Comparison theorems for a subclass of proper splittings of matrices

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\textbf{Abstract}

In this article, a convergence theorem and several comparison theorems are presented for a subclass of proper splittings of matrices introduced recently.

\textbf{1. Introduction and preliminaries}

Iterative methods for solving linear systems

\begin{equation}
Ax = b,
\end{equation}

where $A$ is a real rectangular $m \times n$ matrix and $b$ is a real $m$-vector, are related to the decompositions of $A$ of the form $A = U - V$. A decomposition $A = U - V$ of $A \in \mathbb{R}^{m \times n}$ is called a \textit{proper splitting} [1] if $R(A) = R(U)$ and $N(A) = N(U)$, where $R(A)$ and $N(A)$ stand for the range of $A$ and the kernel of $A$. The asymptotic behavior of the iterative sequence

\begin{equation}
x^{(i+1)} = U^\dagger x^{(i)} + U^\dagger b,
\end{equation}

where $U^\dagger$ is the Moore–Penrose inverse [2] of $U$, is governed by the spectral radius of the iteration matrix $U^\dagger V$. ($X^\dagger$ is the unique matrix which satisfies $XX^\dagger X = X$, $X^\dagger XX^\dagger = X^\dagger$, $(XX^\dagger)^T = XX^\dagger$ and $(X^\dagger X)^T = X^\dagger X$). The spectral radius of a real square matrix $X$ is the maximum modulus of the eigenvalues of $X$, and is denoted by $\rho(X)$. For a proper splitting $A = U - V$, the iteration scheme (1.2) converges to $x = A^\dagger b$ for every initial vector $x^0$ if and only if $\rho(U^\dagger V) < 1$ (see [1, Corollary 1]).

When two decompositions or splittings of $A$ are given, it is of interest to compare the spectral radii of the corresponding iteration matrices. The comparison of asymptotic rates of convergence of the iteration matrix induced by two splittings of a given matrix has been studied by many authors; for example, Csordas and Varga [3], Elsner [4], Song [5], Varga [6] and Woźnicki [7], to name a few. Here, splitting means a decomposition $A = U - V$ of a square matrix $A$ where $U$ is invertible. Woźnicki [8] has considered different types of splittings (such as regular, weak regular and weak nonnegative splittings of different types) of a given monotone matrix and has proved corresponding comparison theorems. Elsner [4] considered weak regular splittings and multisplittings, and proved comparison results. Song [5] studied a comparison theorem for nonnegative splittings and then applied it to study different basic iterative methods.

In this article, we prove comparison results for different iterative schemes arising out of a new decomposition (called $B_1$-splitting) introduced by Mishra and Sivakumar [9]. There, the authors have demonstrated the existence of a $B_1$-splitting
for a class of matrices, using a constructive procedure. Thus, the results that we have obtained in this article are very pertinent. However, applications of our results in the consideration of practical problems are devoted to a future study.

The purpose of this article is to present a convergence theorem (Theorem 2.10) and two comparison theorems (Theorems 2.12 and 2.14) for $B_1$-splittings.

We say that a real matrix $A$ is nonnegative, if it is entry-wise nonnegative, and we write this as $A \geq 0$. The same notation and nomenclature are also used for vectors. If $A$ and $B$ are two real matrices, we write $B \geq A$ if $B - A \geq 0$. A decomposition $A = U - V$ of a real rectangular matrix $A$ is called positive if $U \geq 0$ and $V \geq 0$. Before proving the main results, we recall certain definitions and results which will be used in Section 2. We begin with the definition of a $B_1$-splitting.

**Definition 1.1** ([9, Definition 3.6]). A positive proper splitting $A = U - V$ of $A \in \mathbb{R}^{m \times n}$ is called a $B_1$-splitting if it satisfies the following conditions:

(i) $VU^T \geq 0$, and
(ii) $Ax, \; Ux \in \mathbb{R}^m_+ \cap N(A^T)$ and $x \in R(A^T)$ imply $x \geq 0$.

