X(r)$ is pseudo-finite. □

Definition 2.30. A **pseudo-finite family** of sets is a definable family $(X_a\colon\ a \in A)$, such that A is pseudo-finite.

Lemma 2.31. *Let P be an additive definable property. Let $(X_y\colon\ y \in A)$ be a pseudo-finite family, such that, for every $y \in A$, $P(X_y)$. Then, $P(\bigcup_{y \in A} X_y)$.*

Proof. Let $A \subseteq \mathbb{K}^n$ and $\trianglelefteq$ be the lexicographic order on $\mathbb{K}^n$. Let

$$B := \Bigl\{y \in A\colon\ P\Bigl(\bigcup_{\substack{z \in A,\\ z \trianglelefteq y}} A_z\Bigr)\Bigr\}.$$

Since P is definable, B is a definable subset of A. Hence, B is pseudo-finite, and therefore it has a $\trianglelefteq$-maximum b. It is now easy to see that b is also the $\trianglelefteq$-maximum of A. □

Since "being closed" is an additive definable property, we see that the union of a pseudo-finite family of closed sets is closed, and the intersection of a pseudo-finite family of open sets is open. Similarly, the union of a pseudo-finite family of pseudo-finite sets is pseudo-finite, and the union of a pseudo-finite family of nowhere dense sets is nowhere dense.

The following conjecture is studied (as an open problem) more in detail in [18].

Conjecture 2.32 *(Pigeon hole principle). Let $X \subseteq \mathbb{K}^n$ be pseudo-finite and $f : X \to X$ be definable. If f is injective, then it is surjective.*

2.6. Bad sets

In the following, it will often be necessary to prove that, for a given definable set A, the set of "bad points" of A, is "small" (in some suitable sense: usually, meaning "meager").

Lemma 2.33. *Let $d \leqslant n$, $A \subseteq \mathbb{K}^n$ be definable, $\pi := \prod^n_d$, and*

$$Z := Z(A) := \{a \in A\colon\ \exists U \text{ neighbourhood of } a \text{ s.t. } \pi(A \cap U) \text{ is nowhere dense}\}.$$

Then, Z is definable and open subset of A, and $\pi(Z)$ is meager.

Proof. The fact that Z is definable and open in A is trivial. Let $Z' := Z(\overline{A})$.

Claim 3. *$Z = Z' \cap A$.*

Let $a \in Z$. Then, $\pi(A \cap U)$ is nowhere dense, for some U open neighbourhood of a. Moreover, $A \cap U$ is dense in $\overline{A} \cap U$, because U is open, and thus $\pi(\overline{A} \cap U) \subseteq \overline{\pi(A \cap U)}$. Hence, $\pi(\overline{A} \cap U)$ is nowhere dense, and therefore $a \in Z'$.

Therefore, it suffices to prove the lemma in the case when A is closed. Since Z is open in A, Z is locally closed. Moreover, for every $a \in Z$ there exists U neighbourhood of a, such that $\pi(U \cap Z)$ is nowhere dense; thus, by [9, Corollary 3.8], $\pi(Z)$ is meager. □

Lemma 2.34. *Let $f : \mathbb{K}^n \to \mathbb{K}^m$ be definable. If the graph $\Gamma(f)$ is an $\mathcal{F}_\sigma$ set, then $\mathcal{D}(f)$ is meager.*

Proof. See [9, Lemma 3.10]. □

Lemma 2.35. *Let $f : \mathbb{K}^n \to \mathbb{K}$ be definable. Define*

$$N_f := \{x \in \mathbb{K}^n\colon \text{ x is a strict local minimum for } f\}.$$

Then, N_f is meager in $\mathbb{K}^n$.

Proof. For every $r > 0$, let

$$N(r) := \{x \in \mathbb{K}^n\colon \ \|x\| \leqslant r \,\&\, x \text{ is a strict minimum for } f \text{ in the ball } B(x, 1/r)\}.$$

Note that $N_f = \bigcup_r N(r)$. Moreover, each $N(r)$ is closed and discrete, and hence nowhere dense. Thus, N_f is meager. □

2.7. Locally closed sets

Definition 2.36. Let $X \subseteq \mathbb{K}^n$. Define $\mathrm{lc}(X) := \{x \in X\colon \ X \text{ is locally closed at } x\}$ (that is, $x \in \mathrm{lc}(X)$ iff there exists an open ball B of centre x, such that $X \cap B = \overline{X} \cap B$), and $\ulcorner X \urcorner := X \setminus \mathrm{lc}(X)$.

Define $X^{\ulcorner 0 \urcorner} := X$, and, for each $k \in \mathbb{N}$, $X^{\ulcorner k+1 \urcorner} := \ulcorner X^{\ulcorner k \urcorner} \urcorner$.

Notice that $\mathrm{lc}(X)$ is locally closed, and therefore constructible. Notice also that, if X is definable, then also $\ulcorner X \urcorner$ and each $X^{\ulcorner k \urcorner}$ are definable. Therefore, if X is an $\mathcal{F}_\sigma$, then $\ulcorner X \urcorner$ is also an $\mathcal{F}_\sigma$.

Proposition 2.37. *$\ulcorner A \urcorner = A \cap \partial(\partial A)$. A is the union of m locally closed sets if and only if $A^{\ulcorner m+1 \urcorner}$ is empty.*

Proof. See [1], where ∂A is denoted by $\check{A}$, and $\ulcorner A \urcorner$ by either $\mathcal{B}(A)$ or $H(A)$. □

Remark 2.38. If $U \subseteq \mathbb{K}^n$ is open, then $\mathrm{lc}(A) \cap U = \mathrm{lc}(A \cap U)$, and $\ulcorner A \cap U \urcorner = \ulcorner A \urcorner \cap U$.

Remark 2.39. Let $U \subseteq \mathbb{K}^n$ be open, and $A \subseteq U$ be closed in U. Let $E := \overline{A}$. Then, $E \cap \overline{U} = E$. Therefore, $E \cap \overline{U} = \overline{E \cap U}$.

Proof. The $\subseteq$ inclusion is obvious. The opposite inclusion follows immediately from $E = \overline{E \cap U}$. □

Remark 2.40. Let $A \subseteq \mathbb{K}^n$ be definable and locally closed, and $d \leqslant n$. Let $U \subseteq \mathbb{K}^n$ be definable and open, such that $A = \overline{A} \cap U$. Then, for every $x \in \mathbb{K}^d$,

$$A_x = \mathrm{cl}(A_x) \cap U_x,$$

and in particular A_x is locally closed.

Proof. $A_x \subseteq \mathrm{cl}(A_x) \cap U_x$ is obvious. For the opposite inclusion,

$$\mathrm{cl}(A_x) \cap U_x \subseteq (\overline{A})_x \cap U_x = (\overline{A} \cap U)_x = A_x. \qquad \square$$

3. Structures with o-minimal and locally o-minimal open core

3.1. Basic facts about locally o-minimal structures

In this subsection we will study more in details locally o-minimal structures. Many results proved here will be implicitly used in the study of structures with locally o-minimal open core. More advanced results will be proved in Section 5, when we will have more tools at our disposal. Remember that $\mathbb{K}$ is always a definably complete expansion of an ordered field.

Lemma 3.1. *The following are equivalent*:

(1) *$\mathbb{K}$ is locally o-minimal*;
(2) *for every definable function $f : \mathbb{K} \to \mathbb{K}$, the sign of f is eventually constant* (*that is, there exists $c \in K$ such that the sign of f restricted to $(c, +\infty)$ is constant*);
(3) *for every definable function $f : \mathbb{K} \to \mathbb{K}$, for every $a \in \mathbb{K}$, there exists $c > a$ such that the sign of f restricted to (a, c) is constant*;
(4) *for every X definable subset of $\mathbb{K}$, either X is pseudo-finite, or it has nonempty interior.*

Proof. (2) ⇔ (3) is obvious, since $\mathbb{K}$ is an ordered field.

(3) ⇒ (1). Apply (3) to the characteristic function of X.

(1) ⇒ (3). Let $f : \mathbb{K} \to \mathbb{K}$ be definable, and $a \in \mathbb{K}$. Let $X := f^{-1}(0)$. By (1), there exists $c > a$ such that (a, c) is a subset either of X, or of $\mathbb{K} \setminus X$. In the first case, we are done; in the second, let $Y := f^{-1}((-\infty, 0))$. By decreasing c if necessary, either $(a, c) \subseteq Y$, or $(a, c) \subseteq \mathbb{K} \setminus (X \cup Y)$.

(1) ⇒ (4). Let $X \subset \mathbb{K}$ be definable with empty interior. We must prove that X is pseudo-finite. By replacing X with $f(X)$, where $f : \mathbb{K} \to (0, 1)$ is a definable homeomorphism (see Lemma 2.22), w.l.o.g. we can assume that X is bounded. Let $x \in \mathbb{K}$. By (1), since X has empty interior, there exist $a < x < b$ such that both (a, x) and (x, b) are disjoint from X. Thus, x cannot be an accumulation point for X, and therefore X is pseudo-finite.

