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The colored Jones polynomial and the A-polynomial of Knots ☆

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Abstract

We study relationships between the colored Jones polynomial and the A-polynomial of a knot. The AJ conjecture (of Garoufalidis) that relates the colored Jones polynomial and the A-polynomial is established for a large class of two-bridge knots, including all twist knots. We formulate a weaker conjecture and prove that it holds for all two-bridge knots. Along the way we also calculate the Kauffman bracket skein module of the complements of two-bridge knots. Some properties of the colored Jones polynomial are established.

Keywords: Jones polynomial; A-polynomial; AJ conjecture; Two-bridge knots; Kauffman bracket skein module

0. Introduction

The Jones polynomial was discovered by Jones in 1984 [16] and has made a revolution in knot theory. Despite many efforts little is known about the relationship between the Jones polynomial and classical topology invariants like the fundamental group. The A-polynomial of a knot, introduced in [7], describes more or less the representation space of the knot group into $SL(2, \mathbb{C})$, and has been fundamental in geometric topology.

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In the present paper we study relationships between the Jones polynomial and the A-polynomial. One main goal of the paper is to establish for a large class of two-bridge knots the AJ conjecture (made by Garoufalidis) that relates the colored Jones polynomial and the A-polynomial. This class of knots contains for example all twist knots, and much more. Another main result is the calculation of the Kauffman bracket skein module of all two-bridge knots. This generalizes the work [3,5] where the calculations were carried out for $(2p+1,2)$-torus knots and twists knots, a special class of two-bridge knots. Our method is more geometric and we hope that it can be generalized to all knots.

In a previous paper [12] Garoufalidis and the author proved that for every knot, the colored Jones polynomial satisfies a recurrence relation. The AJ conjecture states that when reducing the quantum parameter to 1, the recurrence polynomial is essentially equal to the A-polynomial (for details see below). The present paper is independent of [12], since we will prove the existence of recurrence relations for two-bridge knots in another way. We also formulate a weaker version of the AJ conjecture (see Conjecture 2) that we believe reflects more accurately the algebra/topology relations between the Jones polynomial and the A-polynomial. We prove that the weaker conjecture holds true for all two-bridge knots.

Some properties of the colored Jones polynomial are established. We also show that for an arbitrary alternating knot, the degree of the recurrence polynomial must be at least 2.

0.1. *The colored Jones polynomial and its recurrence ideal*

0.1.1. *The colored Jones polynomial*

For a knot K in the 3-space $\mathbb{R}^3 \subset S^3$ the colored Jones function (see, for example, [22,26])

$$J_K : \mathbb{Z} \to \mathcal{R} := \mathbb{C}\left[t^{\pm 1}\right]$$

is defined for integers $n \in \mathbb{Z}$; its value $J_K(n)$ is known as the colored Jones polynomial of the knot K with color n. We will recall the definition of $J_K(n)$ in Section 1.

In our joint work with S. Garoufalidis [12] we showed that the function J_K always satisfies a non-trivial recurrence relation as described in the next subsection. Partial results were obtained earlier by Frohman, Gelca, and Lofaro through their theory of non-commutative A-ideal [10,13], which also plays an important role in the present paper.

0.1.2. *Recurrence relations and q-holonomicity*

Consider a function with domain the set of integers, $f : \mathbb{Z} \to \mathcal{R}$, and define the linear operators L and M acting on such functions by:

$$(Mf)(n) = t^{2n} f(n), \qquad (Lf)(n) = f(n+1).$$

It is easy to see that $LM = t^2 ML$, and that $L^{\pm 1}, M^{\pm 1}$ generate the *quantum torus* $\mathcal{T}$, a non-commutative ring with presentation

$$\mathcal{T} = \mathcal{R}\langle M^{\pm 1}, L^{\pm 1}\rangle / \left(LM = t^2 ML\right).$$

We also use the notation $\mathcal{T}_+$ for the subring of $\mathcal{T}$ which consists only of polynomials with non-negative powers of M and L. Traditionally $\mathcal{T}_+$ is called the *quantum plane*.

The *recurrence ideal* of the discrete function f is the left ideal $\mathcal{A}$ in $\mathcal{T}$ that annihilates f:

$$\mathcal{A} = \{P \in \mathcal{T} \mid Pf = 0\}.$$

We say that f is *q-holonomic*, or f satisfies a linear recurrence relation, if $\mathcal{A} \neq 0$. In [12] we proved that for every knot K, the function J_K is q-holonomic. Denote by $\mathcal{A}_K$ the recurrence ideal of J_K.

0.1.3. An example

For the right-handed trefoil, one has

$$J_K(n) = \frac{(-1)^{n-1} t^{2-2n}}{1 - t^{-4}} \sum_{k=0}^{n-1} t^{-4nk} \prod_{i=0}^{k} \bigl(1 - t^{4i-4n}\bigr).$$

The function J_K satisfies $pJ_K = 0$, where

$$P = \bigl(t^4 M^{10} - M^6\bigr) L^2 - \bigl(t^2 M^{10} + t^{-18} - t^{-10} M^6 - t^{-14} M^4\bigr) L + \bigl(t^{-16} - t^{-4} M^4\bigr).$$

Together with the initial conditions $J_K(0) = 0$, $J_K(1) = 1$, this recurrence relation determines $J_K(n)$ uniquely.

0.1.4. Generator of the recurrence ideal

The quantum torus $\mathcal{T}$ is not a principal ideal domain, and $\mathcal{A}_K$ might not be generated by a single element. Garoufalidis [11] noticed that by adding to $\mathcal{T}$ all the inverses of polynomials in M one gets a principal ideal domain $\tilde{\mathcal{T}}$, and hence from the ideal $\mathcal{A}_K$ one can define a polynomial invariant. Formally one can proceed as follows. Let $\mathcal{R}(M)$ be the fractional field of the polynomial ring $\mathcal{R}[M]$. Let $\tilde{\mathcal{T}}$ be the set of all Laurent polynomials in the variable L with coefficients in $\mathcal{R}(M)$:

$$\tilde{\mathcal{T}} = \left\{ \sum_{k \in \mathbb{Z}} a_k(M) L^k \;\middle|\; a_k(M) \in \mathcal{R}(M),\ a_k = 0 \text{ almost everywhere} \right\},$$

and define the product in $\tilde{\mathcal{T}}$ by $a(M)L^k \cdot b(M)L^l = a(M)b(t^{2k}M)L^{k+l}$.

Then it is known that every left ideal in $\tilde{\mathcal{T}}$ is principal, and $\mathcal{T}$ embeds as a subring of $\tilde{\mathcal{T}}$. The extension $\tilde{\mathcal{A}}_K := \tilde{\mathcal{T}} \mathcal{A}_K$ of $\mathcal{A}_K$ in $\tilde{\mathcal{T}}$ is then generated by a single polynomial

$$\alpha_K(t; M, L) = \sum_{i=0}^{n} \alpha_{K,i}(t; M) L^i \in \mathcal{T}_+,$$

where the degree in L is assumed to be minimal and all the coefficients $\alpha_{K,i}(t; M) \in \mathbb{Z}[t^{\pm 1}, M]$ are assumed to be co-prime. That α_K can be chosen to have integer coefficients follows from the fact that $J_K(n) \in \mathbb{Z}[t^{\pm 1}]$. It is clear that $\alpha_K(t; M, L)$ annihilates J_K, and hence it is in the recurrence ideal $\mathcal{A}_K$. Note that $\alpha_K(t; M, L)$ is defined up to a factor $\pm t^a M^b$, $a, b \in \mathbb{Z}$. We will call α_K the *recurrence polynomial* of K. For example, the polynomial P in the previous subsection is the recurrence polynomial of the right-handed trefoil.

Remark 0.1. If P is a polynomial in t and M (no L), and $Pf = 0$ then $P = 0$. Hence adding all the inverses of polynomials in M does not affect the recurrence relations.

0.2. Main results

Let ϵ be the map reducing $t = -1$. Formally, if V is an $\mathcal{R}$-module, then let $\epsilon(V) = \mathbb{C} \otimes_{\mathcal{R}} V$, where $\mathbb{C}$ is considered as an $\mathcal{R}$-module by setting $t = -1$. Also if $x \in V$ then $\epsilon(x)$ is the image of $1 \otimes x$ in $\epsilon(V)$. Thus $\epsilon(\alpha_K)$ is the polynomial obtained from $\alpha_K(t; M, L)$ by putting $t = -1$. For example, when K is the right-handed trefoil, $\epsilon(\alpha_K) = (M^4 - 1)(L - 1)(LM^6 + 1)$.

For non-zero $f, g \in \mathbb{C}[M, L]$, we say that f is *M-essentially equal to g*, and write

$$f \overset{M}{=} g,$$

if the quotient f/g does not depend on L. We say that two algebraic subsets of $\mathbb{C}^2$ with parameters (M, L) are *M-essentially equal* if they are the same up to adding some lines parallel to the L-axis. It is clear that if f is M-essentially equal to g, then $\{f = 0\}$ and $\{g = 0\}$ are M-essentially equal. Here $\{f = 0\}$ is the algebraic set of zero points of f.

0.2.1. The AJ conjecture

Let $A_K \in \mathbb{Z}[L, M]$ be the A-polynomial of K (see [6,7]); we will review its definition in Section 3. Garoufalidis [11] made the following conjecture.

Conjecture 1 *(The AJ conjecture). The polynomials $\epsilon(\alpha_K)$ and $(L - 1)A_K$ are M-essentially equal.*

Actually, this is the strong version. The weak version of the conjecture says that $\{\epsilon(\alpha_K) = 0\}$ and $\{(L - 1)A_K = 0\}$ are M-essentially equal. The algebraic set $\{(L - 1)A_K = 0\}$ is known as the *deformation variety* of the knot group, with the component $\{L - 1 = 0\}$ corresponding to abelian representations of the knot group into $SL_2(\mathbb{C})$, and $\{A_K = 0\}$—to non-abelian ones.