In the case of square nonsingular matrices, the above definition reduces to a $B$-splitting which we recall next.

**Definition 1.2** ([10, Definition 1]). A positive splitting $A = U - V$ of $A \in \mathbb{R}^{n \times n}$ is called a $B$-splitting if it satisfies the following conditions:

(i) $VU^{-1} \geq 0$, and
(ii) $Ax, \; Ux \geq 0$ imply $x \geq 0$.

The next result guarantees the existence of $B_1$-splittings for a certain class of matrices.

**Theorem 1.3** ([9, Theorem 3.10]). Let $A \in \mathbb{R}^{m \times n}$. Suppose that $A^T \geq 0$, $R(A) \cap \text{int}(\mathbb{R}^n_+) \neq \emptyset$ and $A^1A \geq 0$. Then $A$ has a $B_1$-splitting with $\rho(VU^T) < 1$.

Now we recall the convergence theorem for semimonotone (meaning, $A^T \geq 0$) matrices using a $B_1$-splitting of $A$.

**Theorem 1.4** ([9, Theorem 3.8]). Let $A \in \mathbb{R}^{m \times n}$. If $A$ has a $B_1$-splitting, then $A^T \geq 0$ if and only if $\rho(VU^T) < 1$.

The next theorem is a part of the Perron–Frobenius theorem.

**Theorem 1.5** ([6, Theorem 2.20]). Let $A$ be a real square nonnegative matrix. Then we have the following.

(i) $A$ has a nonnegative real eigenvalue equal to its spectral radius.
(ii) There exists a nonnegative eigenvector for its spectral radius.

Another result which relates the spectral radius of two nonnegative matrices is given below.

**Theorem 1.6** ([6, Theorem 2.21]). Let $A \geq B \geq 0$. Then $\rho(A) \geq \rho(B)$.

The result given next characterizes the “reverse order law” for the Moore–Penrose inverse.

**Theorem 1.7** ([11, Theorem 1]). If $A$ and $B$ are real matrices such that $AB$ is defined, then $(AB)^\dagger = B^\dagger A^\dagger$ if and only if $A^\dagger ABB^T A^T = BB^T A^T$ and $BB^T A^T AB = A^\dagger AB$.

The following particular case will be useful in our discussion.

**Corollary 1.8.** If $A$ and $B$ are real matrices such that $AB$ is defined and $B$ is invertible, then $(AB)^\dagger = B^{-\dagger}A^\dagger$ if and only if $A^\dagger ABB^T A^T = BB^T A^T$.

2. Main results

In this section we will prove a convergence theorem and various comparison theorems of $B_1$-splittings. First, let us recall certain basic properties of proper splittings which form part of [1, Theorem 1].

**Theorem 2.1.** Let $A = U - V$ be a proper splitting of $A \in \mathbb{R}^{m \times n}$. Then

(a) $A = U(I - U^1V)$;
(b) $I - U^1V$ is invertible;
(c) $A^\dagger = (I - U^1V)^{-1}U^\dagger$.

It is evident that if $A = U - V$ is a proper splitting of $A$, then $A^T = U^T - V^T$ is a proper splitting of $A^T$. Next, we derive other properties of proper splittings that will be used in the rest of the paper.
Theorem 2.2. Let $A = U - V$ be a proper splitting of $A \in \mathbb{R}^{m \times n}$. Then

(a) $AA^\dagger = UU^\dagger$ and $A^\dagger A = U^\dagger U$;
(b) $U = A(I + A^\dagger V) = (I + VA^\dagger)A$;
(c) $I + A^\dagger V$ and $I + VA^\dagger$ are invertible;
(d) $U^\dagger = (I + A^\dagger V)^{-1}A^\dagger = A^\dagger (I + VA^\dagger)^{-1}$;
(e) $A^\dagger = (I + A^\dagger V)U^\dagger = U^\dagger (I + VA^\dagger)$;
(f) $U^\dagger VA^\dagger = A^\dagger UU^\dagger$;
(g) $U^\dagger VA^\dagger = A^\dagger UU^\dagger$;
(h) $VU^\dagger VA^\dagger = VA^\dagger UU^\dagger$.