(4) ⇒ (1). Let $X \subseteq \mathbb{K}$ be definable, $x \in \mathbb{K}$, $Y := (x, +\infty) \cap X$, and $Z := \mathring{Y}$. Since Z is open, $\mathrm{bd}(Z) := \overline{Z} \setminus \mathring{Z}$ has empty interior, and therefore it is pseudo-finite. Thus, there exists $y > x$ such that (x, y) is contained either in Z or in $\mathbb{K} \setminus Z$. In the first case, we have the conclusion. In the second case, we have that $W := X \cap (x, y)$ has empty interior. Thus, W is pseudo-finite, and therefore, after possibly decreasing $y > x$, we have $(x, y) \subseteq \mathbb{K} \setminus X$. □

Ref. [2, 2.11] proves the following other equivalent formulations of local o-minimality.

Fact 3.2. Let $\mathbb{F}$ be a definably complete structure (not necessarily expanding a field). Then, t.f.a.e.:

(1) for every definable $X \subseteq \mathbb{F}$ and $a \in \mathbb{F}$, there is an open interval I around a, such that $I \cap X$ is a finite union of points and intervals;
(2) for every definable $X \subseteq \mathbb{F}$, either X has interior, or it is closed and discrete;
(3) for every definable $X \subseteq \mathbb{F}$, X is constructible; moreover, every discrete definable set $X \subset \mathbb{F}$ is closed;
(4) for every definable $X \subseteq \mathbb{F}$, $\mathrm{lc}(X)$ is nonempty; moreover, every discrete definable set $X \subset \mathbb{F}$ is closed.

If moreover $\mathbb{F}$ expands an ordered field, then the above conditions are equivalent to our definition of local o-minimality.

Since locally o-minimal structures *a fortiori* have locally o-minimal open core, Theorem 3.3 will apply to them. Later, we will show that if $\mathbb{K}$ is locally o-minimal, then every definable set is constructible and has a well-behaved dimension.

3.2. Structures with (locally) o-minimal open core

Theorem 3.3. *The following are equivalent*:

(1) *Every definable discrete closed subset of $\mathbb{K}$ is pseudo-finite*;
(2) *Every definable discrete closed subset of $\mathbb{K}^n$ is pseudo-finite for every $n \in \mathbb{N}$*;
(3) *Every definable discrete subset of $\mathbb{K}$ is pseudo-finite*;
(4) *Every definable discrete subset of $\mathbb{K}^n$ is pseudo-finite for every $n \in \mathbb{N}$*;
(5) *Every definable nowhere-dense subset of $\mathbb{K}$ is pseudo-finite*;
(6) *Every definable meager subset of $\mathbb{K}$ is pseudo-finite*;
(7) *Every definable nowhere-dense subset of $\mathbb{K}$ is discrete*;
(8) *The **open core** of $\mathbb{K}$ is **locally o-minimal***;
(9) *The union of any definable increasing family of pseudo-finite subsets of $\mathbb{K}$ is pseudo-finite*;
(10) *The union of any definable increasing family of pseudo-finite subsets of $\mathbb{K}^n$ is pseudo-finite, for every $n \in \mathbb{N}$.*

Moreover, if any of the above equivalent conditions is satisfied, then every $\mathcal{F}_\sigma$ subset of $\mathbb{K}^n$ is constructible, $\mathbb{K}$ is Baire, and every meager subset of $\mathbb{K}^n$ is nowhere dense.

We will prove the above theorem in Section 4. In the remainder of this section we will give some consequences.

Definition 3.4. Let $X \subseteq \mathbb{K}^{n+m}$. Define $\mathrm{Fin}_n(X) := \{y \in \mathbb{K}^n : X_y \text{ is finite}\}$.

"Being finite" is not a definable property in general (see Definition 2.14). The following lemma characterises when it *is* a definable property.

Lemma 3.5. *The following are equivalent*:

(1) *Every pseudo-finite set is finite*;
(2) *For every $X \subseteq \mathbb{K}^2$ definable,* $\mathrm{Fin}_1(X)$ *is also definable*;
(3) *"Being finite" is a definable property.*

Proof. (3) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are clear. Assume (2). Let $Z \subset \mathbb{K}^n$ be pseudo-finite; we have to prove that Z is finite. Assume that $n = 1$. Let $r := \sup\{r \in \mathbb{K}\colon\ Z \cap [-r, r] \text{ is finite}\}$. Clearly, $r = +\infty$, and therefore, since Z is bounded, Z is finite. The general case then follows from Corollary 2.24. □

Notice that if $\mathbb{K}$ expands $\mathbb{R}$, then "being finite" *is* a definable property. Notice also that "Uniform Finiteness" (UF), as defined in [2] (also known as "elimination of the quantifier $\exists^\infty$") is a stronger property than "being finite is definable in $\mathbb{K}$", because the former says that "being finite" is definable in the theory of $\mathbb{K}$ (for instance, if $\mathbb{K}$ expands $\mathbb{R}$, then "being finite" is definable for $\mathbb{K}$, but not necessarily for the theory of $\mathbb{K}$).

Corollary 3.6. *The following are equivalent*:

(1) *The open core of $\mathbb{K}$ is locally o-minimal, and every pseudo-finite subset of $\mathbb{K}$ is finite*;
(2) *The open core of $\mathbb{K}$ is locally o-minimal, and every pseudo-finite subset of $\mathbb{K}^n$ is finite, for every $n \in \mathbb{N}$*;
(3) *The open core of $\mathbb{K}$ is locally o-minimal, and for every $X \subseteq \mathbb{K}^2$ definable,* $\mathrm{Fin}_1(X)$ *is definable*;
(4) *The open core of $\mathbb{K}$ is locally o-minimal, and "being finite" is a definable property for $\mathbb{K}$*;
(5) *$\mathbb{K}$ has **o-minimal open core***;
(6) *Every definable closed discrete subset of $\mathbb{K}$ is finite*;
(7) *Every definable closed discrete subset of $\mathbb{K}^n$ is finite, for every $n \in \mathbb{N}$*;
(8) *Every definable discrete subset of $\mathbb{K}$ is finite*;
(9) *Every definable discrete subset of $\mathbb{K}^n$ is finite, for every $n \in \mathbb{N}$.*

The main result of [2] is that, if $\mathbb{K}$ satisfies UF, then it has o-minimal open core (see Corollary 3.8); we want to examine the converse (see [2, 4.2]). In other words: assume that $\mathbb{K}$ has o-minimal open core, and let $\mathbb{K}' \equiv \mathbb{K}$; by Theorem 3.3, we know that the open core of $\mathbb{K}'$ is locally o-minimal: we want to know whether it is also o-minimal.

Remark 3.7. If $\mathbb{K}$ satisfies UF, then $\mathbb{K}$ satisfies Corollary 3.6(6).

Proof. Let $X \subseteq \mathbb{K}$ be definable, discrete and closed. We want to prove that X is finite. Since $\mathbb{K}$ satisfies UF, w.l.o.g. we can assume that $\mathbb{K}$ is ω-saturated. For every $r \in \mathbb{K}$, $X \cap [-r, r]$ is pseudo-finite, and hence finite. Since $\mathbb{K}$ is ω-saturated, X is also finite. □

The following corollary was proven in [2].

Corollary 3.8. *(See [2, Theorem A].) If $\mathbb{K}$ satisfies UF then $\mathbb{K}$ has o-minimal open core.*

Proof. By Remark 3.7 and Corollary 3.6. □

Remark 3.9. By Corollary 3.6(2), if $\mathbb{K}$ has o-minimal open core, then every pseudo-finite set is finite.
Moreover, if $\mathbb{K}$ is an expansion of $\mathbb{R}$, then the open core of $\mathbb{K}$ is locally o-minimal iff it is o-minimal.

Conjecture 3.10. *$\mathbb{K}$ satisfies UF if and only if $\mathbb{K}$ has o-minimal open core.*

3.3. Examples

Every o-minimal structure is locally o-minimal. Since "local o-minimality" is a first-order property, an ultra-product of locally o-minimal structures is also locally o-minimal. The above observation leads us to the following example.

Example 3.11. Let M be a fixed o-minimal structure, in a language $\mathcal{L}$. Let P be a new unary predicate. For every $n \in \mathbb{N}$, let $P_n := \{1, 2, \dots, n\} \subset M$. Let $M_n := \langle M, P_n\rangle$ be the $L(P)$-expansion of M, where P is interpreted by P_n. Let $N := \langle M^*, P^*\rangle$ be a non-principal ultra-product of the M_n. Then, N is locally o-minimal, but not o-minimal (because P^* is pseudo-finite, but not finite).