Garoufalidis [11] verified the conjecture for the trefoil and Fig. 8 knot. Takata [25] gave some evidence to support the conjecture for twist knots, but did not prove it. Both works are based heavily on the computer programs of Wilf and Zeilberger. Hikami [15] verifies the conjecture for torus knots. In all these works direct calculations with explicit formulas are used.

In the present paper we prove the conjecture for a large class of two-bridge knots, using a more conceptual approach. Two-bridge knots $\mathfrak{b}(p, m)$ are parametrized by a pair of odd positive integers $m < p$, with $\mathfrak{b}(p, m) = \mathfrak{b}(p, m')$ if $mm' = 1 \pmod p$ (see [2] and Section 5).

Theorem 1. *Suppose $K = \mathfrak{b}(p, m)$ is a two-bridge knot.*

(a) *The recurrence polynomial α_K has L-degree less than or equal to $(p + 1)/2$.*
(b) *The algebraic set $\{\epsilon(\alpha_K) = 0\}$ is M-essentially equal to an algebraic subset of $\{(L - 1)A_K = 0\}$.*
(c) *The AJ conjecture holds true if*

$$\textit{the } A\textit{-polynomial is } \mathbb{Z}\textit{-irreducible and has } L\textit{-degree } (p - 1)/2. \qquad (*)$$

Here $\mathbb{Z}$-irreducibility means irreducibility in $\mathbb{Z}[M, L]$. There are many two-bridge knots that satisfy condition $(*)$. For example in a recent work [14] Hoste and Shanahan proved that all the twist knots satisfy the condition $(*)$. Hence we have the following corollary.

Corollary 0.2. *The AJ conjecture holds true for twist knots.*

In a separate paper [19] we will prove that if both p and $(p-1)/2$ are prime, then $\mathfrak{b}(p,m)$ satisfies the condition $(*)$. Also knot tables show that many two-bridge knots with small p, m satisfy the condition $(*)$.

0.2.2. The Kauffman Bracket Skein Module of knot complements

Our proof of the main theorem is more or less based on the ideology that the Kauffman Bracket Skein Module (KBSM) is a quantization of the $SL_2(\mathbb{C})$-character variety (see [4,24] and Section 4 below), which has been exploited in the work of Frohman, Gelca, and Lofaro [10] where they defined the non-commutative A-ideal. The calculation of the KBSM of a knot complement is a difficult task. Bullock [3] and recently Bullock and Lofaro [5] calculated the KBSM for the complements of $(2, 2p+1)$ torus knots and twist knots. Another main result of this paper is a generalization of these works: We calculate explicitly the KBSM for complements of all two-bridge knots. We will use another, more geometric approach that allows us to get the results for all two-bridge knots.

0.2.3. Other results

We also prove that the growth of degree (or breadth) of the colored Jones polynomial of an arbitrary knot is at most quadratic with respect to the color, and if the knot is alternating, then the growth is exactly quadratic, given by explicit formula. This is based on the exact estimate of the crossing number, used in the proof (of Kauffman, Murasugi, and Thistlethwaite) of the Tait conjecture on the crossing number of alternating knots. As a corollary, we show that the L-degree of the recurrence polynomial of an alternating knot must be at least 2.

0.3. Plan of the paper

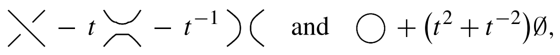
In Section 1 we review the theory of skein modules the colored Jones polynomial. In Section 2 we study the growth of the degree of the colored Jones polynomial and the L-degree of the recurrence polynomial. In Section 3 we review the A-polynomial and introduce a closely related polynomial, B_K. Section 4 is devoted to the "quantum" version of B_K, the peripheral polynomial. We will formulate another weaker version of the AJ conjecture and prove it holds true for two-bridge knots (in Section 6). In Section 5 we calculate the skein module of the complement of two-bridge knots. The last section contains a proof of Theorem 1.

1. The colored Jones polynomial and skein modules

We recall the definition and known facts about the colored Jones polynomial through the theory of Kauffman Bracket Skein Modules which was introduced by Przytycki and Turaev, see the survey [23].

1.1. Skein modules

Recall that $\mathcal{R} = \mathbb{C}[t^{\pm 1}]$. A *framed link* in an oriented 3-manifold Y is a disjoint union of embedded circles, equipped with a non-zero normal vector field. Framed links are considered up to isotopy. In all figures we will draw framed links, or part of them, by lines as usual, with the convention that the framing is blackboard. Let $\mathcal{L}$ be the set of isotopy classes of framed links in the manifold Y, including the empty link. Consider the free $\mathcal{R}$-module with basis $\mathcal{L}$, and factor it by the smallest submodule containing all expressions of the form



where the links in each expression are identical except in a ball in which they look like depicted. This quotient is denoted by $\mathcal{S}(Y)$ and is called the Kauffman bracket skein module, or just skein module, of Y.

When $Y = \Sigma \times [0, 1]$, the cylinder over the surface Σ, we also use the notation $\mathcal{S}(\Sigma)$ for $\mathcal{S}(Y)$. In this case $\mathcal{S}(\Sigma)$ has an algebra structure induced by the operation of gluing one cylinder on top of the other. The operation of gluing the cylinder over ∂Y to Y induces a $\mathcal{S}(\partial Y)$-left module structure on $\mathcal{S}(M)$.

1.2. *Example: $\mathcal{S}(\mathbb{R}^3)$ and the Jones polynomial*

When Y is the 3-space $\mathbb{R}^3$ or the 3-sphere S^3, the skein module $\mathcal{S}(Y)$ is free over $\mathcal{R}$ of rank one, and is spanned by the empty link. Thus if ℓ is a framed link in $\mathbb{R}^3$, then its value in the skein module $\mathcal{S}(\mathbb{R}^3)$ is $\langle \ell \rangle$ times the empty link, where $\langle \ell \rangle \in \mathcal{R}$, known as the Kauffman bracket of ℓ (see [17,21]), and is just the Jones polynomial of *framed links* in a suitable normalization.

1.3. *Example: The solid torus and the colored Jones polynomial*

The solid torus ST is the cylinder over an annulus, and hence its skein module $\mathcal{S}(ST)$ has an algebra structure. The algebra $\mathcal{S}(ST)$ is the polynomial algebra $\mathcal{R}[z]$ in the variable z, which is a knot representing the core of the solid torus.

Instead of the $\mathcal{R}$-basis $\{1, z, z^2, \ldots\}$, two other bases are often useful. The first basis consists of the Chebyshev polynomials $T_n(z)$, $n \geqslant 0$, defined by $T_0(x) = 2$, $T_1(x) = x$, and $T_{n+1}(x) = xT_n - T_{n-1}$. The second basis consists of polynomials $S_n(z)$, $n \geqslant 0$, satisfying the same recurrence relation, but with $S_0(x) = 1$ and $S_1(x) = x$. Extend both polynomials by the recurrence relation to all indices $n \in \mathbb{Z}$. Note that $T_{-n} = T_n$, while $S_{-n} = -S_{n-2}$.

For a framed knot K in a 3-manifold Y we define the nth power K^n as the link consisting of n parallel copies of K. Using these powers of a knot, $S_n(K)$ is defined as an element of $\mathcal{S}(Y)$. In particular, if $Y = \mathbb{R}^3$ one can calculate the bracket $\langle S_n(K) \rangle \in \mathcal{R}$, and it is essentially the colored Jones polynomial. More precisely, we will define the colored Jones polynomial $J_K(n)$ by the equation

$$J_K(n+1) := (-1)^n \times \big\langle S_n(K) \big\rangle.$$

The $(-1)^n$ sign is added so that when K is the trivial knot,

$$J_K(n) = [n] := \frac{t^{2n} - t^{-2n}}{t^2 - t^{-2}}.$$

Then $J_K(1) = 1$, $J_K(2) = -\langle K \rangle$. We extend the definition for all integers n by $J_K(-n) = -J_K(n)$ and $J_K(0) = 0$. In the framework of quantum invariants, $J_K(n)$ is the sl_2-quantum invariant of K colored by the n-dimensional simple representation of sl_2.

We will always assume K has 0 framing. In this case $J_K(n)$ contains only even powers of t, i.e., $J_K(n) \in \mathbb{Z}[t^{\pm 2}]$. Hence the recurrence polynomial α_K can be assumed to have only even powers of t.

1.4. Example: Cylinder over the torus and the non-commutative torus

A pair of *oriented* meridian and longitude on the torus $\mathbb{T}^2$ will define an *algebra* isomorphism Φ between $\mathcal{S}(\mathbb{T}^2)$ and the symmetric part of the quantum torus $\mathcal{T}$ as follows.

For a pair of integers a, b let $(a,b)_T = T_d((a',b')_T)$, where d is the greatest common divisor of a and b, with $a = da'$, $b = db'$, and $(a',b')_T$ is the closed curve without self-intersection on the torus that is homotopic to a' times the meridian plus b' times the longitude. Here T_d is the Chebyshev polynomial defined above; and the framing of a curve on $\mathbb{T}^2$ is supposed to be parallel to the surface $\mathbb{T}^2$. Note that in the definition of skein modules we use *non-oriented links*, hence $(a,b)_T = (-a,-b)_T$. As an $\mathcal{R}$-module, $\mathcal{S}(\mathbb{T}^2)$ is the quotient of the free $\mathcal{R}$-module spanned by $\{(a,b)_T, (a,b) \in \mathbb{Z}^2\}$ modulo the relations $(a,b)_T = (-a,-b)_T$.