Proof. The proofs of (a), (g), (h) are trivial.

(b) The fact that $R(A) = R(U)$ implies that $R(V) \subseteq R(A)$. Hence $V = AA^\dagger V$. Also, $R(A^\dagger) = R(U^\dagger)$ implies $R(V^\dagger) \subseteq R(A^\dagger)$. It then follows, as above, that $V^\dagger = A^\dagger (A^\dagger)^T V^\dagger = (A^\dagger A)^T V^\dagger$ which yields $V = VA^\dagger A$. So, $U = A + V = A(I + A^\dagger V) = (I + VA^\dagger)A$.

(c) Now, we show that $I + A^\dagger V$ is invertible. Suppose that $(I + A^\dagger V)x = 0$. Then $-x = A^\dagger Vx \in R(A^\dagger) = R(U^\dagger)$. Hence $Vx = U^\dagger x$. So $-x = A^\dagger Vx = A^\dagger (U - A)x = A^\dagger Ux - A^\dagger Ax = A^\dagger Ux - x$. Thus $A^\dagger Ux = 0$, so that $Ux \in N(V) = N(A^\dagger) = N(U^\dagger) = N(U)$. Therefore, $x = U^\dagger Ux = 0$. Hence $I + A^\dagger V$ is invertible. By applying the first part to the decomposition $A^\dagger = U^\dagger - V^\dagger$ one gets that $I + (A^\dagger)^T V^\dagger = (I + VA^\dagger)^T$ is invertible. Thus $I + VA^\dagger$ is invertible.

(d) Since $R(A^\dagger) = R(U^\dagger)$, we have $R((I + A^\dagger V)(I + A^\dagger V)^T A^\dagger) = R(U^\dagger)$. Thus $A^\dagger(A^\dagger + V^\dagger)(I + A^\dagger V)^T A^\dagger = (I + A^\dagger V)(I + A^\dagger V)^T A^\dagger$. Then by Corollary 1.8, $A^\dagger = (I + A^\dagger V)^{-1} A^\dagger$. Similarly, $U^\dagger = (I + VA^\dagger)^{-1}$.

(e) $A^\dagger = U^\dagger (I + VA^\dagger)$ and $A^\dagger = (I + A^\dagger V)U^\dagger$ follow from (d).

(f) We have $A^\dagger = U^\dagger (I + VA^\dagger) = U^\dagger U^\dagger VA^\dagger$ and $A^\dagger = (I + A^\dagger V)U^\dagger = U^\dagger + A^\dagger UU^\dagger$. Hence $U^\dagger VA^\dagger = A^\dagger UU^\dagger$. □

Corollary 2.3. Let $A = U - V$ be a proper splitting of $A \in \mathbb{R}^{m \times n}$. Then the matrices $U^\dagger V$ and $A^\dagger V$ (or $VU^\dagger$ and $VA^\dagger$) have the same eigenvectors.

Theorem 2.2 is a generalization of the next result which holds for nonsingular matrices.

Corollary 2.4 ([7, Lemma 1.1]). Let $A = U - V$ be a splitting of $A \in \mathbb{R}^{n \times n}$. Suppose that $A$ and $U$ are nonsingular matrices. Then

(a) $U^\dagger VA^\dagger = A^{-1} U^{-1}$;
(b) $U^\dagger VA^\dagger = A^{-1} U^{-1}$;
(c) $VA^\dagger U^\dagger = A^{-1} U^{-1}$.

Corollary 2.5 ([7, Corollary 1.1]). Let $A = U - V$ be a splitting of $A \in \mathbb{R}^{n \times n}$. If $A$ and $U$ are nonsingular matrices, then the matrices $U^\dagger V$ and $A^\dagger V$ (or $VU^\dagger$ and $VA^\dagger$) have the same eigenvectors.