If M is an expansion of $\mathbb{R}$, then the above structure can be considered a restriction of an ultra-product of $\bar{\mathbb{R}}(\mathbb{N})$. However, we can take for M the counter-example of Hrushovski and Peterzil [11]: M^* is an elementary extension of M, and therefore also satisfies a formula that cannot be true in any expansion of the reals, and the same holds for N.

Since every locally o-minimal structure is also Baire (we are assuming that everything is definably complete), then the above is also a non-trivial example of definably complete and Baire structure.

Along the same lines, we can also consider the following example: let M be as before, and M' be an elementary extension of M, such that M is dense in M' and different from M'. Define P_n as before, let $M_n := \langle M', M, P_n\rangle$, and $N := \langle M'^*, M^*, P^*\rangle$

be a non-principal ultra-product of the M_n. Each M_n has o-minimal open core; thus, N has locally o-minimal open core, but does not have o-minimal open core. Again, if M is the counter-example of Hrushovski and Peterzil, we see that N is not the restriction of an elementary extension of an expansion of $\bar{\mathbb{R}}$.

Regarding structures with o-minimal open core, [2, §8] ask the following question: Suppose that $\mathbb{K}$ is Archimedean and has o-minimal open core; does $\mathrm{Th}(\mathbb{K})$ have a model over $\mathbb{R}$ that is unique up to isomorphism? While we do not have an answer regarding the existence, it is easy to see that uniqueness may fail. In fact, consider the case when $\mathbb{K}$ is given by the dense pair $\langle \bar{\mathbb{R}}, \mathbb{R}^{\mathrm{alg}} \rangle$, where $\mathbb{R}^{\mathrm{alg}}$ is the field of real algebraic numbers. Let S be any real-closed field, such that $\mathbb{R}^{\mathrm{alg}} \subsetneq S \subsetneq \mathbb{R}$. Then, $\langle \bar{\mathbb{R}}, S \rangle$ is elementarily equivalent to $\mathbb{K}$, but it is not isomorphic to $\mathbb{K}$ (see [21]). A stronger question is: if $\mathbb{K}$ is Archimedean and has o-minimal open core, does $\mathbb{K}$ have an elementary extension over the reals? Again, uniqueness cannot be expected. For instance, let S and U be real-closed fields, such that $\mathbb{R}^{\mathrm{alg}} \subsetneq S \subsetneq \mathbb{R}$, $\mathbb{R}^{\mathrm{alg}} \subsetneq U \subsetneq \mathbb{R}$, and U and S are free over $\mathbb{R}^{\mathrm{alg}}$; then, both $\langle \bar{\mathbb{R}}, S \rangle$ and $\langle \bar{\mathbb{R}}, \mathbb{R}^{\mathrm{alg}} \rangle$ are elementary extensions of $\langle U, \mathbb{R}^{\mathrm{alg}} \rangle$ (see [21, Corollary 2.7]).

4. Proof of Theorem 3.3

(3) $\Rightarrow$ (1) is obvious, while (1) $\Rightarrow$ (3) is Lemma 2.28.

(1) $\Rightarrow$ (9) Let $(X(r))_{r \in \mathbb{K}}$ be a definable increasing family of pseudo-finite subsets of $\mathbb{K}$, and $X := \bigcup_r X(r)$. Assume, for a contradiction, that X is not pseudo-finite. W.l.o.g., we can assume that $X \subseteq (0,1)$, and that 0 is an accumulation point of X. For every $r \in \mathbb{K}$, let $z(r) := \min(X(r))$, $Z := \{z(r)\colon r \in \mathbb{K}\}$, $Y := \{1/z(r)\colon r \in \mathbb{K}\}$.

Claim 4. *The only accumulation point of Z in $\mathbb{K}$ is* 0.

In fact, suppose, for a contradiction, that $c > 0$ is an accumulation point of Z. Since 0 is an accumulation point for X; there exists $r_0 \in \mathbb{K}$ such that $X(r_0) \cap (0, c/2) \neq \emptyset$. Thus, $z(r_0) < c/2$, and, since $z(x)$ is a decreasing function, $z(r) < c/2$ for every $r \geqslant r_0$. Let $Z(r_0) := \{z(r)\colon r < r_0\}$, and $Z' := Z \setminus Z(r_0)$. Since $Z(r_0) \subseteq X(r_0)$, $Z(r_0)$ is pseudo-finite. Moreover, since $Z' \subseteq (0, c/2)$, c is not an accumulation point of Z', and thus it is not an accumulation point of Z, absurd.

By the claim, Y is discrete and closed, and therefore, by hypothesis, it is pseudo-finite. Hence, Z is also pseudo-finite; therefore, 0 cannot be an accumulation point for Z, absurd.

(9) $\Leftrightarrow$ (10) is clear from Corollary 2.24.

(9) $\Rightarrow$ (3) follows from Lemma 2.29.

Hence, we have the equivalence (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (9) $\Leftrightarrow$ (10).

We now prove that (1) $\Rightarrow$ (6). Let $X \subseteq \mathbb{K}$ be meager; thus, X is the union of a definable increasing family $(X(r))_{r \in K}$ of nowhere dense subsets of $\mathbb{K}$. We want to prove that X is pseudo-finite. W.l.o.g., we can assume that each $X(r)$ is d-compact. By (9), it suffices to prove that each $X(r)$ is pseudo-finite. Thus, we fix $r \in \mathbb{K}$, and prove that $Y := X(r)$ is pseudo-finite, knowing that it is d-compact and has empty interior. Assume, for a contradiction, that Y has an accumulation point in $\mathbb{K}$. W.l.o.g., we can assume that $Y \subset (0,1)$, and that 0 is an accumulation point of Y. Since $\mathbb{K}$ is definably complete and Y is closed, $(0,1) \setminus Y$ is a union of disjoint open intervals; let D be the set of centres of those intervals, that is:

$$D := \bigl\{z \in (0,1)\colon \exists r > 0,\ z - r \in Y,\ z + r \in Y,\ (z - r, z + r) \cap Y = \emptyset\bigr\}.$$

Note that D is discrete. By (3), D is pseudo-finite; let $a := \min(D)$, and $r > 0$ such that $a - r, a + r \in Y$ and $(a - r, a + r) \cap Y = \emptyset$. Thus, $(0, a - r) \subseteq Y$. Since Y has empty interior, $a - r = 0$. However, this contradicts the fact that 0 is an accumulation point for Y.

(6) $\Rightarrow$ (5) $\Rightarrow$ (1) and (5) $\Rightarrow$ (7) are clear.

(7) $\Rightarrow$ (1). Let $X \subseteq \mathbb{K}$ be discrete and closed. Assume, for a contradiction, that X is not bounded; let $Y := \{1/x\colon 0 \neq x \in X\} \cup \{0\}$. Since Y is nowhere dense, Y is also discrete, contradicting the fact that X is unbounded. Hence, we have the equivalence (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7) $\Leftrightarrow$ (9) $\Leftrightarrow$ (10).

It is clear that (6) implies that $\mathbb{K}$ is Baire.

Now, we prove that (6) also implies that every $\mathcal{F}_\sigma$ subset of $\mathbb{K}^n$ is constructible. This in turns implies that the open core of $\mathbb{K}$ is locally o-minimal, since then every set definable in the open core will be constructible (see Fact 2.4 and Remark 2.6), and constructible subsets of $\mathbb{K}$ with empty interior are meager, and thus pseudo-finite.

Let $X \subseteq \mathbb{K}^n$ be an $\mathcal{F}_\sigma$.

If $n = 1$, then $X = \mathring{X} \cup (X \setminus \mathring{X})$, and therefore X is the union of an open set and a pseudo-finite set; thus, X is constructible.

If $n > 1$, we proceed, as in the proof of [2, 3.4], by induction on n and $\langle d, k \rangle := \mathrm{fdim}(X)$ (the full dimension of X). Note that if $d = 0$, then X is pseudo-finite, and hence constructible.

If $d = n$, then $X = \mathring{X} \cup (X \setminus \mathring{X})$. $\mathring{X}$ is open, and hence constructible, while $X \setminus \mathring{X}$ has dimension less than n, and therefore, by induction, it is also constructible; thus, X is constructible.

If $0 < d < n$, w.l.o.g. we can assume that $\pi(X)$ has nonempty interior, where $\pi := \prod_d^n$. Let $A := \{a \in \mathbb{K}^d\colon \dim(X_a) > 0\}$. By Lemma 2.17, A is an $\mathcal{F}_\sigma$ of dimension $< d$; therefore, by inductive hypothesis, A is constructible. Let $Y := \pi^{-1}(A) \cap X$. Since $\mathrm{fdim}(Y) < \mathrm{fdim}(X)$, by induction, Y is constructible. Thus, it suffices to prove that $X \setminus Y$ is constructible, and hence we can reduce to the case when $A = \emptyset$.