Recall that the quantum torus $\mathcal{T}$ is defined as $\mathcal{T} = \mathcal{R}\langle L^{\pm 1}, M^{\pm 1}\rangle/(LM = t^2 ML)$. Let $\mathcal{T}^\sigma$ be the subalgebra of $\mathcal{T}$ invariant under the involution σ, where $\sigma(M^a L^b) = M^{-a} L^{-b}$. Frohman and Gelca in [9] showed that the map

$$\Phi : \mathcal{S}\big(\mathbb{T}^2\big) \to \mathcal{T}^\sigma, \quad \Phi\big((a,b)_T\big) = (-1)^{a+b} t^{ab}\big(M^a L^b + M^{-a} L^{-b}\big)$$

is an isomorphism of *algebras*.

1.5. Two-punctured disk

Let F be the rectangle (in $\mathbb{R}^2$)

$$\big\{(x_1,x_2) \in \mathbb{R}^2 \;\big|\; 0 \leqslant x_1 \leqslant 6,\ 0 \leqslant x_2 \leqslant 2\big\},$$

without 2 interior points $U = (1,1)$ and $U' = (5,1)$. Then $\mathcal{S}(F)$ is the polynomial algebra $\mathcal{R}[x, x', y]$, where x is a small loop around U, x' a small loop around U', and y a loop circling both U, U' (see Fig. 1). Note that in this case $\mathcal{S}(F)$ is a commutative algebra, which is not true in general when F is replaced by an arbitrary surface, see [23].

We will identify F with the section $F \times 1/2$ in $F \times [0,1]$. The lines $x_1 = 2$ and $x_1 = 4$ divide F into 3 parts: the left part F_l (containing U), the right F_r (containing U'), and the middle part F_m. Let β be a braid on $2k$ strands in $F_m \times [0,1]$ with boundary points on the lines $x_1 = 2$ and $x_1 = 4$ (on F), i.e., β consists of $2k$ connected paths, each begins at a point on $x_1 = 2$ (on F) and goes monotonously to the right until ending at a point on $x_1 = 4$, see Fig. 2. Let the closure $\hat{\beta}$

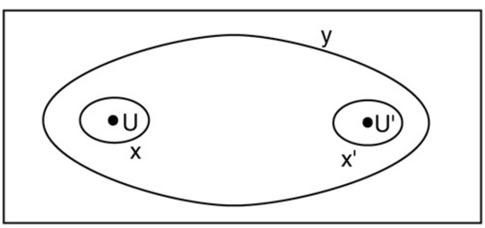


Fig. 1. The 2-punctured disk.

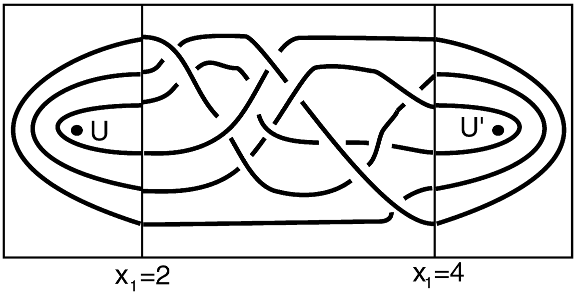


Fig. 2. The braid β (in the middle) and its closure.

be the link in $F \times [0,1]$ obtained by closing β using k parallel rainbow arcs in F_l and k parallel rainbow arcs in F_r, as shown in Fig. 2.

Lemma 1.1. *As an element of the skein module $\mathcal{S}(F) = \mathcal{R}[x, x', y]$ the closure link $\hat{\beta}$ of a braid on $2k$ strands is a polynomial having y-degree k, and the coefficient of y^k is invertible and of the form $\pm t^l$, $l \in \mathbb{Z}$.*

Proof. Consider the diagram D of $\hat{\beta}$ on F. At every crossing point there are 2 ways to smooth the diagram, one positive that gives coefficient t in the skein relation, and the other is negative. A *state* is the result of smoothing all the crossing; what one has is a bunch of non-intersecting circles, each is one of x, x', y, or the trivial loop. The link ℓ, considered as an element of $\mathcal{S}(F) \equiv \mathcal{R}[x, x', y]$, is the sum over all states, each with coefficient a power of t. It is clear that for the closure $\hat{\beta}$ of a braid, there is only one state (the one with horizontal resolutions everywhere) that gives the maximal power y^k, and its coefficient is a power of t. □

2. Some properties of the colored Jones polynomial

2.1. One of the best known applications of the Jones polynomial is a proof (Kauffman, Murasugi, and Thistlethwaite) of the Tait conjecture on the crossing number of alternating links, based on an exact estimate of the crossing number using the *breadth* of the Jones polynomial. We will need a generalization of this estimate for the colored Jones polynomial.

For a Laurent polynomial $P(t) \in \mathbb{Z}[t^{\pm 1}]$ let $d_+(P)$ and $d_-(P)$ be respectively the maximal and minimal degree of t in P. The difference $br := d_+ - d_-$ is called the breadth of P. For a link diagram D, let $s_+(D)$ and $s_-(D)$ be the number of circles obtained by positively (respectively, negatively) smoothing all the double points.

Proposition 2.1. (a) *Suppose K is a knot with a knot diagram D having k crossings. Assuming the framing is black-board. Then*

$$d_+\big(J_K(n)\big) \leqslant k(n-1)^2 + 2(n-1)s_+(D),$$
$$d_-\big(J_K(n)\big) \geqslant -k(n-1)^2 - 2(n-1)s_-(D).$$

Hence the breadth of $J_K(n)$ grows at most as a quadratic function in n.

(b) *If K is a non-trivial alternating knot with k crossings. Then the breadth of $J_K(n) \in \mathbb{Z}[t^{\pm 1}]$ is a quadratic polynomial in n. Moreover*

$$br\big(J_K(n)\big) = 2k(n-1)^2 + 2(n-1)(k+2).$$

Proof. (a) The n-parallel D^n of D will have kn^2 double points. In addition, it is easy to see that $s_\pm(D^n) = ns_\pm(D)$. Hence Lemma 5.4 of [21] says that

$$d_+\big\langle D^n\big\rangle \leqslant f(n) := k(n-1)^2 + 2(n-1)s_+.$$

Note that here we have taken into account the fact that we shift the index $n \to n+1$, and use the normalization in which the unknot takes value $-t^2 - t^{-2}$. Note that $f(n)$ is a strictly increasing function, $f(n+1) > f(n)$. Recall that $S_n(K) = D^n +$ terms of lower degrees in D. Hence one has

$$d_+\big\langle S_n(D)\big\rangle \leqslant k(n-1)^2 + 2(n-1)s_+.$$

The proof for d_- is similar.

(b) For an alternating knot diagram one has $s_+ + s_- = k+2$. Note that the diagram D^n is adequate in the sense of [21, Chapter 5]. For adequate diagrams the estimates of $d_\pm$ in part (a) are exact (see the proof of Lemma 5.4 of [21]) and the result follows immediately. □

2.2. The L-degree of the recurrence polynomial

Proposition 2.2. *Suppose K is a non-trivial alternating knot. Then the recurrence polynomial α_K has L-degree greater than* 1.

Proof. Assume the contrary that $\alpha_K = P(t;M)L + P_0(t;M)$, where $P, P_0 \in \mathbb{Z}[t^{\pm 1}, M^{\pm 1}]$. Garoufalidis [11] showed that the polynomial $\sigma(\alpha_K) = P(t;M^{-1})L^{-1} + P_0(t;M^{-1})$ is also in the recurrence ideal. Since α_K is *the* generator, it follows that for some $\gamma(t;M) \in \mathcal{R}(M)$

$$L\sigma(\alpha_K) = \gamma(t;M)\alpha_K.$$

One can then easily show that, after normalizing both P, P_0 by a same power of M, one has

$$P_0(t;M) = P\big(t;t^{-2}M^{-1}\big).$$

The equation $\alpha_K J_K = 0$ can now be rewritten as

$$J_K(n+1) = -\frac{P(t;t^{-2-2n})}{P(t;t^{2n})} J_K(n).$$

It is easy to see that for n big enough, the difference of the breadths $br(P(t;t^{-2-2n})) - br(P(t;t^{2n}))$ is a constant depending only on the polynomial $P(t;M)$, but not on n. From the above equation it follows that the breadth of $J_K(n)$, for n big enough, is a linear function on n. This contradicts Proposition 2.1. □

2.3. More on the recurrence polynomial of knots

The following proposition was known to Garoufalidis.

Proposition 2.3. *When reduced by $t = -1$, the recurrence polynomial α_K is divisible by $L - 1$. In other words, $\frac{\epsilon(\alpha_K)}{L-1} \in \mathbb{Z}[M, L]$.*

Proof. For a function $f(t^2, n)$ of two variables $t^2 \in \mathbb{C}$ and $n \in \mathbb{Z}$ let $\bar{f}(z)$ be the limit of $f(t^2, n)$ when

$$t^2 \to 1 \text{ and } t^{2n} \text{ is kept equal to } z \text{ all the time.} \tag{$\dagger$}$$

The Melvin–Morton conjecture [22], proved by Bar-Natan and Garoufalidis [1], showed that $h(z) := \overline{J_K(n)}$ is the inverse of the Alexander polynomial. In particular, $h(z) \neq 0$.

Lemma 2.4 below shows that $\overline{J_K(n+k)} = \overline{J_K(n)}$ for any fixed k. Hence the operator L becomes the identity after taking the limit (†). Thus applying the limit (†) to the equation $\alpha_K J_K = 0$ we see that

$$\alpha_K|_{t^2=1, M=z, L=1} h(z) = 0.$$

Since $h(z) \neq 0$, one has $\alpha_K|_{t^2=1, M=z, L=1} = 0$, which is equivalent to the lemma. □

Lemma 2.4. *For every fixed integer k, one has $\overline{J_K(n+k)} = \overline{J_K(n)}$.*

Proof. For a knot K of framing 0, $J_K(n)/[n]$ is a Laurent polynomial in t^4, and (see [22])

$$\left.\frac{J_K(n)}{[n]}\right|_{t^4=\exp\hbar} = \sum_{l=0}^{\infty} P_l(n)\hbar^l,$$

where $P_l(n)$ is a polynomial in n of degree at most l:

$$P_l(n) = P_{l,l}n^l + P_{l,l-1}n^{l-1} + \cdots + P_{l,1}n + P_{l,0}.$$

The limit (†) is the same as the limit $n \to \infty$, with $\hbar = 2\ln z/n$. Under this limit,

$$(n+k)^i \hbar^k \to \begin{cases} 0 & \text{if } i < l, \\ (2\ln z)^l & \text{if } i = l, \end{cases}$$

which does not depend on k. □

3. The A-polynomial and its sibling

We briefly recall here the definition of the A-polynomial and introduce a sibling of it. We will say that f is *M-essentially divisible by g* if f is M-essentially equal to a polynomial divisible by g.