Lemma 2.6. Let $A = U - V$ be a proper splitting of $A \in \mathbb{R}^{m \times n}$. Let $\mu_i$, $1 \leq i \leq s$ and $\lambda_j$, $1 \leq j \leq s$ be the eigenvalues of the matrices $U^\dagger V$ (or $VU^\dagger$) and $A^\dagger V$ (or $VA^\dagger$) respectively. Then for every $j$, we have $1 + \lambda_j \neq 0$. Also, for every $i$, there exists $j$ such that $\mu_i = \frac{\lambda_j}{1 + \lambda_j}$ and for every $j$, there exists $i$ such that $\lambda_j = \frac{\mu_i}{1 - \mu_i}$.

Proof. Let $x$ be an eigenvector corresponding to the eigenvalue $\mu_i$ of the matrix $U^\dagger V$. Then from Corollary 2.3, we have $\mu_i x = U^\dagger Vx = (I + A^\dagger V)^{-1}A^\dagger Vx = \frac{\lambda_j}{1 + \lambda_j} x$ for some eigenvalue $\lambda_j$ of $A^\dagger V$. The second part follows similarly. □

Let $A = U_1 - V_1 = U_2 - V_2$ be two different decompositions of $A \in \mathbb{R}^{m \times n}$. Define $S := \{ j \in N : (A^\dagger V_1)^j A^\dagger \geq (A^\dagger V_2)^j A^\dagger \}$.

Lemma 2.7. Suppose that $A^\dagger \geq 0$, $V_1 \geq 0$ and $V_2 \geq 0$. Then $S$ is closed under addition.

Proof. Let $j, k \in S$ and $j \geq 1, k \geq 1$. Then $(A^\dagger V_1)^j A^\dagger \geq (A^\dagger V_2)^j A^\dagger$ and $(A^\dagger V_2)^k A^\dagger \geq (A^\dagger V_1)^k A^\dagger$. Post-multiplying the first inequality by $V_2(A^\dagger V_2)^{k-1} A^\dagger$, we get $(A^\dagger V_2)^{j+k} A^\dagger \geq (A^\dagger V_1)^j V_2(A^\dagger V_2)^{k-1} A^\dagger$. Pre-multiplying the second inequality by $(A^\dagger V_1)^j$, we get $(A^\dagger V_1)^j V_2(A^\dagger V_2)^{k-1} A^\dagger \geq (A^\dagger V_1)^{j+k} A^\dagger$. Combining, we get $(A^\dagger V_2)^{j+k} A^\dagger \geq (A^\dagger V_1)^{j+k} A^\dagger$ so that $j + k \in S$. □

The next theorem provides sufficient conditions under which $S$ is shown to be nonempty.

Theorem 2.8. Let $A = U_1 - V_1 = U_2 - V_2$ be two proper splittings of $A \in \mathbb{R}^{m \times n}$.

(a) Let $U_1^\dagger \geq 0$, $V_1^\dagger \geq 0$ and $V_2 \geq V_1$. Then $U_1^\dagger \geq U_2^\dagger$.
(b) Let $A^\dagger \geq 0$, $V_2 \geq 0$ and $V_2 \geq V_1$. If $U_1^\dagger \geq U_2^\dagger$ then $(A^\dagger V_2)^j A^\dagger \geq (A^\dagger V_1)^j A^\dagger$, for each positive integer $j$. \hfill □
Proof. (a) Since $A = U_1 - V_1 = U_2 - V_2$ are two proper splittings of $A$, we have $R(U_1) = R(U_2) = \rho(A)$, $R(U_1^\perp) = R(U_2^\perp) = 1$. $U_2 = U_1 U_1^\perp U_2$ and $U_1 = U_1 U_1^\perp U_2$. Further, $U_1 U_1^\perp U_1 = U_1^\perp U_2 U_1 = U_1^\perp U_2 U_1 = 0$. Also, $U_2 - U_1 = (A + V_1) - (\rho(A) + V_2) = V_2 - V_1 \geq 0$. So, $0 \leq U_2 - U_1 = U_1 U_1^\perp U_2 - U_1 U_1^\perp U_1 = U_1 U_1^\perp U_2 - U_1 U_1^\perp U_2 = U_1 U_1^\perp U_2$. Pre-multiplying by $U_1^\perp$, we get $0 \leq U_1 U_1^\perp U_2 = (U_1^\perp - U_2 U_2)$. Post-multiplying by $U_2$, we get $0 \leq (U_1^\perp - U_2 U_2) U_2 = U_1^\perp - U_2$. Therefore $U_1^\perp \geq U_2$.