Thus, we can reduce to the case when $d < n$, $B := \pi(X)$ has nonempty interior, and, for every $y \in B$, X_y is pseudo-finite.

Here the proof will diverge from [2], because we do not have the hypothesis UF, and therefore we cannot proceed by induction on a uniform bound N on the cardinality of the fibres X_y.

We claim that $\mathrm{fdim}(\ulcorner X \urcorner) < \mathrm{fdim}(X)$. If the claim is true, then, by inductive hypothesis, $\ulcorner X \urcorner$ is constructible, and therefore X is constructible.

If, for a contradiction, $\mathrm{fdim}(\ulcorner X \urcorner) = \mathrm{fdim}(X)$, then $\pi(\ulcorner X \urcorner)$ contains a nonempty open ball B'; by shrinking X, we can assume than $B = B'$.

By hypothesis, there exists a definable increasing family $(X(t))_{t \in K}$ of d-compact subsets of $\mathbb{K}^n$, such that $X = \bigcup_t X(t)$. Let $Y(t) := \pi(X(t))$; note that each $Y(t)$ is d-compact, and $B = \bigcup_t Y(t)$. Since $\mathbb{K}$ is Baire, $\mathbb{K}^d$ is also Baire, and therefore there exists $t_0 \in K$ such that $Y(t_0)$ has nonempty interior. Let $B' \subseteq Y(t_0)$ be a d-compact box with nonempty interior; by shrinking X, we can assume that $B = B'$, and, by redefining the family $(X(t))_{t \in \mathbb{K}}$, that $B = Y(t)$ for every t.

The fact that, for every $y \in B$, X_y is pseudo-finite easily implies the following:

Remark 4.1. For every $y \in B$ there exists $t \in \mathbb{K}$ such that $\{y\} \times X_y \subseteq X(t)$.

Lemma 4.2. *There exist $B' \subseteq B$ nonempty open box, and $t_0 \in \mathbb{K}$, such that $X \cap (B' \times \mathbb{K}^{n-d}) \subseteq X(t_0)$.*

Proof. For every $t \in \mathbb{K}$, define $Z(t) := X \setminus X(t)$, and $Y(t) := B \setminus \pi(Z(t))$. Note that each $Z(t)$ is an $\mathcal{F}_\sigma$, and therefore $Y(t)$ is a $\mathcal{G}_\delta$. By induction on n, $Y(t)$ is constructible. Moreover, by the remark, $B = \bigcup_t Y(t)$. Since B is not meager, there exists $t_0 \in \mathbb{K}$ such that $\overline{Y(t_0)}$ has nonempty interior. However, since $Y(t_0)$ is constructible, this implies that $Y(t_0)$ itself has nonempty interior, and therefore contains an open box B'. $\square$

Hence, $X \cap (B' \times \mathbb{K}^{n-d}) = X(t_0) \cap (B' \times \mathbb{K}^{n-d})$, and therefore $X \cap (B' \times \mathbb{K}^{n-d}) \subseteq \mathrm{lc}(X)$. Since the same reasoning can be made for every open box $\tilde{B} \subseteq B$ instead of B, we conclude that $\pi(\ulcorner X \urcorner)$ has empty interior, absurd.

Therefore, we have proved (6) $\Rightarrow$ (8), and (1) $\Rightarrow$ (8) also follows.

For (8) $\Rightarrow$ (1), let $D \subset \mathbb{K}$ be definable, discrete, and closed. Since D is closed, it is definable in the open core of $\mathbb{K}$. Moreover, D has empty interior; thus, by Lemma 3.1, it is pseudo-finite.

(4) $\Rightarrow$ (2) $\Rightarrow$ (1) is clear: thus, to conclude it remains to prove (1) $\Rightarrow$ (4). Let $X \subseteq \mathbb{K}^n$ be discrete and definable. Since we have seen that $\mathbb{K}$ is Baire, this implies that every projection $\mu(X)$ of X on a coordinate axis has empty interior; however, $\mu(X)$ is an $\mathcal{F}_\sigma$-set, and thus $\mu(X)$ is pseudo-finite for every coordinate axis. Thus, X is pseudo-finite.

The fact that every meager definable subset of $\mathbb{K}^n$ is nowhere dense follows from the fact that, if X is meager, then it is contained in a meager $\mathcal{F}_\sigma$ set Y; we have seen that Y is constructible; hence, Y is nowhere dense. $\square$

4.1. Further results on structures with locally o-minimal open core

From the above proof, we can extract the following results.

Proviso. For the remainder of this subsection, $\mathbb{K}$ has locally o-minimal open core.

Lemma 4.3. *Let $C \subseteq \mathbb{K}^n$ be definable and constructible (or, equivalently, an $\mathcal{F}_\sigma$) set of dimension d. Then, $\dim(\ulcorner C \urcorner) < d$. Therefore, C is the union of $d+1$ locally closed definable sets.*

For a general definably complete structure, we do not know if "being constructible" is definable. In fact, a constructible set X is a finite union of locally closed sets; however, if we do not have a bound (independent from the parameters of definition of X) on the number of locally closed sets C_i such that $X = \bigcup_i C_i$, we are not able to express the constructibility of X in a definable way.

However, the above lemma implies that, when the open core of $\mathbb{K}$ is locally o-minimal, "being constructible" is a definable property. In fact (given our assumption on $\mathbb{K}$) $X \subseteq \mathbb{K}^n$ is constructible iff $X^{\ulcorner n+1 \urcorner}$ is empty. Thus, Lemma 2.31 implies the following:

Corollary 4.4. *Let A be pseudo-finite, and $(X_a)_{a \in A}$ be a definable family of constructible subsets of $\mathbb{K}^n$. Then, $\bigcup_{a \in A} X_a$ is also constructible.*

Moreover, the dimension is well-behaved for *constructible* sets definable in $\mathbb{K}$. We have already seen that $\dim(C \cup C') = \max(\dim C, \dim C')$ for constructible definable sets (Lemma 2.18).

Lemma 4.5. *Let C be a definable constructible subset of $\mathbb{K}^n$. Then, $\dim(C) \leqslant 0$ iff C is pseudo-finite.*

Proof. By Corollary 2.24 and Lemma 3.1. $\square$

In Lemma 5.12, we will show that, for C definable, constructible, and nonempty subset of $\mathbb{K}^n$, we have $\dim(\partial C) < \dim(C)$ if C is nonempty.

Lemma 4.6. *Let $C(t)$ be a definable increasing family of subsets of $\mathbb{K}^n$, such that each $C(t)$ is constructible (or, equivalently, an $\mathcal{F}_\sigma$), and $\dim(C(t)) < d$. Let $C := \bigcup_t C(t)$. Then, $\dim(C) < d$.*

Proof. Assume, for a contradiction, that $\pi(C)$ contains a nonempty open set U, where $\pi := \prod^n_d$. For every t, let $D(t) := \pi(C(t))$. By hypothesis, each $D(t)$ has empty interior and is an $\mathcal{F}_\sigma$, and therefore it is meager. Thus, since the open core of $\mathbb{K}$ is locally o-minimal, each $D(t)$ is nowhere dense. However, $\bigcup_t D(t) \supseteq U$, and thus U is meager, which is absurd. □

We can prove a different version of Corollary 4.4, albeit with a longer proof.

Lemma 4.7. *Let $(A(t))_{t\in\mathbb{K}}$ be a definable increasing family of constructible subsets of $\mathbb{K}^n$. Then, $X := \bigcup_{t\in\mathbb{K}} A(t)$ is also constructible.*

Proof. We will proceed by induction on $\langle d, k\rangle := \operatorname{fdim}(X)$. If $d = 0$, then each $A(t)$ is pseudo-finite (by Lemma 4.5); therefore, X is pseudo-finite, and hence constructible. If $d = n$, then $X := \mathring{X} \sqcup (X \setminus \mathring{X})$. Define $B(t) := A(t) \setminus \mathring{X}$; notice that $X \setminus \mathring{X} = \bigcup_t B(t)$, and $(B(t))_{t\in\mathbb{K}}$ is a definable increasing family of constructible sets. Thus, by inductive hypothesis, $X \setminus \mathring{X}$ is constructible, and therefore X is constructible.

It remains to treat the case $0 < d < n$. W.l.o.g., $\pi(X)$ has nonempty interior, where $\pi := \prod^n_d$. By inductive hypothesis, $\pi(X)$ is constructible. For each $t \in \mathbb{K}$, define $C(t) := \{y \in \mathbb{K}^d\colon \dim(A(t)_y) > 0\}$. Each $C(t)$ is definable in the open core of $\mathbb{K}$, and thus constructible.