3.1. The character variety of a group

The set of representations of a finitely presented group π into $SL_2(\mathbb{C})$ is an algebraic set defined over $\mathbb{C}$, on which $SL_2(\mathbb{C})$ acts by conjugation. The naive quotient space, i.e. the set of orbits, does not have a good topology/geometry. Two representations in the same orbit (i.e. conjugate) have the same character, but the converse is not true in general. A better quotient, the algebro-geometric quotient denoted by $\chi(\pi)$ (see [8,20]), has the structure of an algebraic set. There is a bijection between $\chi(\pi)$ and the set of all characters of representations of π into $SL_2(\mathbb{C})$, hence $\chi(\pi)$ is usually called the *character variety* of π. For a manifold Y we use $\chi(Y)$ also to denote $\chi(\pi_1(Y))$.

Suppose $\pi = \mathbb{Z}^2$, the free abelian group with 2 generators. Every pair of generators λ, μ will define an isomorphism between $\chi(\pi)$ and $(\mathbb{C}^*)^2/\tau$, where $(\mathbb{C}^*)^2$ is the set of non-zero complex pairs (L, M) and τ is the involution $\tau(M, L) = (M^{-1}, L^{-1})$, as follows: Every representation is conjugate to an upper diagonal one, with L and M being the upper left entry of λ and μ, respectively. The isomorphism does not change if one replaces (λ, μ) with (λ^{-1}, μ^{-1}).

3.2. The A-polynomial

Let X be the closure of S^3 minus a tubular neighborhood $N(K)$ of a knot K. The boundary of X is a torus whose fundamental group is free abelian of rank two. An orientation of K will define a unique pair of an oriented meridian and an oriented longitude such that the linking number between the longitude and the knot is 0. The pair provides an identification of $\chi(\pi_1(\partial X))$ and $(\mathbb{C}^*)^2/\tau$ which actually does not depend on the orientation of K.

The inclusion $\partial X \hookrightarrow X$ induces the restriction map

$$\rho : \chi(X) \to \chi(\partial X) \equiv (\mathbb{C}^*)^2/\tau.$$

Let Z be the image of ρ and $\hat{Z} \subset (\mathbb{C}^*)^2$ the lift of Z under the projection $(\mathbb{C}^*)^2 \to (\mathbb{C}^*)^2/\tau$. The Zariski closure of $\hat{Z} \subset (\mathbb{C}^*)^2 \subset \mathbb{C}^2$ in $\mathbb{C}^2$ is an algebraic set consisting of components of dimension 0 or 1. The union of all the 1-dimension components is defined by a single polynomial $A'_K \in \mathbb{Z}[M, L]$, whose coefficients are co-prime. Note that A'_K is defined up to ± 1. It is known that A'_K is divisible by $L - 1$, hence $A'_K = (L - 1)A_K$, where $A_K \in \mathbb{C}[M, L]$ is called the A-polynomial of K. It is known that $A_K \in \mathbb{Z}[M^2, L]$. By definition, A_K does not have repeated factor, and is not divisible by $L - 1$.

Question 1. Can $A_K(L, M)$ have a factor a non-constant polynomial depending on M only?

3.3. The dual picture

It is also instructive and convenient to see the dual picture in the construction of the A-polynomial. For an algebraic set Y let $R[Y]$ denote the ring of regular functions on Y. For example, $R[(\mathbb{C}^*)^2/\tau] = \mathfrak{t}^\sigma$, the σ-invariant subspace of $\mathfrak{t} := \mathbb{C}[L^{\pm 1}, M^{\pm 1}]$, where $\sigma(M^a L^b) := M^{-a} L^{-b}$.

The map ρ in the previous subsection induces an algebra homomorphism

$$\theta : R\big[\chi(\partial X)\big] \equiv \mathfrak{t}^\sigma \to R\big[\chi(X)\big].$$

We will call the kernel $\mathfrak{p}$ of θ the *classical peripheral ideal*; it is an ideal of $\mathfrak{t}^\sigma$. Let $\hat{\mathfrak{p}} := \mathfrak{t}\mathfrak{p}$ be the ideal extension of $\mathfrak{p}$ in $\mathfrak{t}$. The set of zero points of $\hat{\mathfrak{p}}$ is the closure of $\hat{Z}$ in $\mathbb{C}^2$.

3.4. A sibling of the A-polynomial

The ring $\mathfrak{t} = \mathbb{C}[M^{\pm 1}, L^{\pm 1}]$ embeds naturally into the principal ideal domain $\tilde{\mathfrak{t}} := \mathbb{C}(M)[L^{\pm 1}]$, where $\mathbb{C}(M)$ is the fractional field of $\mathbb{C}[M]$. The ideal extension of $\hat{\mathfrak{p}}$ in $\tilde{\mathfrak{t}}$, which is $\tilde{\mathfrak{t}}\hat{\mathfrak{p}} = \tilde{\mathfrak{t}}\mathfrak{p}$, is thus generated by a single polynomial $B_K \in \mathbb{Z}[M, L]$ which has co-prime coefficients and is defined up to a factor $\pm M^a$ with $a \in \mathbb{Z}$. Again B_K can be chosen to have integer coefficients because everything can be defined over $\mathbb{Z}$.

From the definitions one has immediately

Proposition 3.1. *The polynomial B_K is M-essentially divisible by $A'_K = (L-1)A_K$. The two algebraic sets $\{B_K = 0\}$ and $\{A'_K = 0\}$ are M-essentially equal.*

Note that B_K might not be M-essentially equal to A'_K because B_K might contain repeated factors. If the answer to Question 1 is negative, then the two algebraic sets $\{B_K = 0\}$ and $\{A'_K = 0\}$ are equal, and we have a natural way to define the multiplicity of factors of the A-polynomial, using the B-polynomial.

4. The quantum peripheral ideal and the peripheral polynomial

4.1. Skein modules as quantum deformations of character varieties

Recall that ϵ is the map reducing $t = -1$. One important result (Bullock, Przytycki, and Sikora [3,24]) in the theory of skein modules is that $\epsilon(\mathcal{S}(Y))$ has a natural algebra structure and, when factored by its nilradical, is canonically isomorphic to $R[\chi(Y)]$, the ring of regular functions on the character variety of $\pi_1(Y)$. The product of 2 links in $\epsilon(\mathcal{S}(Y))$ is their union. Using the skein relation with $t = -1$, it is easy to see that the product is well-defined, and that the value of a knot in the skein module depends only on the homotopy class of the knot in Y. The isomorphism between $\epsilon(\chi(Y))$ and $R[\chi(Y)]$ is given by $K(r) = -\operatorname{tr} r(K)$, where K is a homotopy class of a knot in Y, represented by an element, also denoted by K, of $\pi_1(Y)$, and $r : \pi_1(Y) \to SL_2(\mathbb{C})$ is a representation of $\pi_1(Y)$.

In many cases the nilradical of $\epsilon(\mathcal{S}(Y))$ is trivial, and hence $\epsilon(\mathcal{S}(Y))$ is exactly equal to the ring of regular functions on the character variety of $\pi_1(Y)$. For example, this is the case when Y is a torus, or when Y is the complement of a two-bridge knots (see Section 5).

In light of this fact, one can consider $\mathcal{S}(Y)$ as a quantization of the character variety.

4.2. The quantum peripheral ideal

Recall that X is the closure of the complement of a tubular neighborhood $N(K)$ in S^3. The boundary ∂X is a torus, and using the preferred meridian and longitude we will identify $\mathcal{S}(\partial X)$ with $\mathcal{T}^\sigma$, see Section 1.4.

The embedding of ∂X into X gives us a map $\Theta : \mathcal{S}(\partial X) \equiv \mathcal{T}^\sigma \to \mathcal{S}(X)$, which can be considered as a quantum analog of θ. One has the following commutative diagram:

$$\begin{array}{ccc} \mathcal{T}^\sigma & \xrightarrow{\Theta} & \mathcal{S}(X) \\ {\scriptstyle\epsilon}\big\downarrow & & {\scriptstyle\epsilon}\big\downarrow \\ \mathfrak{t}^\sigma & \xrightarrow{\theta} & R[\chi(X)]. \end{array}$$

The kernel of Θ, denoted by $\mathcal{P}$, is called the *quantum peripheral ideal*; it is a left ideal of $\mathcal{T}^\sigma$ and can be considered as a quantum analog of the classical peripheral ideal $\mathfrak{p} = \ker\theta$. The ideal $\mathcal{P}$ was introduced by Frohman, Gelca, and Lofaro in [10] and there it is called simply the peripheral ideal. From the commutative diagram it is clear that $\epsilon(\mathcal{P}) \subset \mathfrak{p}$. The following question is important.

Question 2. Is it true that $\epsilon(\mathcal{P})$ can never be 0?