(b) Let $A \geq 0$, $V_1 \geq 0$, and $V_2 \geq 0$. If $U_1^\perp \geq U_2$, then (from Theorem 2.2 (d)) $(I + A V_1) - A \geq A (I + V_2 A)$, hence $A (I + V_2 A) \geq A (I + V_1 A)$, i.e., $A^\perp \geq A^\perp$. Thus $A^\perp A \geq A^\perp A$. Hence $\rho(A^\perp A^\perp) \geq (A^\perp A^\perp)^\perp$, for each positive integer $j$.

Now, we present a comparison theorem for a $B_1$-splitting.

Theorem 2.10. Let $A = U - V$ be a $B_1$-splitting of $A \in \mathbb{R}^{m \times n}$. If $A^\perp \geq 0$, then we have the following.

(a) $A^\perp \geq U^\perp$.

(b) $\rho(VA^\perp) \geq \rho(VU^\perp)$.

(c) $\rho(VU^\perp) = \rho(U^\perp V) = \frac{\rho(A^\perp V)}{1 + \rho(A^\perp V)} < 1$.

Proof. Let $A = U - V$ be a $B_1$-splitting. Then $R(A) = R(U), N(A) = N(U), U \geq 0, V \geq 0$ and $V U^\perp \geq 0$. Also, $Ax, Ux \in \mathbb{R}^n + N(A^\perp)$ and $x \in R(A^\perp)$ imply $x \geq 0$.

(a) We have $A^\perp - U^\perp = (I + A V) U^\perp - A^\perp V \geq 0$, where the last inequality is valid due to $A^\perp \geq 0$. So, $A^\perp \geq U^\perp$.

(b) Since $V \geq 0$, so pre-multiplying the inequality $A^\perp \geq U^\perp$ by $V$, we have $VA^\perp \geq VU^\perp \geq 0$. By Theorem 1.6, it then follows that $\rho(VA^\perp) \geq \rho(VU^\perp)$.

(c) Let $\lambda$ be any eigenvalue of $VA^\perp$. Let $f(\lambda) = \frac{\lambda}{1 + \lambda}, \lambda \geq 0$. Then $f$ is a strictly increasing function. There exists an eigenvalue of $VU^\perp$, say $\mu$, such that $\mu = \frac{\lambda}{1 + \lambda}$. So $\mu$ attains its maximum when $\lambda$ is maximum. But $\lambda$ is maximum when $\lambda = \rho(A^\perp V) = \rho(VA^\perp)$. (Note that by Theorem 1.5, $\rho(A^\perp V)$ is an eigenvalue of $VA^\perp$.) As a result, the maximum value of $\mu$ is $\rho(U^\perp V) = \frac{\rho(A^\perp V)}{1 + \rho(A^\perp V)} < 1$. □

Note that (c) of Theorem 2.10 is already known for outer generalized inverses. (See, for instance, [12, Theorem 3.5.9].)

Corollary 2.11. Let $A = U - V$ be a $B$-splitting of $A \in \mathbb{R}^{m \times n}$. If $A^\perp \geq 0$, then we have the following.

(a) $A^\perp \geq U^\perp$.

(b) $\rho(\rho(A) \geq U^\perp - V^\perp - 1$.

(c) $\rho(VU^\perp) = \rho(U^\perp V) = \frac{\rho(A^\perp V)}{1 + \rho(A^\perp V)} < 1$.