Claim 5. *$C(t)$ has empty interior, for every t.*

Assume, for a contradiction, that, for some $t_0 \in \mathbb{K}$, $C(t_0)$ has nonempty interior, and let $t \geqslant t_0$. Then, since $\dim(A(t)) \leqslant \dim(X) = d$, $\pi(A(t))$ has empty interior, and thus it is nowhere dense. Hence, $\pi(X)$ is meager, a contradiction.

Let $Y := \bigcup_t C(t)$. By inductive hypothesis, Y is constructible; moreover, by Claim 5, it is meager, and therefore $\dim(Y) < d$. Let $X_1 := X \setminus (Y \times \mathbb{K}^{n-d})$ and $X_2 := X \cap (Y \times \mathbb{K}^{n-d})$. We have seen that $\operatorname{fdim}(X_2) < \operatorname{fdim}(X)$, and thus, by inductive hypothesis, X_2 is constructible. Thus, w.l.o.g. we can assume that $X = X_1$, that is, $A(t)_y$ is pseudo-finite for every $t \in \mathbb{K}$ and $y \in \mathbb{K}^d$. Let $y \in \mathbb{K}^d$; by Lemma 4.6, $\dim(X_y) \leqslant 0$, and, by Lemma 4.5, X_y is pseudo-finite.

Notice that, for every $t \in \mathbb{K}$, $X \setminus A(t) = \bigcup_s (A(s) \setminus A(t))$, and therefore, by induction on n, $\pi(X \setminus A(t))$ is constructible. Thus, reasoning as in Lemma 4.2, we can prove the following claim.

Claim 6. *For every nonempty open box B there exists $t \in \mathbb{K}$ and $B' \subseteq B$ nonempty open box, such that $X \cap (B' \times \mathbb{K}^{n-d}) = A(t) \cap (B' \times \mathbb{K}^{n-d})$.*

Fix B' and t as in the above claim, and let $\tilde{A}(t) := A(t) \cap (B' \times \mathbb{K}^{n-d})$. By Lemma 4.3, $\dim(\tilde{A}(t) \setminus \operatorname{lc}(\tilde{A}(t))) < \dim(\tilde{A}(t)) \leqslant d$. Moreover, $\ulcorner X \urcorner \cap (B' \times \mathbb{K}^{n-d}) = \ulcorner \tilde{A}(t) \urcorner$. Therefore, by Claim 6, $\pi(\ulcorner X \urcorner)$ has empty interior. Thus, by inductive hypothesis, $\ulcorner X \urcorner$ is constructible, and hence X is constructible. □

Lemma 4.8. *Let $C \subseteq \mathbb{K}^n$ be definable. T.f.a.e.:*

(1) *C is a.o.*;
(2) *$C \setminus \mathring{C}$ is nowhere dense*;
(3) *∂C is nowhere dense*;
(4) *$\operatorname{bd}(C)$ is nowhere dense.*

Proof. (4) ⇒ (3) ⇒ (1) and (4) ⇒ (2) ⇒ (1) are clear (and are true even without the hypothesis on the open core).

For (1) ⇒ (4), assume that C is a.o. Then, $C = U \mathbin{\Delta} F$, for some definable meager set F, and some definable open set U. By Remark 2.9,

$$\operatorname{bd}(C) \subseteq \operatorname{bd}(U) \cup \operatorname{bd}(F) \subseteq \partial U \cup \overline{F}.$$

Since the open core of $\mathbb{K}$ is locally o-minimal, both $\overline{F}$ and ∂U are nowhere dense, and hence $\operatorname{bd}(C)$ is also nowhere dense. □

Lemma 4.9. *Let $(A(t))_{t\in\mathbb{K}}$ be a definable increasing family of subsets of $\mathbb{K}^n$, and $C := \bigcup_t A(t)$. If each $A(t)$ is meager, then C is meager (and thus nowhere dense). If each $A(t)$ is a.o., then C is a.o.*

Proof. Since the open core of $\mathbb{K}$ is locally o-minimal, if each $A(t)$ is meager, then each $A(t)$ is nowhere dense, and therefore C is meager.

If each $A(t)$ is a.o., let $U := \mathring{C}$, and $B(t) := A(t) \setminus U$. Notice that $B(t) \subseteq A(t) \setminus \operatorname{int}(A(t))$; thus, by Lemma 4.8, each $B(t)$ is nowhere dense. Therefore, $C \setminus U = \bigcup_t B(t)$ is meager, and hence C is a.o. □

We don't know whether the above lemma remains true without the assumption that $\mathbb{K}$ has locally o-minimal open core.

5. Locally o-minimal structures revisited

Now that we have proved a few results about structures with locally o-minimal open core (most notably, Theorem 3.3), let us use those results to study more in details locally o-minimal structures.

Proviso. In this section, $\mathbb{K}$ is a definably complete and locally o-minimal expansion of an ordered field.

5.1. The Monotonicity Theorem

Theorem 5.1 *(Monotonicity Theorem). Let $I \subseteq \mathbb{K}$ be an open interval and $f : I \to \mathbb{K}$ be a definable function. Then, there exists a pseudo-finite set $A \subseteq I$, such that on each open sub-interval of $I \setminus A$ the function f is either constant, or strictly monotone and continuous.*

The proof proceeds as in [22, Theorem 3.1.2], by substituting everywhere "finite" with "pseudo-finite", and "infinite" with "non-pseudo-finite"; the "difficult case" in the analogue of [22, ¶3.1.5, Lemma 2] follows immediately from Lemma 2.35. For an alternative proof of the above theorem, see [19].

We also have the following consequence:

Corollary 5.2. *Let $(a, b) \subseteq \mathbb{K}$ be an open interval, $f : (a, b) \to \mathbb{K}$ be definable, and $c \in (a, b)$. The limits $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^+} f(x)$ exist in $\mathbb{K}_\infty := \mathbb{K} \cup \{\pm\infty\}$. Also the limits $\lim_{x\to a^+} f(x)$ and $\lim_{x\to b^-} f(x)$ exist in $\mathbb{K}_\infty$.*

5.2. Constructibility and partition into multi-cells

In this subsection we will prove the following theorem.

Theorem 5.3. *Every definable subset of $\mathbb{K}^n$ is constructible. Therefore, $\mathbb{K}$ coincides with its open core.*

The above theorem was also proved in [19]; however, we obtain it as a corollary of a stronger result.

Definition 5.4. Let $X \subseteq \mathbb{K}^n$ be definable and of dimension d, and let $\pi : \mathbb{K}^n \to L$ be a projection onto a coordinate space of dimension d. For notational convenience, assume that $\pi = \prod_d^n$. Assume that $X \subseteq \mathbb{K}^d \times (0, 1)^{n-d}$. A point $\langle a, b\rangle \in \mathbb{K}^n$ is **(X, π)-normal** (see [22, 3.1.7]) if there exist a neighbourhood A of a and a neighbourhood B of b, such that either $A \times B$ is disjoint from X, or $(A \times B) \cap X = \Gamma(f)$ for some (unique and definable) continuous function $f : A \to B$.[1] A point $a \in \mathbb{K}^d$ is **(X, π)-bad** if it is the projection of a non-(X, π)-normal point; otherwise, a is **(X, π)-good**.

If X is unbounded, let $\phi : \mathbb{K} \to (0, 1)$ be a definable homeomorphism, and $\psi := \mathrm{id}^d \times \phi^{n-d} : \mathbb{K}^d \times \mathbb{K}^{n-d} \to \mathbb{K}^d \times (0, 1)^{n-d}$ be the corresponding homeomorphism. Then, we say that a is **(X, π)-good** if it is $(\psi(X), \pi)$-good: the definition does not depend on the particular choice of ϕ, and it is equivalent to the one in [22].

Note that the set of (X, π)-normal points is definable. Note that if $a \in L$ is (X, π)-good, then X_a is pseudo-finite (because it is discrete); thus, by Lemma 2.31, there exists A open neighbourhood of a, such that for every $b \in \mathbb{K}$ there exists B open neighbourhood of b as in Definition 5.4. Note also that if $L = \mathbb{K}^n$ and id is the identity function on $\mathbb{K}^n$, then the set of (X, id)-bad points is equal to $\operatorname{bd} X$.

Definition 5.5. A **multi-cell** of dimension d in $\mathbb{K}^n$, with respect to a coordinate space L, is a definable subset $X \subseteq \mathbb{K}^n$ of dimension equal to $\dim(L)$, such that every point of $\pi(X)$ is (X, π)-good, where π is the orthogonal projection onto L.