4.3. The peripheral polynomial

Let us adapt the construction of the B_K polynomial to the quantum setting. Recall that $\tilde{\mathcal{T}}$ (see introduction) is a principal left-ideal domain that contains $\mathcal{T}^\sigma$ as a subring. The left-ideal extension $\tilde{\mathcal{P}} := \tilde{\mathcal{T}}\mathcal{P}$ in $\tilde{\mathcal{T}}$ is generated by a polynomial

$$\beta_K(t; M, L) = \sum_{i=0}^{s} \beta_{K,i}(t, M)\, L^i \in \mathcal{T}_+,$$

where s is assumed to be minimum and all the coefficients $\beta_{K,i}(t, M) \in \mathbb{Z}[t^{\pm 1}, M^{\pm 1}]$ are co-prime. We call β_K the *peripheral polynomial* of K, which is defined up to $\pm t^a M^b$ with $a, b \in \mathbb{Z}$.

Proposition 4.1. *$\epsilon(\beta_K)$ is M-essentially divisible by B_K, and hence is M-essentially divisible by $A'_K = (L-1)A_K$.*

Proof. The proposition follows the fact that $\epsilon\mathcal{P} \subset \mathfrak{p}$ and Proposition 3.1. □

Proposition 4.2. *Suppose $\epsilon(\mathcal{P}) = \mathfrak{p}$. Then $\epsilon(\beta_K) \overset{M}{=} B_K$.*

Proof. For the extensions $\hat{\mathcal{P}} := \mathcal{T}\mathcal{P}$ and $\hat{\mathfrak{p}} := \mathfrak{t}\mathfrak{p}$ we also have $\epsilon(\hat{\mathcal{P}}) = \hat{\mathfrak{p}}$, since $\epsilon\hat{\mathcal{P}} = \epsilon(\mathcal{T}\mathcal{P}) = \epsilon(\mathcal{T})\epsilon(\mathcal{P}) = \mathfrak{t}\mathfrak{p}$.

From the definition we have that $h(M)\mathcal{B}_K \in \hat{\mathfrak{p}}$ for some non-zero polynomial $h(M) \in \mathbb{Z}[M]$. Hence $\epsilon^{-1}(h(M)B_K) \subset \hat{\mathcal{P}}$ is not empty. Take an element $u \in \epsilon^{-1}(h(M)B_K)$; it is M-essentially divisible by β_K, the generator. Applying the map ϵ, one gets B_K is M-essentially divisible by $\epsilon(\beta_K)$. Combining with Proposition 4.1 one has $\epsilon(\beta_K) \overset{M}{=} B_K$. □

Conjecture 2. *For every knot we have $\epsilon(\mathcal{P}) = \mathfrak{p}$, and hence $\epsilon(\beta_K) \overset{M}{=} B_K$.*

Later we will show that for all two-bridge knots holds true the conjecture, which is closely related to the AJ conjecture. A sufficient condition for the conjecture to hold true is given in Section 6.

4.4. The orthogonal ideal and recurrence relations

There is a bilinear pairing

$$\mathcal{S}\big(N(K)\big) \otimes \mathcal{S}(X) \to \mathcal{R}, \quad \text{with} \quad \ell \otimes \ell' \to \langle \ell, \ell' \rangle := \langle \ell \cup \ell' \rangle \in \mathcal{S}\big(S^3\big) = \mathcal{R}, \tag{1}$$

where ℓ and ℓ' are framed links in $N(K)$ and X, respectively. The *orthogonal ideal* $\mathcal{O}$ is defined by

$$\mathcal{O} := \{\ell' \in \mathcal{S}(\partial X) \mid \langle \ell, \Theta(\ell') \rangle = 0 \text{ for every } \ell \in \mathcal{S}(N(K))\}.$$

It is clear that $\mathcal{O}$ is a left ideal of $\mathcal{S}(\partial X) \equiv \mathcal{T}^\sigma$ and $\mathcal{P} \subset \mathcal{O}$. In [10], where $\mathcal{O}$ was first introduced, $\mathcal{O}$ was called the formal ideal. What is important for us is the following

Proposition 4.3. *The orthogonal ideal is in the recurrence ideal of a knot, $\mathcal{O} \subset \mathcal{A}_K$. As a consequence, $\mathcal{P} \subset \mathcal{A}_K$.*

This was proved by Garoufalidis [11]. Frohman, Gelca, and Lofaro [10] proved that every element ℓ' in the orthogonal ideal $\mathcal{O}$ gives rise to a linear recurrence relation for the colored Jones polynomial. The idea is simple and beautiful: ℓ' annihilates everything in $\mathcal{S}(N(K))$, in particular, $\langle T_n(z), \Theta(\ell') \rangle = 0$; but this equation, after some calculation, can be rewritten as a linear recurrence relation for the colored Jones polynomial. Garoufalidis, using the Weyl symmetry, further simplified the recurrence relation, and obtained that $\mathcal{O} = \mathcal{A}_K \cap \mathcal{T}^\sigma$, which is stronger than the proposition.

Conjecture 3. *The right kernel of the bilinear form* (1) *is trivial.*

This conjecture implies that $\mathcal{O} = \mathcal{P}$, from which, due to an argument of A. Sikora and the author, one can show that the colored Jones polynomial distinguishes the unknot from other knots.

4.5. Relation between the peripheral and recurrence polynomials

Lemma 4.4. *The peripheral polynomial β_K is divisible by the recurrence polynomial α_K in the sense that there are polynomials $g(t, M) \in \mathbb{Z}[t, M]$ and $\gamma(t, M, L) \in \mathcal{T}_+$ such that*

$$\beta_K(t, M, L) = \frac{1}{g(t, M)} \gamma(t, M, L) \alpha_K(t, M, L). \tag{2}$$

Moreover $g(t, M)$ and $\gamma(t, M, L)$ can be chosen so that $\epsilon g \neq 0$.

Proof. From Proposition 4.3 we have that $\mathcal{P} \subset \mathcal{A}$. Hence the left-ideal extension $\hat{\mathcal{P}} := \mathcal{T}\mathcal{P}$ is also a subset of $\mathcal{A}$, since both are left ideals of $\mathcal{T}$. It follows that β_K, as the generator of the extension of $\hat{\mathcal{P}}$ in $\tilde{\mathcal{T}}$, is divisible by the generator of the extension of $\mathcal{A}$, and (2) follows.

We can assume that $t + 1$ does not divide both $g(t, M)$ and $\gamma(t, M, L)$ simultaneously. If $\epsilon g = 0$ then g is divisible by $t + 1$, and hence γ is not. But then from the equality

$$g\beta_K = \gamma\alpha_K,$$

it follows that α_K is divisible by $t + 1$, which is impossible, since all the coefficients of powers of L in α_K are supposed to be co-prime. $\square$

5. Two-bridge knots and their skein modules

5.1. Two-bridge knots

A two-bridge knot is a knot $K \subset S^3$ such that there is a 2-sphere $S^2 \subset S^3$ that separates S^3 into 2 balls B_1 and B_2, and the intersection of K and each ball is isotopic to 2 trivial arcs in the ball. The branched double covering of S^3 along a two-bridge knot is a lens space $L(p,m)$, which is obtained by doing a p/m surgery on the unknot. Such a two-bridge knot is denoted by $\mathfrak{b}(p,m)$. It is known that both p, m are odd. One can always assume that $p > m \geqslant 1$. It is known that $\mathfrak{b}(p,m) = \mathfrak{b}(p,m')$ if $mm' \equiv 1 \pmod{p}$.

We will present the ball B_1 as the rectangular parallelepiped, see Fig. 3,

$$B_1 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 0 \leqslant x_1 \leqslant 6,\ 0 \leqslant x_2 \leqslant 2,\ 0 \leqslant x_3 \leqslant 1\}.$$

We suppose that the knot intersects the interior of B_1 in two vertical (i.e. parallel to the x_3-axis) straight intervals UV and $U'V'$, where $U = (1,1,1)$, $U' = (5,1,1)$, and $V = (1,1,0)$, $V' = (5,1,0)$. After an isotopy, we assume that the part of K outside the interior of B_1 are 2 non-intersecting arcs $\mathfrak{u}$ and $\mathfrak{v}$ on $S^2 := \partial B_1$, where $\mathfrak{u}$ connects U and U', and $\mathfrak{v}$ connects V and V'. Later we will describe explicitly the arc $\mathfrak{u}$. If one cuts S^2 along the arc $\mathfrak{u}$, then one obtains a disk, hence the other arc $\mathfrak{v}$ is uniquely determined by $\mathfrak{u}$, up to isotopy.

5.2. Skein module of complements of two-bridge knots

Let W be the top rectangle of B_1, $W = \{(x_1, x_2, x_3) \in X_1 \mid x_3 = 1\}$. Note that $X_1 := B_1 \setminus (UV \cup U'V')$ is the cylinder over a two-punctured disk $W \setminus \{U, U'\}$. Hence $\mathcal{S}(X_1)$ is isomorphic to the commutative algebra $\mathcal{R}[x, x', y]$, as described in Section 1.5. Here x is a small loop circling U, x' a small loop circling U', and $y = \partial W$. One of our main results is

Theorem 2. *The skein module $\mathcal{S}(S^3 \setminus \mathfrak{b}(p,m))$ is free over $\mathcal{R}$ with basis $\{x^a y^b, 0 \leqslant a, 0 \leqslant b \leqslant (p-1)/2\}$.*

The remaining part of the section is devoted to a proof of this theorem. Moreover, we will present more explicit structures of the skein module $\mathcal{S}(S^3 \setminus \mathfrak{b}(p,m))$.

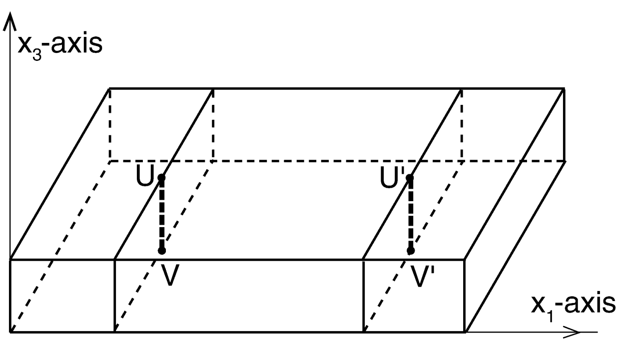


Fig. 3. The ball B_1.