Next, we present comparision theorems for $B_1$-splittings.

Theorem 2.12. Let $A = U_1 - V_1 = U_2 - V_2$ be $B_1$-splittings of $A \in \mathbb{R}^{m \times n}$. If $V_1 \leq V_2$ and $A^\perp \geq 0$, then $\rho(U_1^\perp V_1) \leq \rho(U_2^\perp V_2) < 1$.

Proof. We have $\rho(U_1^\perp V_1) < 1$ for $i = 1, 2$ by Theorem 2.10. Observe that $A^\perp V_1 \geq A^\perp V_2 \geq 0$. Then $\rho(A^\perp V_2) \geq \rho(A^\perp V_1)$. As in Theorem 2.10, it follows that $\rho(U_1^\perp V_1) < 1$. □

Corollary 2.13. Let $A = U_1 - V_1 = U_2 - V_2$ be $B$-splittings of $A \in \mathbb{R}^{m \times n}$. If $V_1 \leq V_2$ and $A^\perp \geq 0$, then $\rho(U_1^\perp V_1) \leq \rho(U_2^\perp V_2) < 1$.

Theorem 2.14. Let $A = U_1 - V_1 = U_2 - V_2$ be $B_1$-splittings of $A \in \mathbb{R}^{m \times n}$, where $A^\perp \geq 0$. If $U_1^\perp \geq U_2^\perp$, then $\rho(U_1^\perp V_1) \leq \rho(U_2^\perp V_2) < 1$.

Proof. By Theorem 2.10, we have $\rho(U_1^\perp V_1) < 1$ for $i = 1, 2$. Also $V_2 A^\perp \geq 0$ for $i = 1, 2$. Clearly, it suffices to show that $\rho(A^\perp V_2) \geq \rho(A^\perp V_1)$. We have $(I + A^\perp V_1) - A \geq (I + A^\perp V_2) - A$. Thus $A^\perp (I + V_2 A) \geq A^\perp (I + V_1 A)$. Post-multiplying by $V_2$, we have $(A^\perp V_2)^2 \geq A^\perp V_1 A^\perp V_2$. On the other hand, post-multiplying by $V_1$, we get $A^\perp V_2 A^\perp V_1 \geq (A^\perp V_1)^2$. Therefore $\rho((A^\perp V_2)^2) \geq \rho(A^\perp V_2 A^\perp V_1) \geq \rho((A^\perp V_1)^2)$. Hence $\rho(A^\perp V_2) \geq \rho(A^\perp V_1)$. □
Corollary 2.15. Let $A = U_1 - V_1 = U_2 - V_2$ be $B$-splittings of $A \in \mathbb{R}^{n \times n}$, where $A^{-1} \geq 0$. If $U_1^{-1} \geq U_2^{-1}$, then $\rho(U_1^{-1}V_1) \leq \rho(U_2^{-1}V_2) < 1$.

Finally, we impose still weaker hypotheses yielding generalizations of Theorems 2.12 and 2.14, using Theorem 2.8. The proof of the first one is trivial while the other one is a particular case of the first.

Corollary 2.16. Let $A = U_1 - V_1 = U_2 - V_2$ be $B$-splittings of $A \in \mathbb{R}^{m \times n}$, where $A^\dagger \geq 0$. If there exists a positive integer $j$ such that $(A^\dagger V_2)^j A^\dagger \geq (A^\dagger V_1)^j A^\dagger$, then $\rho(U_1^j V_1) \leq \rho(U_2^j V_2) < 1$.

Corollary 2.17. Let $A = U_1 - V_1 = U_2 - V_2$ be $B$-splittings of $A \in \mathbb{R}^{n \times n}$, where $A^{-1} \geq 0$. If there exists a positive integer $j$ such that $(A^{-1} V_2)^j A^{-1} \geq (A^{-1} V_1)^j A^{-1}$, then $\rho(U_1^{-1}V_1) \leq \rho(U_2^{-1}V_2) < 1$.

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