Note that the multi-cells of dimension 0 are the pseudo-finite sets, while the multi-cells of dimension n in $\mathbb{K}^n$ are the definable open subsets of $\mathbb{K}^n$. Note that if X is bounded, then the set of (X, π)-normal points is a multi-cell. Note also that if X is a multi-cell, then, locally around every point of X, X is a cell, and $\pi(X)$ is open (where π is as in the above

[1] Since we assumed that $X \subseteq \mathbb{K}^d \times (0, 1)^{n-d}$, we do not need to worry about the behaviour at infinity.

definition). However, a multi-cell X might not be a cell around points not of X. For instance, let $D \subseteq (0,1)$ be an infinite pseudo-finite set, and let

$$X := \left\{ \langle x, y \rangle \in \mathbb{K}^2 \colon x > 0,\ y > 0,\ (\exists m \in D)\ y = mx \right\}.$$

Then, X is a multi-cell, but, for every neighbourhood V of 0, $X \cap V$ has infinitely many definably connected components, and thus cannot be a cell, nor a finite union of cells.

Theorem 5.6. *Let $X \subseteq \mathbb{K}^n$ be definable. Then, there exists a finite partition of X into multi-cells. The number of multi-cells is bounded by a function of n.*

Since every multi-cell is locally closed, it is clear that the above theorem implies Theorem 5.3.

Proof of Theorem 5.6. For this proof, we call a definable set "strongly constructible" if it is a finite union of multi-cells. Let $X \subseteq \mathbb{K}^n$ be definable: we have to prove that X is strongly constructible.

If $n = 1$, then ∂X has empty interior (by Remark 2.9) and therefore is pseudo-finite. The same is true for $\mathbb{K} \setminus X$, and therefore, (also by Remark 2.9) $\operatorname{bd}(X)$ is pseudo-finite. Thus, by Lemma 2.11, it is clear that X is the union of a pseudo-finite set and an open cell, and therefore X is the union of 2 multi-cells.

If $n = 2$, we consider first the case when, for every $a \in \mathbb{K}$, X_a is pseudo-finite. W.l.o.g., we can assume that $X \subseteq (0,1) \times (0,1)$.

Let $\pi := \prod_1^2$, and $\mathcal{N} := \{\langle a, b \rangle \in \mathbb{K}^2 \colon \langle a, b \rangle \text{ is } (X, \pi)\text{-normal}\}$. Note that $Z := \mathbb{K}^2 \setminus \mathcal{N}$ is contained in $[0,1] \times [0,1]$. We call $\mathcal{B} := \pi(Z) \subseteq \mathbb{K}$ the set of "bad" points.

Claim 7. *Z is d-compact.*

In fact, it is bounded, and, by definition, its complement Y is open. Hence, $\mathcal{B}$ is d-compact.

Claim 8. *$\mathcal{B}$ is pseudo-finite.*

Proof. If not, let $I \subseteq \mathcal{B}$ be a nonempty open interval. For every $a \in I$; let $\beta(a) := \min(Z_a)$. Conclude as in the proof of [22, 3.1.7]. □

$\mathcal{N}$ is a multi-cell, and, by Lemma 2.31, Z is pseudo-finite. Thus, $X = \mathcal{N} \cup Z$ is strongly constructible.

We treat now the case when there exists some $a \in \mathbb{K}$ such that X_a is not pseudo-finite. Let $A := \{a \in \mathbb{K} \colon X_a \text{ is not pseudo-finite}\}$.

Lemma 5.7. *If A is not meager, then X has nonempty interior.*

Proof. If A is not meager, then it contains an open interval I. So, w.l.o.g. we can assume $A = I$. For every $a \in A$, there exist $b \in \mathbb{K}$ accumulation point of X_a. Let $f(a) \in \mathbb{K}$ be the smallest such b. Note that there exists $b' > f(a)$ such that $(f(a), b') \subseteq X_a$; let $g(a)$ be the greatest such b'. Let $I' \subseteq I$ be an open interval, such that f and g are continuous on I' (I' exists by the Monotonicity Theorem). Then, the set $\{\langle a, b \rangle \colon a \in I,\ f(a) < a < g(a)\}$ is an open set contained in X. □

Note that $X = \mathring{X} \cup (X \setminus \mathring{X})$. Since $X \setminus \mathring{X}$ has empty interior and $\mathring{X}$ is a multi-cell, w.l.o.g. we can assume that X has empty interior, and therefore A is pseudo-finite. As in the case $n = 1$, we have that, for every $a \in \mathbb{K}$, X_a is the union of its interior and a pseudo-finite set. Thus, $X \cap \pi^{-1}(A) = \bigcup_{a \in A} \{a\} \times X_a$ is strongly constructible. Since we have seen that $X \setminus \pi^{-1}(A)$ is also strongly constructible, we are done.

For the general case, let us prove, by induction on n, the following 2 statements:

(I_n) Every definable $X \subseteq \mathbb{K}^n$ is strongly constructible;
(II_n) If $A \subseteq \mathbb{K}^n$ is open and definable, and $f : A \to \mathbb{K}$ is a definable function, then there exists a nonempty open box $B \subseteq A$ such that $f \restriction B$ is continuous.

Note that II_n implies that the set of discontinuity points of f is nowhere dense.

We have already proved I_1 and II_1. So, assume that we have already proved I_m and II_m, for every $m \leqslant n$, and let us prove them for $n + 1$.

We will prove I_{n+1} by induction on $\langle d, k \rangle := \operatorname{fdim}(X)$, the full dimension of X. Assume we have already proved the statement for any Z of full dimension less than $\langle d, k \rangle$. W.l.o.g., $X \subseteq (0,1)^{n+1}$.

If $d = n + 1$, note that $X = \mathring{X} \sqcup (X \setminus \mathring{X})$. $\mathring{X}$ is open, and therefore a multi-cell, and $(X \setminus \mathring{X})$ has empty interior, and thus dimension less than $n + 1$; therefore, by the inductive hypothesis, $X \setminus \mathring{X}$ is strongly constructible, and we are done.

If $d = 0$, then X is pseudo-finite, and we are done.

If $0 < d \leqslant n$, w.l.o.g. we can assume that $Y := \pi(X)$ has nonempty interior, where $\pi := \prod_d^{n+1}$. Let $W := X \cap \pi^{-1}(Y \setminus \mathring{Y})$. Note that $\operatorname{fdim}(W) < \operatorname{fdim}(X)$, and therefore W is strongly constructible. Thus, we can assume that Y is open.

Let $\mathcal{N} := \{\langle a, b\rangle \in \mathbb{K}^d \times \mathbb{K}^{n+1-d}\colon \langle a, b\rangle \text{ is } (X, \pi)\text{-normal}\}$. Note that $Z := \mathbb{K}^n \times \mathbb{K} \setminus \mathcal{N}$ is contained in $[0, 1]^n \times [0, 1]$.

We call $\mathcal{B} := \pi(Z) \subseteq \mathbb{K}^n$ the set of "bad" points.

Again, $\mathcal{N}$ is open by definition, and thus $\mathcal{B}$ is d-compact. Thus, it suffices to prove that $\operatorname{fdim}(Z) < \operatorname{fdim}(X)$ to obtain that X is strongly constructible. If, for a contradiction, $\operatorname{fdim}(Z) = \operatorname{fdim}(X)$, then $\pi(Z)$ has nonempty interior; let $B \subseteq \pi(Z)$ be a nonempty open box. For every $a \in B$, let $\beta(a) := \operatorname{lex\,min} Z_a$. Proceeding as in the case $n = 2$, and using the inductive hypothesis II_n, we get a contradiction.

Let us prove now II_{n+1}. We use the same technique in [22, 3.2.17]. Let $f : A \to \mathbb{K}$ be a definable function, with $A \subseteq \mathbb{K}^n \times \mathbb{K}$ open and definable. Define the set A^* of well-behaved points for f as in [22, 3.2.17]. By the same proof as in [22], A^* is dense in A. Since A^* is definable, by I_{n+1} A^* is (strongly) constructible, and therefore it contains a nonempty open box B. Moreover, by [22, Lemma 3.2.16], f is continuous on B.

Thus, X is a finite union of multi-cells. From the above argument, we can see that the number N of multi-cells partitioning X is bounded by a function of n, d, and k. However, d and k are bounded by functions of n, and thus N is bounded by a function of n.

Open Problem 5.8. We would like to have a structure theorem for *open* definable sets. Such a theorem should also give a further refinement of Theorem 5.6.

Schoutens [19] tackles the above problem, and proves that every definable set has a "quasi-cell decomposition"; it would be interesting to combine his results with Theorem 5.6 to solve the above problem.

5.3. Continuity and differentiability

Remark 5.9. Let $f : \mathbb{K}^n \to \mathbb{K}^m$ be definable. Since, by Theorem 5.3, $\Gamma(f)$ is constructible, then, by Lemma 2.34, $\mathcal{D}(f)$, the set of discontinuity points of f, is meager, and thus nowhere dense.