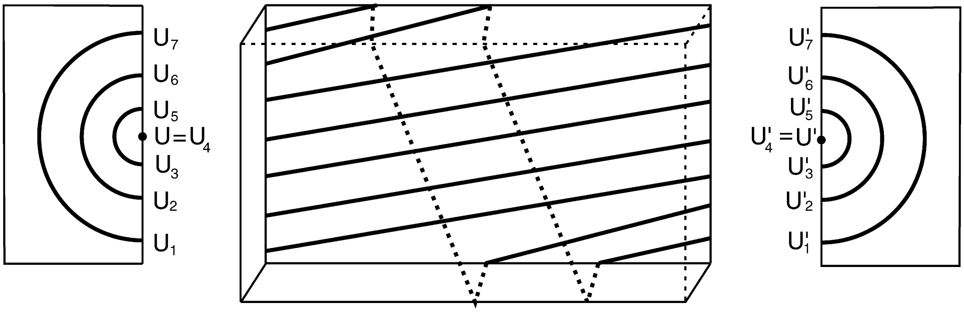


Fig. 4. $\mathfrak{u}_l$, $\mathfrak{u}_m$, and $\mathfrak{u}_r$ for $p=7$, $m=5$.

5.3. Description of the two-bridge knot: The curve $\mathfrak{u}$

Let ψ be the rotation by 180° about the axis $\{x_1=3, x_2=1\}$ (which is parallel to the x_3-axis and passing through the center of the rectangle W). One has $\psi(B_1)=B_1$.

For a set $Z\subset\mathbb{R}^3$ let $Z[a,b]$ be the part of Z in the strip $\{a\leqslant x_1\leqslant b\}$, i.e. $Z[a,b]:=Z\cap\{(x_1,x_2,x_3)\mid a\leqslant x_1\leqslant b\}$. We will consider 3 pieces $B_1[0,1]$, $B_1[1,5]$, and $B_1[5,6]$ of B_1.

Recall that W is the top rectangle of the rectangular parallelepiped X_1. On $W[0,1]$ let $\mathfrak{u}_l$ be the collection of $(p-1)/2$ half-circles centered at U, with radii $\varepsilon, 2\varepsilon, \dots, (p-1)\varepsilon/2$, where $\varepsilon = 2/(p+1)$, see Fig. 4. The end points of the half-circles, together with the center U, are on a straight line; there are exactly p of them, including U. Mark them from bottom to top by $U_1,\dots,U_p$. (Thus $U=U_{(p+1)/2}$.)

Similarly, on $W[5,6]$ let $\mathfrak{u}_r$ be the collection of $(p-1)/2$ half-circles centered at U', with radii $\varepsilon, 2\varepsilon, \dots, (p-1)\varepsilon/2$, where $\varepsilon=2/(p+1)$. Mark the end-points of the half-circles and U' from bottom to top by $U'_1,\dots,U'_p$. (Thus $U'=U'_{(p+1)/2}$.)

Recall that $S^2=\partial B_1$. On $S^2[1,5]$ we construct $\mathfrak{u}_m$ as follows. Note that $S^2[1,5]$ is a cylinder over $S^2[1,1]$: $S^2[1,5]=[1,5]\times S^2[1,1]$. Informally $\mathfrak{u}_m$ is the braid on the cylinder $S^2[1,5]=[1,5]\times S^2[1,1]$ representing the rotation by $(p-m)\pi/p$. Formally, we first connect U_1 with $U'_{1+(p-m)/2}$ by a straight interval. Then for all i, $2\leqslant i\leqslant p$, connect U_i with $U'_{i+(p-m)/2}$ by non-intersecting arcs on $S^2[1,5]$; up to isotopy there is a unique way to do so, see Fig. 4. There are in total p arcs; denote them by $\mathfrak{u}_m$. We can assume that each arc in $\mathfrak{u}_m$ always travel from left to right (no backwards traverse, just like in the case of braids), and moreover, $\mathfrak{u}_m$ is invariant under ψ.

Let $\mathfrak{u}$ be the arc on S^2 obtained by combining $\mathfrak{u}_l$, $\mathfrak{u}_m$ and $\mathfrak{u}_r$; it connects U and U' and is invariant under ψ, see Fig. 5. Up to isotopy there is a unique arc $\mathfrak{v}$ on S^2 connecting V and V'.

Proposition 5.1. *The knot K consisting of the arcs $\mathfrak{u},\mathfrak{v}$, the straight intervals UV and $U'V'$ is the two-bridge knot $\mathfrak{b}(p,m)$.*

Proof. It is clear that K is a two-bridge knot. Let $\tilde{S}^2$ be the 2-fold covering of $S^2=\partial B_1$ branched along the 4 points U, U', V, V'. One can recognize the knot type by looking at the homology class of the lift of $\mathfrak{u}$. Note that $\tilde{S}^2$ is a torus, with the following preferred meridian and longitude. The plane passing through U, U', V, V' intersects $S^2[0,1]$ in an arc $\mathfrak{m}$ that connect U and V. The total lift $\tilde{\mathfrak{m}}$ of $\mathfrak{m}$ is a closed curve on the torus $\tilde{S}^2$ which will serve as the meridian. The total lift $\tilde{\mathfrak{l}}$ of the straight interval UU' is another closed curve serving as the longitude. It is easy to

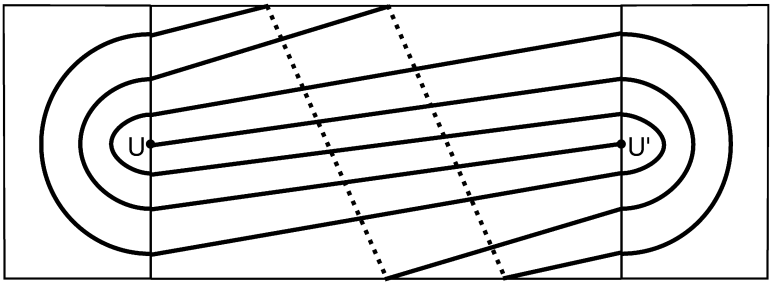


Fig. 5. The arc $\mathfrak{u}$.

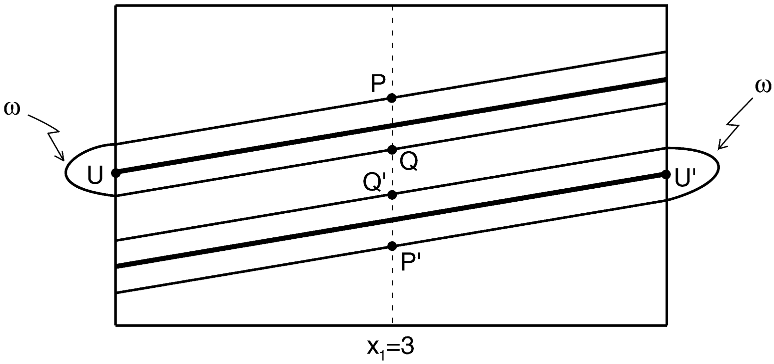


Fig. 6. The ball X_1.

see that $\tilde{\mathfrak{m}}$ and $\tilde{\mathfrak{l}}$ form a basis of $H_1(\tilde{S}^2, \mathbb{Z})$, and that the total lift of the curve $\mathfrak{u}$, as a homology class, is equal to $p\,\tilde{\mathfrak{m}} + (p-m)\tilde{\mathfrak{l}}$. According to [2, Chapter 12], K is a two-bridge knot of type (p, m). $\square$

5.4. From X_1 to the knot complement X

Let ω be the boundary curve of a small normal neighborhood of the arc $\mathfrak{u}$ in $S^2 \equiv \partial B_1$. We can assume that ω is invariant under ψ. Then $X = S^3 \setminus N(K)$ is obtained from B_1 by gluing a 2-handle to along ω.

Recall that on $S^2[1,5]$ the curve $\mathfrak{u}$ consists of p arcs, and we assume that the part of ω on $S^2[1,5]$ consists of $2p$ arcs parallel and close to those of $\mathfrak{u}$. For example, the 2 arcs of $\mathfrak{u}$ on $S^2[1,5]$ containing U and U' are drawn in Fig. 6 by bold line, and the part of ω near them are drawn by lighter lines. Let the plane $x_1 = 3$ intersect these lighter lines (which are parts of ω) at the points P, Q, Q', P' (order from top to bottom), as in Fig. 6. We have $\psi(P) = P'$, $\psi(Q) = Q'$.

5.5. Relative skein module

Let us recall the definition of the relative skein module $\mathcal{S}(X_1; P, Q')$ (see [5]). A *type* 1 *tangle* is the disjoint union of a framed link and a framed arc in X_1 such that the parts of the arc near the two end points are on the boundary ∂X_1, and the framing on these parts are given by vectors normal to ∂X_1. Type 1 tangles are considered up to isotopy relative the endpoints.

Then $\mathcal{S}(X_1; P, Q')$ is the $\mathcal{R}$-vector space generated by type 1 tangles with endpoints at P, Q' modulo the usual skein relations, like in the definition of $\mathcal{S}(X)$. One defines in a similar way the relative Kauffman bracket skein module $\mathcal{S}(\Sigma; P, Q') := \mathcal{S}(\Sigma \times [0,1]; P, Q')$, where we identify $\Sigma \times [0,1]$ with a collar of Σ in X_1.

Since $\partial X_1 = \Sigma$, there is a natural bilinear map $\mathcal{S}(\Sigma; P, Q') \otimes \mathcal{S}(X_1) \to \mathcal{S}(X_1; P, Q')$, where $\ell \otimes \ell' \to \ell \star \ell'$, which is the disjoint union of ℓ and ℓ'.