Lemma 5.10. *Let $I \subseteq \mathbb{K}$ be an open interval and $f : I \to \mathbb{K}$ be a definable function. Then, there exists a pseudo-finite set $A \subset I$, such that on each sub-interval of $I \setminus A$ the function f is $\mathcal{C}^1$.*

Proof. Proceed as in the proof of [22, Proposition 7.2.5], substituting "finite" with "pseudo-finite" and "infinite" with "non-pseudo-finite". □

Theorem 5.11. *Let $U \subseteq \mathbb{K}^n$ be a definable open set, $f : U \to \mathbb{K}$ be a definable function, and p be a natural number. Then, there exists a closed definable nowhere dense set $D \subset U$, such that f restricted to $U \setminus D$ is $\mathcal{C}^p$.*

Proof. The case $p = 0$ is Remark 5.9. By induction on p, it suffices to prove the case $p = 1$. The case $n = 1$ is Lemma 5.10. The proof of the case $n > 1$ is done by induction on n, proceeding as in the proof of [14, Theorem 3.3]. □

5.4. Additional results on locally o-minimal structures

Since $\mathbb{K}$ is locally o-minimal, every definable subset of $\mathbb{K}^n$ is constructible. Hence, we can apply Lemmas 2.18, 4.5, and 4.6, and obtain the following lemma (cf. Example 2.19).

Lemma 5.12. *Let C and C' be definable subsets of $\mathbb{K}^n$. Then,* $\dim(C \cup C') = \max(\dim C, \dim C')$, *and* $\dim(\partial C) < \dim(C)$ *if C is nonempty; besides,* $\dim(C) \leqslant 0$ *iff C is pseudo-finite.*

Proof. We only need to show that $\dim(\partial C) < \dim(C)$.

First, we claim that $\operatorname{fdim}(\bar{C}) = \operatorname{fdim}(C)$. In fact, $\prod_m^n(\bar{C}) \subseteq \overline{\prod_m^n C}$, and therefore, by local o-minimality, if the latter has nonempty interior, also the former has nonempty interior. Hence, $\operatorname{fdim}(\partial C) \leqslant \operatorname{fdim}(C)$.

Assume, for a contradiction, that $d := \dim(C) = \dim(\partial C)$; w.l.o.g., $\pi(\partial C)$ contains a nonempty open box U, where $\pi := \prod_d^n$. Let $\mathcal{B}$ be the set of $\langle C, \pi\rangle$-bad points: $\mathcal{B}$ is nowhere dense, and therefore there exists a nonempty open box $V \subseteq U \setminus \mathcal{B}$. However, $\pi(\partial(C \cap \pi^{-1}(V))) \subseteq \partial V$, contradicting the definition of U. □

The above lemma was proved also in [19]. We can now give another criterion for a structure to be locally o-minimal.

Proposition 5.13. *Let $\mathbb{F}$ be a definably complete expansion of a field. The following are equivalent*:

(1) *$\mathbb{F}$ is locally o-minimal*;

(2) *for every* $X \subseteq \mathbb{F}$ *definable and nonempty,* $\dim(\partial X) < \dim(X)$;
(3) *for every* $X \subseteq \mathbb{F}^n$ *definable and nonempty,* $\dim(\partial X) < \dim(X)$.

Proof. (3) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1). First, let us prove that the open core of $\mathbb{F}$ is locally o-minimal. Let $X \subseteq [0,1)$ definable, discrete and closed in $[0,1)$. We have to prove that F is pseudo-finite. If not, 1 is an accumulation point for X. Thus, $\partial X = \{1\}$, and therefore $\dim(\partial X) = \dim(X) = 0$, absurd.

Then, we prove that X is locally o-minimal. It suffices to prove that if $X \subseteq \mathbb{K}$ has dimension 0, then it is pseudo-finite. Since $\dim(X) = 0$, ∂X is empty, and therefore X is closed; thus, X is nowhere dense, and hence, by Theorem 3.3(5), X is pseudo-finite.

(1) $\Rightarrow$ (3) follows from Lemma 5.12. $\square$

Lemma 5.14. *Let* $C(t)$ *be a definable increasing family of subsets of* $\mathbb{K}^n$, *such that* $\dim(C(t)) \leqslant d$ *for every* t. *Then,* $\dim(\bigcup_t C(t)) \leqslant d$.

Lemma 5.15. *Every locally o-minimal structure has definable Skolem functions.*

Proof. The same as in [22, Proposition 6.1.2]. See also [19] for a different proof. $\square$

Lemma 5.16 *(Curve selection). If* $a \in \partial X$, *where* $X \subseteq \mathbb{K}^n$ *is definable, then there exists a definable continuous injective map* $\gamma : (0,1) \to X$, *such that* $\lim_{t\to 0} \gamma(t) = a$.

Proof. The same as in [22, Corollary 6.1.5]. See also [19] for a different proof. $\square$

Example 5.17. A locally o-minimal structure does not necessarily satisfy the NIP. In fact, fix an o-minimal structure M (expanding a field) in a language L. Let P be a new binary predicate symbol. For every $0 < n \in \mathbb{N}$, let $f_n : \{1, \ldots, 2^n\} \to \mathcal{P}(n)$ be an enumeration of the subsets of $\{1, 2, \ldots, n\}$, such that f_n extends f_m for every $m \leqslant n$. Let $P_n := \bigcup_{k=1}^{2^n} \{k\} \times f_n(k) \subseteq M^2$, and $N_n := (M, P_n)$ be the $L(P)$-expansion of M, where P is interpreted by P_n. Let $N^* = (M^*, P^*)$ be a non-principal ultraproduct of the N_m. Since M is o-minimal, M^* and each N_m are o-minimal, and N^* is locally o-minimal. However, it is clear that N^* is not o-minimal, and does have the Independence Property. Moreover, if L is countable, then N^* is ω-saturated, and therefore, by the proof [2, 1.17], N^* does not have the Exchange Property.

The definitions and the proof in [12] work almost verbatim for locally o-minimal structures. Hence, we have the following theorem (see [12] for the relevant definitions).

Theorem 5.18. *If* $\mathbb{K}$ *is a locally o-minimal structure, then it is either* ***power-bounded****, or it is* ***exponential*** *(that is defines an exponential function). If* $\mathbb{K}$ *is power bounded, then for every ultimately non-zero definable function* $f : \mathbb{K} \to \mathbb{K}$ *there exist* $0 \neq c \in \mathbb{K}$ *and* r *in the field of exponents of* $\mathbb{K}$, *such that* $f \sim cx^r$.

Example 5.19. For every n, let R_n be the expansion of the real field by the function

$$f_n(x) := \underbrace{\exp\bigl(\ldots \exp}_{n \text{ times}}(x)\ldots\bigr).$$

Let $\mathbb{K} := \langle K, +, \cdot, <, f\rangle$ be a non-principal ultraproduct of the structures R_n. Then, since each structure R_n is o-minimal, $\mathbb{K}$ is locally o-minimal. Moreover, the function f is not power-bounded, and therefore $\mathbb{K}$ defines an exponential function $\exp^*$: it is easy to see that $\langle K, \exp^*\rangle$ is precisely the ultrapower of $\langle \mathbb{R}, \exp\rangle$. We don't know if $\mathbb{K}$ is o-minimal or not.

5.5. One-types in locally o-minimal structures

If $\mathbb{K}$ is an o-minimal structure, then a 1-type over $\mathbb{K}$ is either realised in $\mathbb{K}$, or determined by a cut of $\mathbb{K}$ (that is, if $\mathbb{K}^* \succ \mathbb{K}$ and $c, c' \in \mathbb{K}^* \setminus \mathbb{K}$, then $c \equiv_{\mathbb{K}} c'$ iff c and c' determine the same cut of $\mathbb{K}$). As the next example shows, this is no longer true if $\mathbb{K}$ is only locally o-minimal (in general, there will be some cut of $\mathbb{K}$ that can be completed to more than one complete type).

Example 5.20. Let $\mathbb{K}$ be any ω-saturated locally o-minimal non-o-minimal structure. By Corollary 3.6, there exists $X \subset \mathbb{K}$ which is pseudo-finite but not finite. Thus, there exists Y (non-definable!) convex subset of X such that Y is order-isomorphic to either $\langle \mathbb{N}, <\rangle$ or to $\langle -\mathbb{N}, <\rangle$; w.l.o.g., we can assume that Y is isomorphic to $\langle \mathbb{N}, <\rangle$. Let Λ be the cut of $\mathbb{K}$ given by the supremum of Y. We claim that there are at least two ways of completing the cut Λ to a 1-type $p(x)$: we can require either $x \in X$ or $x \notin X$; as it is easily seen, both conditions are consistent with the cut Λ.

However, by definition of local o-minimality, for every $a \in \mathbb{K}$ the cut a^+ determines a unique complete type. Therefore, the cuts of the form a^+, a^- (where $a \in \mathbb{K}$), $+\infty$, and $-\infty$ determine a unique complete type, which we will designate by the same symbol.

Remark 5.21. The definable 1-types over $\mathbb{K}$ are exactly the types of the form a, a^+, a^- (where $a \in \mathbb{K}$), $+\infty$, and $-\infty$.

Open Problem 5.22. Give a characterisation of definable n-types in locally o-minimal structures, along the lines of Marker–Steinhorn theorem.

5.6. Definable groups

R. Wencel in [23] proves a generalisation to weakly o-minimal structures of A. Pillay's theorem that a group definable in an o-minimal structure can be equipped with a topology, making it a topological group and a definable manifold. His main result can be applied to locally o-minimal structures.

Definition 5.23. (See [23].) $\mathbb{K}$ has the continuity property (with respect to dim) if for any set $A \subseteq \mathbb{K}$, A-definable nonempty sets $X \subseteq Y \subseteq \mathbb{K}^n$ satisfying $\dim(X) = \dim(Y)$ and any A-definable function $f : Y \to \mathbb{K}$, there exists an A-definable set $V \subseteq X$, such that $\dim(X \setminus V) < \dim(X)$, V is open in Y, and the function f restricted to V is continuous.

Lemma 5.24. *If $\mathbb{K}$ is locally o-minimal, then $\mathbb{K}$ has the continuity property.*

Proof. Let X, Y and f be as in the definition. Let X' be the interior of X inside Y: notice that $\dim(X \setminus X') < \dim(Y)$; thus, w.l.o.g. we can assume that $X = X'$. Let $D \subseteq Y$ be the set of discontinuity points of f, and D' be the closure of D inside Y. Notice that $\dim(D') < \dim(Y)$. Define $V := X \setminus D'$. □

Theorem 5.25. *Let $\mathbb{K}$ be a locally o-minimal structure, and $\langle G, \cdot \rangle$ be a group which is definable with parameters A, such that $G \subseteq \mathbb{K}^m$ and $\dim(G) = d$. Then, there exist an A-definable set $V \subseteq G$ and a topology τ on G, such that:*

(1) *$\langle G, \cdot, \tau \rangle$ is a topological group;*
(2) *$\dim(G \setminus V) < d$, and V is τ-open in G;*
(3) *the topology τ restricted to V coincides with the product topology induced from $\mathbb{K}^m$;*
(4) *some $d + 1$ right translates of V cover G.*

Proof. We want to apply [23, Theorem 3.5]. Notice that dim is a dimension function (according to the definition in [23]). By Lemma 5.24, the hypothesis of [23, Theorem 3.5] holds, and we are done. □

A version of Theorem 5.25 holds in d-minimal structures (see [8]); however, Lemma 5.24 is false in any d-minimal not locally o-minimal structure.

6. Dense pairs and dense tuples of locally o-minimal structures

C. Miller gave already a definition of d-minimal structures: in [4] we prove that $\mathbb{K}$ is topologically d-minimal (see Definition 1.4) iff it is d-minimal in Miller's sense (under the Proviso that $\mathbb{K}$ is a definably complete expansion of a field). Here we will give the proof for the case when $\mathbb{K}$ is locally o-minimal.

Theorem 6.1. *If $\mathbb{K}$ is locally o-minimal, then $\mathbb{K}$ is a d-minimal topological structure.*

Proof. (DM1) is trivial. For (DM2), let $X \subset \mathbb{K}^n$ be definable and discrete. Thus, by Theorem 3.3, X is pseudo-finite, and therefore, by Lemma 2.22, $\prod_1^n(X)$ has empty interior.

Let us prove (DM3). Let X and U be as in the assumption of (DM3). Since $\mathbb{K}$ is locally o-minimal, X is constructible; thus, we can apply Fact 2.5, and obtain that X is nonmeager. Hence, by Corollary 2.10, X has nonempty interior. □

Hence, we can apply the results in [6], and obtain the following result.

Definition 6.2. Let T be the theory of $\mathbb{K}$. Let T^d be the theory of pairs $\langle B, A \rangle$, where $B \models T$ and A is a proper dense elementary substructure of B; the language of T^d is the expansion of the language of T by a new unary predicate P, denoting the set A. More generally, given $1 \leqslant n \in \mathbb{N}$, let T^{nd} be the theory of $(n+1)$-tuples $\langle B_{n+1}, B_n, \ldots, B_1 \rangle$, where $B_1 \prec B_2 \prec \cdots \prec B_{n+1} \models T$, each B_i is a proper elementary substructure of B_{i+1}, and B_1 is dense in B_{n+1}; the language of T^{nd} is the language of T expanded by n unary predicates $P_1, \ldots, P_n$.

Corollary 6.3. *(See [6, 13.5, 13.7 and 13.11].) For every $n \geqslant 1$, T^{nd} is consistent and complete. If $\mathbb{F} := \langle B_{n+1}, B_n, \ldots, B_1 \rangle$ is a model of T^{nd}, then B_{n+1} is the open core of $\mathbb{F}$; in particular, $\mathbb{F}$ has a locally o-minimal open core.*

References

[1] Jean-Paul Allouche, Note on the constructible sets of a topological space, in: Papers on General Topology and Applications, Gorham, ME, 1995, in: Ann. N.Y. Acad. Sci., vol. 806, New York Acad. Sci., New York, 1996, pp. 1–10, MR MR1429640 (98a:54032).

[2] Alfred Dolich, Chris Miller, Charles Steinhorn, Structures having o-minimal open core, Trans. Amer. Math. Soc. 362 (2010) 1371–1411.

[3] Randall Dougherty, Chris Miller, Definable Boolean combinations of open sets are Boolean combinations of open definable sets, Illinois J. Math. 45 (4) (2001) 1347–1350, MR MR1895461 (2003c:54018).

[4] Antongiulio Fornasiero, D-minimal structures, preprint, 2010.

[5] Antongiulio Fornasiero, Definably complete structures are not pseudo-enumerable, Arch. Math. Logic (2011) 1–13, http://dx.doi.org/10.1007/s00153-011-0235-x.

[6] Antongiulio Fornasiero, Dimensions, matroids, and dense pairs of first-order structures, in: Logic Colloquium 2009, Ann. Pure Appl. Logic 162 (7) (2011) 514–543.

[7] Antongiulio Fornasiero, Expansions of the reals which do not define the natural numbers, preprint, 2011.

[8] Antongiulio Fornasiero, Groups and rings definable in d-minimal structures, preprint, 2012.

[9] Antongiulio Fornasiero, Tamara Servi, Definably complete Baire structures, Fund. Math. 209 (3) (2010) 215–241, MR 2720211.

[10] Philipp Hieronymi, An analogue of the Baire category theorem, preprint, 2011.

[11] Ehud Hrushovski, Ya'acov Peterzil, A question of van den Dries and a theorem of Lipshitz and Robinson; not everything is standard, J. Symbolic Logic 72 (1) (2007) 119–122, MR MR2298474 (2008d:03033).

[12] Chris Miller, A growth dichotomy for o-minimal expansions of ordered fields, in: Logic: From Foundations to Applications, Staffordshire, 1993, in: Oxford Sci. Publ., Oxford Univ. Press, New York, 1996, pp. 385–399, MR MR1428013 (98a:03052).

[13] Chris Miller, Expansions of dense linear orders with the intermediate value property, J. Symbolic Logic 66 (4) (2001) 1783–1790, MR MR1877021 (2003j:03044).

[14] Chris Miller, Tameness in expansions of the real field, in: Logic Colloquium '01, in: Lect. Notes Log., vol. 20, Assoc. Symbol. Logic, Urbana, IL, 2005, pp. 281–316, MR MR2143901 (2006j:03049).

[15] Chris Miller, Patrick Speissegger, Expansions of the real line by open sets: o-minimality and open cores, Fund. Math. 162 (3) (1999) 193–208, MR MR1736360 (2001a:03083).

[16] Bruno Poizat, Cours de théorie des modèles: Une introduction à la logique mathématique contemporaine, Bruno Poizat, Lyon, 1985, MR MR817208 (87f:03084).

[17] Alex Rennet, The non-axiomatizability of o-minimality, preprint, 2012.

[18] Hans Schoutens, O-minimalism, preprint, 2011.

[19] Hans Schoutens, Prolegomena to o-minimalism: definable completeness, type completeness, and tameness, preprint, 2011.

[20] Carlo Toffalori, Kathryn Vozoris, Notes on local o-minimality, MLQ Math. Log. Q. 55 (6) (2009) 617–632, MR 2582162.

[21] Lou van den Dries, Dense pairs of o-minimal structures, Fund. Math. 157 (1) (1998) 61–78, MR MR1623615 (2000a:03058).

[22] Lou van den Dries, Tame Topology and o-Minimal Structures, London Math. Soc. Lecture Note Ser., vol. 248, Cambridge University Press, Cambridge, 1998, MR MR1633348 (99j:03001).

[23] Roman Wencel, Groups, group actions and fields definable in first-order topological structures, preprint, 2011.