The pair P, Q divide ω into two arcs, the one that is fully drawn in Fig. 6 (and that goes around point U) is denoted by $\mathfrak{n}_l$. Similarly, the pair P', Q' divide ω into two arcs, the one that is fully drawn in Fig. 6 (and that goes around point U') is denoted by $\mathfrak{n}_r$. For G, H among P, Q, P', Q' let $\mathfrak{d}(GH)$ be the straight interval connecting G and H, whose interior is slightly pushed inside the interior of B_1 (to avoid intersections with other arcs on the boundary ∂B_1) and whose framing is given by vectors normal to ∂B_1. Let $\mathfrak{a}_1$ be $\mathfrak{d}(P, Q')$; $\mathfrak{a}_2$ be $\mathfrak{n}_l$ followed by $\mathfrak{d}(QQ')$; $\mathfrak{a}_3$ be $\mathfrak{d}(PP')$ followed by $\mathfrak{n}_r$; and $\mathfrak{a}_4$ be $\mathfrak{n}_l$ followed by $\mathfrak{d}(QP')$ then followed by $\mathfrak{n}_r$. In all cases the framing is given by vectors normal to ∂B_1.

Using the skein relations one can simplify the arc part of elements in $\mathcal{S}(X_1; P, Q)$, showing that the arc part is one of the four $\mathfrak{a}_i$, $i = 1, 2, 3, 4$. More precisely one has the following lemma, which is a slightly different version of [5, Lemma 3.1]. The simple proof in [5] works for our version as well.

Lemma 5.2. *The relative skein $\mathcal{S}(X_1; P, Q)$ is equal to the union $\bigcup_{i=1}^4 (\mathfrak{a}_i \star \mathcal{S}(X_1))$.*

5.6. *From $\mathcal{S}(X_1)$ to $\mathcal{S}(X)$ through sliding*

5.6.1. Let $\Sigma = S^2 \setminus \{U, U', V, V'\}$. Then $\mathcal{S}(\Sigma)$ is an algebra, and $\mathcal{S}(X_1) = \mathcal{R}[x, x', y]$ is a left $\mathcal{S}(\Sigma)$-module, since $\partial X_1 = \Sigma$: For $\ell \in \mathcal{S}(\Sigma)$ and $\ell' \in \mathcal{S}(X_1)$ the action $\ell \star \ell' \in \mathcal{S}(X_1)$ is the disjoint union of ℓ and ℓ'. It is easy to see that the action descends to an action of $\mathcal{S}(\Sigma)$ on $\mathcal{R}[x, y] \equiv \mathcal{R}[x, x', y]/(x = x')$.

5.6.2. Sliding

Recall that X is obtained from X_1 by attaching a 2-handle along the curve ω. The embedding of X_1 into X gives rise to a linear map from $\mathcal{S}(X_1) \equiv \mathcal{R}[x, x', y]$ to $\mathcal{S}(X)$. It is known that the map is surjective, and its kernel $\mathcal{K}$, see [5,23], can be described through slides as follows.

Suppose $\mathfrak{a}$ is a type 1 tangle whose 2 endpoints are on ω such that outside a small neighborhood of the 2 endpoints $\mathfrak{a}$ is in the interior of X_1 and in a small neighborhood of the endpoints $\mathfrak{a}$ is on the boundary $\partial B_1 = S^2$. The two end points of $\mathfrak{a}$ divide ω into 2 arcs ω_l and ω_r. The loop ω partitions $S^2 = \partial B_1$, which is a sphere, into 2 parts; the one not containing U, U' is called the *outside one*. Let us isotope $\mathfrak{a}$ (relatively to the endpoints) to $\mathfrak{a}'$ so that in a small neighborhood of the endpoints $\mathfrak{a}'$ is in the outside part of ω.

Let $sl(\mathfrak{a})$ be $\mathfrak{a}'.\omega_l - \mathfrak{a}'.\omega_r$, considered as an element of the skein module $\mathcal{S}(X_1)$. Here $\mathfrak{a}'.\omega_l$ is the framed link obtained by combining $\mathfrak{a}'$ and ω_l. Note that $sl(\mathfrak{a})$ is defined up to a factor $\pm t^{3n}$, $n \in \mathbb{Z}$: The exchange $\omega_l \leftrightarrow \omega_r$ changes the sign, and isotopies in neighborhoods of the endpoints change the framing, which results in a factor equal to a power of $(-t^3)$.

It is clear that as framed links in X, $\mathfrak{a}'.\omega_l$ is isotopic to $\mathfrak{a}'.\omega_r$, since one is obtained from the other by sliding over the 2-handle attached to the curve ω. Hence we always have $sl(\mathfrak{a}) \in \mathcal{K}$. It was known that $\mathcal{K}$ is spanned by all possible $sl(\mathfrak{a})$, where $\mathfrak{a}$ can be chosen among all type 1 tangles with pre-given two endpoints on ω.

5.6.3. The kernel

Now we choose and fix the two end points of $\mathfrak{a}$: P and Q'; i.e. $\mathfrak{a} \in \mathcal{S}(X_1; P, Q')$. We will assume that ω_l contains $\mathfrak{n}_l$. The kernel $\mathcal{K}$ is spanned over $\mathcal{R}$ by $sl(\mathfrak{a})$, $\mathfrak{a} \in \mathcal{S}(X_1; P, Q')$. From the description of $\mathcal{S}(X_1; P, Q)$ in Lemma 5.2 we have

Lemma 5.3. *The kernel $\mathcal{K}$ is equal to the $\mathcal{R}$-span of* $\{sl(\mathfrak{a}_i) \star \mathcal{S}(X_1),\ i = 1, 2, 3, 4\}$.

We will call an element in $sl(\mathfrak{a}_i) \star \mathcal{S}(X_1)$ a *relation of type i*. Here $i = 1, 2, 3, 4$.

Lemma 5.4. *One has*

$$sl(\mathfrak{a}_1) = sl\big(\mathfrak{d}(PQ')\big), \qquad sl(\mathfrak{a}_2) = sl\big(\mathfrak{d}(QQ')\big),$$
$$sl(\mathfrak{a}_3) = sl\big(\mathfrak{d}(PP')\big), \qquad sl(\mathfrak{a}_4) = sl\big(\mathfrak{d}(P'Q)\big).$$

Proof. The first identity is a tautology. The last three follows from trivially a simple isotopy of the links involved. □

5.6.4. Simplifying the kernel

Let $\mathcal{K}'$ be the $\mathcal{R}$-span of $sl(\mathfrak{a}_1) \star \mathcal{S}(X_1)$ and $(x - x')\mathcal{S}(X_1)$.

Lemma 5.5. *For every $\ell \in \mathcal{S}(X_1)$ one has $\ell - \psi(\ell) \in (x - x')\mathcal{S}(X_1)$.*

Proof. Note that $\psi(x) = x'$, $\psi(x') = x$, and $\psi(y) = y$. Hence for any link ℓ, the skein $\psi(\ell)$, as an element in $\mathcal{S}(X_1) = \mathcal{R}[x, x', y]$, is obtained from the skein of ℓ by the involution $x \to x'$, $x' \to x$, $y \to y$. It follows that for any framed link ℓ, and one has $\ell - \psi\ell \in (x - x')\mathcal{S}(X_1) \subset \mathcal{K}'$. □

Lemma 5.6. *One has $\mathcal{K} = \mathcal{K}'$.*

Proof. First we prove that $\mathcal{K}' \subset \mathcal{K}$. Since $sl(\mathfrak{a}_1) \star \mathcal{S}(X_1)$ is already in $\mathcal{K}$, we need to show $(x - x')\,\mathcal{S}(X_1)$ is in $\mathcal{K}$. Note that $sl(\mathfrak{d}(PQ))$ is exactly $x - x'$. Moreover, for any link ℓ in the interior of X_1 one has $sl(\mathfrak{d}(PQ) * \ell) = (x - x')\ell$. Hence $(x - x')\,\ell \in \mathcal{K}$, and hence $\mathcal{K}' \subset \mathcal{K}$.

Now we prove $\mathcal{K} \subset \mathcal{K}'$.

By Lemma 5.4 one has $sl(\mathfrak{a}_3) = sl(\mathfrak{d}(PP'))$. Since both $\mathfrak{d}(PP')$ and ω is invariant under ψ, we have $\psi(\mathfrak{d}(PP).\omega_l(PP')) = \mathfrak{d}(PP).\omega_r(PP')$, where $\omega_l(PP')$ and $\omega_r(PP')$ are the two arcs of $\mathfrak{d}$ obtained by dividing ω using the two points P, P'. Hence $sl(\mathfrak{d}(PP')) \star \mathcal{S}(X_1) = (\mathfrak{d}(PP).\omega_l(PP') - \mathfrak{d}(PP).\omega_r(PP')) \star \mathcal{S}(X_1)$ is in $(x - x')\mathcal{S}(X_1) \subset \mathcal{K}'$ by Lemma 5.5. The proof that all relations of type 2 belongs to $\mathcal{K}'$ is similar.

For type 4, by Lemma 5.4 we have $sl(\mathfrak{a}_4) = sl(\mathfrak{d}(P'Q))$. Since $\psi(P) = P'$, $\psi(Q) = Q'$, one has $\psi(sl(\mathfrak{d}(P'Q))) = sl(\mathfrak{d}(PQ')) = sl(\mathfrak{a}_1)$. Hence $[sl(\mathfrak{a}_4) - sl(\mathfrak{a}_1)] \star \mathcal{S}(X_1)$ belongs to $(x - x')\mathcal{S}(X_1)$ by Lemma 5.5. Thus $sl(\mathfrak{a}_4) \star \mathcal{S}(X_1)$ is in $\mathcal{K}'$. □

5.7. Proof of Theorem 2

We have $\mathcal{S}(X) = \mathcal{R}[x, x', y]/\mathcal{K}'$. Note that $\mathcal{R}[x, x', y]/(x - x')\mathcal{S}(X_1) = \mathcal{R}[x, y]$. Hence $\mathcal{S} = \mathcal{R}[x, y]/\mathcal{K}''$, where $\mathcal{K}''$ is the $\mathcal{R}$-span of $sl(\mathfrak{a}_1) \star \mathcal{R}[x, y]$. Note that there is a natural

$\mathcal{R}[x]$-module structure on $\mathcal{S}(X)$: Here x is a meridian, thus belongs to the boundary of X. Over $\mathcal{R}[x]$, $\mathcal{R}[x, y]$ is spanned by $1, y, y^2, \dots$. Hence $\mathcal{K}''$, as a $\mathcal{R}[x]$ module, is spanned by $sl(\mathfrak{a}_1) \star y^k = (\mathfrak{a}_1.\omega_l - \mathfrak{a}_1.\omega_r) \star y^k$, $k = 0, 1, 2, \dots$.

Note that $\mathfrak{a}_1.\omega_r$ is the closure in the sense of Section 1.5 of a braid on $(p+1)$ strands, while $\mathfrak{a}_1.\omega_l$ is the closure of a braid on $(p-1)$ strands. Moreover, $\mathfrak{a}_1.\omega_r \star y^k$ is the closure of a braid on $(p+1)+2k$ strands, while $\mathfrak{a}_1.\omega_l \star y^k$ is the closure of a braid on $(p-1)+2k$ strands. Lemma 1.1 shows that $(\mathfrak{a}_1.\omega_l - \mathfrak{a}_1.\omega_r) \star y^k$, as an element of $\mathcal{R}[x, y]$, has y-degree $(p+1)/2 + k$, with highest coefficient invertible and of the form a power of t. Hence when we factor out $\mathcal{R}[x, y]$ by $\mathcal{K}''$, we get a free $\mathcal{R}[x]$-module with representatives y^l, $l = 0, 1, 2, \dots, (p-1)/2$, as a basis. This completes the proof of Theorem 2.

Remark 5.7. The same proof shows that the theorem still holds true if we replace the ground ring $\mathcal{R} = \mathbb{C}[t^{\pm 1}]$ by $\mathcal{R}_{\mathbb{Z}} := \mathbb{Z}[t^{\pm 1}]$.

Corollary 5.8. *For two-bridge knots one has $\epsilon(\mathcal{S}(X)) = R(\chi(X))$, the ring of regular functions on the character variety.*

Proof. By the result of [18], the ring $R(\chi(X))$ is $\mathbb{C}[\bar{x}, \bar{y}]/(\varphi(\bar{x}, \bar{y}))$, where $\varphi(\bar{x}, \bar{y})$ is a polynomial of $\bar{y}$-degree $(p+1)/2$, with leading coefficient 1. Here $\bar{x}$, $\bar{y}$ are respectively the traces of the loop x, y. The corollary follows immediately. □

6. Proof of Theorem 1

6.1. The peripheral polynomial of two-bridge knots

Theorem 2 about the structure of the skein module of complements of two-bridge knots is used to prove the following

Proposition 6.1. *For the two-bridge knot $K = \mathfrak{b}(p, m)$ the peripheral polynomial β_K is never* 0 *and has L-degree less than or equal to $(p+1)/2$.*

Proof. By definition one has the following exact sequence of $\mathcal{R}[x]$-modules

$$0 \to \mathcal{P} \hookrightarrow \mathcal{T}^\sigma \xrightarrow{\Theta} \mathcal{S}(X). \tag{3}$$

When $x = M + M^{-1}$, the field $\mathcal{R}(M)$ of rational functions in M is a flat $\mathcal{R}[x]$-module, since $\mathcal{R}(M)$ contains the fractional field of $\mathcal{R}[x]$ as a subfield. Hence the following sequence, which is obtained from (3) by tensoring with $\mathcal{R}(M)$, is exact

$$0 \to \mathcal{R}(M) \otimes_{\mathcal{R}[x]} \mathcal{P} \hookrightarrow \mathcal{R}(M) \otimes_{\mathcal{R}[x]} \mathcal{T}^\sigma \xrightarrow{\mathrm{id} \otimes \Theta} \mathcal{R}(M) \otimes_{\mathcal{R}[x]} \mathcal{S}(X). \tag{4}$$

Note that the first module $\mathcal{R}(M) \otimes_{\mathcal{R}[x]} \mathcal{P}$ is exactly $\tilde{\mathcal{P}}$, the left-ideal extension of $\mathcal{P}$ from $\mathcal{T}^\sigma$ to $\tilde{\mathcal{T}}$. It is easy to check that the second module $\mathcal{R}(M) \otimes_{\mathcal{R}[x]} \mathcal{T}^\sigma$ is $\tilde{\mathcal{T}}$. One can now rewrite (4) as

$$0 \to \tilde{\mathcal{P}} \hookrightarrow \tilde{\mathcal{T}} \xrightarrow{\mathrm{id} \otimes \Theta} \mathcal{R}(M) \otimes_{\mathcal{R}[x]} \mathcal{S}(X). \tag{5}$$

The third module is a finite-dimensional $\mathcal{R}(M)$-vector space; in fact, its basis is $\{y^i,\ 0 \leqslant i \leqslant (p-1)/2\}$, since $\mathcal{S}(X)$ is $\mathcal{R}[x]$-free with the same basis, by Theorem 2. The middle module $\tilde{\mathcal{T}}$ is an $\mathcal{R}(M)$-vector space of infinite dimension; in fact, its basis is $\{L^a,\ a \in \mathbb{Z}\}$. Thus the kernel $\tilde{\mathcal{P}}$ is never 0, and hence its generator β_K is not 0. Moreover the image of $(p+1)/2+1$ elements $1, L, L^2, \dots, L^{(p+1)/2}$ are linearly dependent. Hence there must be a non-trivial element in the kernel of L-degree less than or equal to $(p+1)/2$. □

Corollary 6.2. *Every two-bridge knot $\beta(p,m)$ satisfies a recurrence relation with L-degree less than or equal to $(p+1)/2$.*

This is because β_K is divisible by the recurrence polynomial α_K. Note that the existence of recurrence relations for *arbitrary* knots was established in [12] by another method. But in [12] the L-degree is much larger.

6.2. Conjecture 2 holds true for two-bridge knots

Proposition 6.3. *Suppose for a knot K the skein module $\mathcal{S}(X)$ is free over $\mathcal{R}$ and $\epsilon(\mathcal{S}(X)) = R(\chi(X))$. Then $\epsilon(\mathcal{P}) = \mathfrak{p}$, and hence $\epsilon(\beta_K) \stackrel{M}{=} B_K$.*

Proof. Consider again the exact sequence (3), but now as sequence of modules over $\mathcal{R} = \mathbb{C}[t^{\pm 1}]$, a principle ideal domain. By assumption, the last module $\mathcal{S}(X)$ is free over $\mathcal{R}$. Hence when tensoring (3) with any $\mathcal{R}$-module, one gets an exact sequence. In particular, tensoring with $\mathbb{C}$, considered as $\mathcal{R}$-module by putting $t=-1$, one has the exact sequence

$$0 \to \epsilon(\mathcal{P}) \hookrightarrow \epsilon\big(\mathcal{T}^\sigma\big) \xrightarrow{\epsilon(\Theta)} \epsilon\big(\mathcal{S}(X)\big).$$

Notice that $\epsilon(\mathcal{T}^\sigma) = \mathfrak{t}^\sigma$, $\epsilon(\mathcal{S}(X)) = R(\chi(X))$, and $\epsilon(\Theta) = \theta$. Thus $\mathfrak{p}$, being the kernel of θ, is equal to $\epsilon(\mathcal{P})$. The second statement follows from Proposition 4.2. □

From Theorem 2, Corollary 5.8 and Proposition 6.3 we get

Theorem 3. *Conjecture* 2 *holds true for two-bridge knots*: $\epsilon(\mathcal{P}) = \mathfrak{p}$ *and* $\epsilon(\beta_K) = B_K$.

6.3. Proof of Theorem 1

(a) is Proposition 6.1.

(b) One has $\epsilon(\beta_K) \stackrel{M}{=} B_K$ by Theorem 3. Thus the algebraic set $\{\epsilon(\beta_K)=0\} \stackrel{M}{=} \{B_K=0\}$ is M-essentially equal to $\{A'_K=0\}$, by Proposition 3.1. Applying ϵ to (2) we get

$$\epsilon(\beta_K) \stackrel{M}{=} \epsilon(\gamma)\epsilon(\alpha_K), \tag{6}$$

which means $\epsilon(\beta_K)$ is M-divisible by $\epsilon(\alpha_K)$. Hence $\{\epsilon(\alpha_K)=0\}$ is M-essentially an algebraic subset of $\{\epsilon(\beta_K)=0\} \stackrel{M}{=} \{A'_K=0\}$.

(c) Suppose A_K has L-degree $(p-1)/2$. Then $A'_K = (L-1)A_K$ has L-degree $(p+1)/2$. By Proposition 6.1, β_K has L-degree less than or equal to $(p+1)/2$. But $\epsilon(\beta_K)$ is M-essentially divisible by A'_K. Hence the L-degree of β_K must be exactly $(p+1)/2$, and also

$$\epsilon(\beta_K) \overset{M}{=} (L-1)A_K.$$

Combining with (6) we have

$$A_K \overset{M}{=} \epsilon(\gamma)\frac{\epsilon(\alpha_K)}{L-1}. \tag{7}$$

Recall that $\frac{\epsilon(\alpha_K)}{L-1}$ is a polynomial by Proposition 2.3. From (2) and the fact that the L-degree of α_K is bigger than 1 (Proposition 2.2) it follows that the L-degree of γ is less than $(p-1)/2$, which is the L-degree of A_K. Hence if A_K is $\mathbb{Z}$-irreducible, from (7) one must have $\epsilon(\gamma) \overset{M}{=} 1$ and $\frac{\epsilon(\alpha_K)}{L-1} \overset{M}{=} A_K$. This completes the proof of Theorem 1.

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